Perfect Matchings in Edge-Transitive Graphs

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ABSTRACT. We find recursive formulae for the number of perfect matchings in a graph G by splitting G into subgraphs H and Q. We use these formulas to count perfect matching of P hypercube Q_n. We also apply our formulas to prove that the number of perfect matching in an edge-transitive graph is $Pm(G) = (2q/p)Pm(G\backslash\{u,v\})$, where $Pm(G)$ denotes the number of perfect matchings in G, $G\backslash\{u,v\}$ is the graph constructed from G by deleting edges with an end vertex in $\{u,v\}$ and $uv \in E(G)$.

Keywords: Perfect matching, edge–transitive graph.

1. INTRODUCTION

Counting the number of perfect matchings in a graph is a much-studied topic in graph theory. Some perfect matching enumeration methods are algebraic and others use matrix theory [3–8]. Enumerating the number of perfect matchings and making algorithms by these methods are really complicated (see [3]), but in some special cases it is possible to find good algorithms. In this paper we derive a recursive formula for counting perfect matchings in arbitrary graphs.

All graphs considered in this paper will be simple and connected. Let $V(G)$ and $E(G)$ be the sets of vertices and edges of the graph G, respectively. The degree of a vertex $u \in V(G)$ is denoted by $\deg(u)$ and the set of all neighbors of u is denoted by $N(u)$. A *matching* in G is a collection M of edges of G such that none of edges in M has a vertex in common. If every vertex from $V(G)$ is incident with exactly one edge from M, the matching M is *perfect*. The number of perfect matchings in a given graph G is denoted by $Pm(G)$. A distance between two vertices $u, v \in V(G)$ in a graph G is denoted by $d(u, v)$, the number of edges in a shortest path connecting them.

A graph G is called edge-transitive if the group of automorphisms acts transitively on the edges. In other words, for each edge $e, w \in E(G)$, an automorphism g exists such that $g(e) = w$.

The hypercube Q_n is recursively defined as:

$$Q_n = \begin{cases} K_2 & n = 1 \\ Q_{n-1} \times K_2 & n = 2 \end{cases}$$

Suppose $T = \{e_1, \ldots, e_t\}$ is a set of edges in G. T separates G into H and Q ,*i.e.* $G = H \overline{\underline{T}} Q$ if by omitting T, G separate to subgraphs H and Q. In figure 1, T is the set of edges that separate $G = H \overline{\underline{T}} Q$.

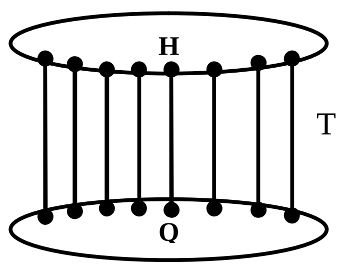

Figure 1. The Graph G is separated into H and Q by T.

2. MAIN RESULTS

We start this section by finding a recursive formula for the number of perfect matching in a graph G that can be separated into subgraphs H and Q.

Theorem 1. Suppose $G = H \overline{\underline{T}} Q$. Then,

$$Pm(G) = Pm(H) * Pm(Q) + \sum_{t=1}^{|T|} \sum_{\substack{u_1v_1,\ldots,u_tv_t \in T \\ u_i \neq u_j \quad i \neq j \\ v_i \neq v_j \quad i \neq j}} Pm(H \backslash v_1, \ldots, v_t) * Pm(Q \backslash u_1, \ldots, u_t)$$

Proof. We define the relation « ~ » as follows:

$M_1 \sim M_2$ if and only if $|M_1 \cap T| = |M_2 \cap T|$ where M_1 and M_2 are two erfect matchings of G.Therefore, ~ is an equivalence relation. For the set of all perfect matchings of G we have:

$$M(G) = \cup_{M\in M(G)}[M]_{\sim} = \cup_{i=1}^{t}[M_i]_{\sim} \implies Pm(G) = |M(G)| = |\cup_{i=1}^{t}[M_i]_{\sim}|,$$

where $M(G)$ is the set of all perfect matchings of G. Since $\sim$ is an equivalence relation, one can decompose $M(G)$ into equivalence classes of $\sim$, i.e.

$$|\cup_{i=1}^{t}[M_i]_{\sim}| = \sum_{i=1}^{t}|[M_i]_{\sim}| = Pm(G).$$

We also have,

$$\sum_{i=1}^{t}|[M_i]_{\sim}| = \sum_{j=0}^{|T|}|\{M: |M\cap T| = j\}|.$$

We know that for every $0 \le k \le |T|$:

$$|\{M: |M\cap T| = k\}| = \sum_{\substack{u_1v_1,\dots,u_kv_k\in T \\ u_i\neq u_j \quad i\neq j \\ v_i\neq v_j \quad i\neq j}} Pm(H\backslash v_1,\dots,v_k) * Pm(Q\backslash u_1,\dots,u_k).$$

where $v_1,\dots,v_k \in V(H)$ and $u_1,\dots,u_k \in V(Q)$. This completes the proof. ■

Corollary 1. For an arbitrary graph G, the number of perfect matchings of $G\times K_2$ is:

$$Pm(G\times K_2) = Pm^2(G) + \sum_{uv\in T} Pm^2(G\backslash u) + \sum_{u_1v_1,u_2v_2\in T} Pm^2(G\backslash u_1,u_2) + \cdots$$
$$+ \sum_{u_1v_1,\dots,u_tv_t\in T} Pm^2(G\backslash u_1,\dots,u_t)$$

where $G\times K_2 = G\,\overline{\underline{T}}\,G$.

Proof. It is sufficient to put $H = Q = G$ in Theorem 1. ■

Example 1. Apply Theorem 1 to count the number of perfect matchings of Q_3 and Q_4. Harary and Graham in [3] resolved this problem by using an algebraic method.

$$Pm(Q_3) = Pm^2(Q_2) + \sum_{u_1v_1,u_2v_2\in T} Pm^2(Q_2\backslash\{u_1,u_2\}) + 1. \quad (1)$$

To compute $Pm(Q_3)$, we have to compute $Pm(Q_2)$ and $Pm(Q_2\backslash\{u_1,u_2\})$. $Pm(Q_2) = 2$ and to compute $Pm(Q_2\backslash u_1,u_2)$, we omit two vertices of Q_2, resulting in, two remained vertices. If these vertices are neighbors, there will be <u>one</u> perfect matching, otherwise, there will be no perfect matching. Hence, $Pm(Q_3) = 4 + 4 + 1 = 9$.

To compute

$$Pm(Q_4) = Pm(Q_3\times K_2) = Pm^2(Q_3) + \sum_{u_1v_1,u_2v_2\in T} Pm^2(Q_3\backslash u_1,u_2)$$
$$+ \sum_{u_1v_1,\dots,u_4v_4\in T} Pm^2(Q_3\backslash u_1,\dots,u_4) + \sum_{u_1v_1,\dots,u_6v_6\in T} Pm^2(Q_3\backslash u_1,\dots,u_6) + 1.$$

We apply a similar argument and computation and we can compute $Pm(Q_4)$ as follows: $Pm(Q_4) = 81 + 4*4 + 9*12 + 4*6 + 2*12 + 6 + 12 + 1 = 272$.

Theorem 2. Suppose G is an arbitrary graph and $e = uv$ is its edge $(u, v \in V(G))$. Then the number of perfect matchings of G is:

$$Pm(G) = Pm(G\backslash u, v) + \sum_{\substack{w \in N(u) \\ z \in N(V) \\ z \neq w}} Pm(G\backslash u, v, z, w).$$

Proof. Suppose T is a set of edges that each element of T has just one head in $\{u, v\}$, as depicted in Figure 2.

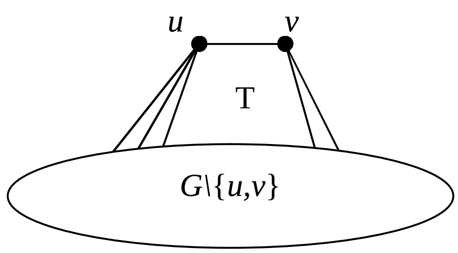

Figure 2: Separating G to e and $G\backslash\{u, v\}$.

Then $G = \{e\} \overline{\underline{T}} G\backslash\{u, v\}$, and therefore theorem 1 implies that the number of perfect matchings of G is $Pm(G\backslash u, v) + \sum_{\substack{w \in N(u) \\ z \in N(V) \\ z \neq w}} Pm(G\backslash u, v, z, w)$. ■

Now we're changing the graph H in Theorem 1 to a vertex of G.

Theorem 3. Suppose G is an arbitrary graph. The number of perfect matchings of G can be computed with omitting a vertex of G.

Proof. In Theorem 1, put H a vertex of G likes v i.e. $H \coloneqq \{v\}$. Then the number of perfect matchings of G is $Pm(G) = \sum_{u \in N(v)} Pm(G\backslash u, v)$. ■

This theorem is more practical than Theorem 1. We use this theorem to find a necessary condition for edge–transitive graphs using two lemmas.

Lemma 1. Suppose G and H are isomorphic. Then $Pm(G) = Pm(H)$.

Proof. Assume $f: G \to H$ is an isometric function and M is a perfect matching of G. Then $f(M)$ is a perfect matching of H because $V(f(G)) = V(H)$ and since f is an isometric function then the edges of $f(M)$ cover all vertices of H, therefore, $f(M)$ is a perfect matching of H. Similarly, the pre image of each perfect matching of H is a perfect matching of G. So, the numbers of perfect matchings of isomorphic graphs are the same. ■

Lemma 2. Suppose that G is an edge–transitive graph and $e = uv \in E(G)$ then:

$$PM(G) = \frac{2q}{p} Pm(G \backslash u, v)$$

where $p = |V(G)|$, $q = |V(G)|$.

Proof. Because G is an edge-transitive graph, omitting each one of edges with its vertices makes isomorphism graphs .Using lemma 1, the graphs $G \backslash u, v$ that $uv \in E(G)$ have the same number of perfect matchings. Now we're counting the number of (M, e) where M is a perfect matching of G and e belongs to edges of M. Counting can be done in two ways. One way is counting the number of perfect matchings of G which is $Pm(G)$, then counting its edges that is $p/2$. Another way is counting the number of edges of G, which is q, then counting the number of perfect matchings that has a special edge, which is x .The number of perfect matchings that has a special edge are the same for each edge. Assume that $e = uv$ is an edge of G then $x = Pm(G \backslash u, v)$. So $\frac{p}{2} * Pm(G) = q * Pm(G \backslash u, v)$. Hence $Pm(G) = \frac{2q}{p} Pm(G \backslash u, v)$. ■

Here we prove a necessary condition for edge-transitive graphs:

Theorem 4. Suppose G is an edge-transitive graph and $v \in V(G)$ then: $\deg(v) \,|Pm(G)$.

Proof. Suppose G is an edge–transitive graph and $v \in V(G)$ then:

$$Pm(G) = \sum_{u \in N(v)} Pm(G \backslash u, v)$$

Assume w and u are neighbors of v and $e = uv, q = vw$. Since G is an edge-transitive graph so there is an isometric function $g: G \to G$ such that $g(e) = q$ .It means that $g(u) = v$ and $g(v) = w$, and therefore, $g(G \backslash u, v) = G \backslash v, w$. So, $Pm(G \backslash u, v) = Pm(G \backslash v, w)$ so theorem 3 concludes that $Pm(G) = deg(v) * Pm(G \backslash u, v)$. ■

Corollary 2. If G is an edge–transitive graph and $V(G) = \{v_1, \dots, v_n\}$ then

$$\left[\deg(v_1), \dots, \deg(v_n), \frac{2q}{(2q, 2q - p)}\right] |Pm(G)$$

where $p = |V(G)|$, $q = |V(G)|$.

Proof. Assume that $e = uv \in E(G)$ then by theorem 2 and lemma 2 we have:

$$Pm(G) = \frac{2q}{2q-p} \sum_{\substack{w \in N(u) \\ z \in N(v) \\ z \neq w}} Pm(G \backslash u, v, z, w)$$

By applying Theorem 4,

$$\deg(v), \frac{2q}{(2q, 2q-p)} | Pm(G) \quad \forall v \in V(G)$$

Hence $\left[\deg(v_1), \dots, \deg(v_n), \frac{2q}{(2q, 2q-p)}\right] | Pm(G)$. ■

Corollary 3. If there is a vertex in graph G such that $\deg(v) \nmid Pm(G)$ then G is not edge–transitive.

Proof. By Theorem 4, it is straightforward. ■

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REFERENCES

1. L. Lovasz and M. D. Plummer, Matching theory, North–Holland Mathematics Studies, 121, *Annals of Discrete Mathematics*, **29**. *North–Holland Publishing Co., Amsterdam; Akadémiai Kiadó* (*Publishing House of the Hungarian Academy of Sciences*), *Budapest*, 1986.
2. N. Graham and F. Harary, The number of perfect matching in a hypercube, *Appl. Math. Lett.* **1** (1998) 45–48.
3. W. Yan and F. Zhang, Enumeration of perfect matching of graphs with reflective symmetry by pfafians, *Adv. Appl. Math.* **32** (2004) 655–668.
4. A. Behmaram and S. Friedland, Upper bounds for perfect matchings in pfafian and planar graphs, *Electron. J. Combin.* **20** (1) (2013) paper 64, 16pp.
5. M. Ciucu, Enumeration of perfect matchings in graphs with reflective symmetry, *J. Combin. Theory, Ser. A* **77** (1997) 67–97.

6. B. N. Cyvin and S. J. Cyvin, Enumeration of Kekule structures: pentagon–shaped benzenoids–part IV: seven–tier pentagons and related classes, *MATCH Commun.Math. Comput. Chem.* **22** (1987) 157–173.
7. J. Qian and F. Zhang, On the number of Kekule structures in capped zig–zag nanotubes, *J. Math. Chem.* **38** (2005) 233–246.
8. W. Yan, Y. Yeh and F. Zhang, Dimer problem on the cylinder and torus, *Physica A* **387** (2008) 6069–6078.