The Laplacian Polynomial and Kirchhoff Index of the k–th Semi Total Point Graphs

ZEINAB MEHRANIAN•

(COMMUNICATED BY ALI REZA ASHRAFI)

Department of Mathematics, University of Qom, Qom, I. R. Iran

ABSTRACT. The k–th semi total point graph of a graph G, $R^k(G)$, is a graph obtained from G by adding k vertices corresponding to each edge and connecting them to the endpoints of edge considered. In this paper, a formula for Laplacian polynomial of $R^k(G)$ in terms of characteristic and Laplacian polynomials of G is computed, where G is a connected regular graph. The Kirchhoff index of $R^k(G)$ is also computed.

Keywords: Resistance distance, Kirchhoff index, Laplacian spectrum, derived graph.

1. INTRODUCTION

Let $G = (V(G), E(G))$ be a simple connected (n,m)–graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \dots, e_n\}$. The adjacency and incidence matrices of G are denoted by $A(G)$ and $B(G)$, respectively. The eigenvalues $\lambda_1(G) \ge \lambda_2(G) \ge \dots \ge \lambda_n(G)$ of G are the eigenvalues of $A(G)$. Let d_i be the degree of vertex $v_i \in V(G)$ and $D(G) = diag(d_1, d_2, \dots, d_n)$ be the diagonal matrix of G. The matrix $L(G) = D(G) - A(G)$ is called the Laplacian matrix of G and its eigenvalues are called the Laplacian eigenvalues of G. By a well–known result in algebraic graph theory it is possible to order the Laplacian eigenvalues of G as $\mu_1(G) \ge \mu_2(G) \ge \dots \ge \mu_n(G) = 0$. Also, the polynomials $\phi_G(\lambda) = det(\lambda I_n - A(G))$ and $\mu_G(\lambda) = det(\lambda I_n - L(G))$ are called the characteristic and Laplacian polynomials of G, respectively. Moreover, the distance between vertices v_i and v_j, denoted by d_{ij}, is the length of a shortest path connecting them. The Wiener index is the first graph

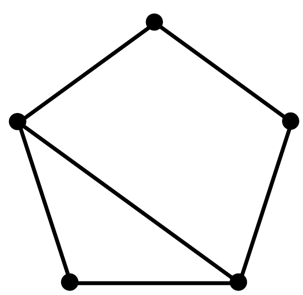
• E-mail: mehranian.z@gmail.com

Received: June 4, 2014; Accepted: November 1, 2014.

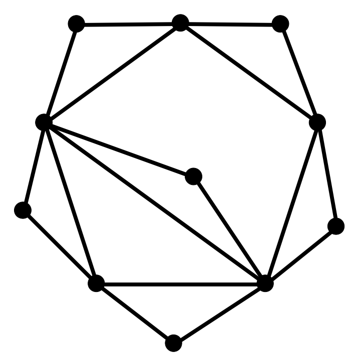
invariant applicable in chemistry based on distance in a graphs [10], which counts the sum of distances between pairs of vertices in the graph.

In 1993, Klein and Randić defined a new distance function named resistance distance in terms of electrical network theory [6]. If v_i and v_j are vertices of G then the resistance distance between these vertices are denoted by r_{ij}. This new distance is an effective resistance between nodes v_i and v_j according to Ohm's law. Notice that all the edges of G are considered to be unit resistors. The summation of all resistance distances between pair of vertices, $Kf(G)=\sum_{i<j} r_{ij}$, is called the Kirchhoff index of G [1].

Suppose $R(G)$ denotes a graph constructed from G by adding a new vertex corresponding to each edge and connecting it to the endpoints of edge considered. This graph is called the semi total point graph. In Figure 1, a graph G and its semi total graph are depicted. Jog et al. [5], introduced a k–step generalization of $R(G)$, denoted by $R^k(G)$. To define, we assume that G is a simple graph of order n possessing m edges and k is a natural number. The $k-$th semi total point graph of G, denoted by $R^k(G)$, is the graph obtained by adding k vertices to each edge of G and joining them to the endpoints of the respective edge. Obviously, this is equivalent to adding k triangle to each edge of G. Clearly, this graph has order $n + mk$ containing $(1 + 2k)m$ edges. In Figure 2, the graphs G and $R^3(G)$ are depicted.



(a)



(b)

Figure 1. (a) The Graph G. (b) The Graph R(G).

2. THE LAPLACIAN POLYNOMIAL OF $R^k(G)$

Let G be a regular graph. In [9], the Laplacian polynomial $R(G)$ is determined by the characteristic and the Laplacian polynomials of G. The characteristic polynomial of $R^k(G)$ calculated in [5]. In this section, we use a similar method to calculate the

Laplacian polynomial of $R^k(G)$, for $k \geq 2$. The following two results are crucial throughout this paper.

Theorem 1. ([5]) If G is a regular graph of order n and degree r, then for any $k \geq 1$, the characteristic polynomial of the $k-$th semi total pointgraph $R^k(G)$ is given by $\phi(R^k(G),\lambda) = \lambda^{mk-n}(\lambda+k)^n \phi(G, \frac{\lambda^2 - kr}{\lambda + k})$,

where $m = \frac{nr}{2}$ is the number of edges of G.

Lemma 2. ([2]) Let M be a non–singular square matrix. Then

$$\det \begin{pmatrix} M & N \\ P & Q \end{pmatrix} = \det M \det(Q - PM^{-1}N).$$

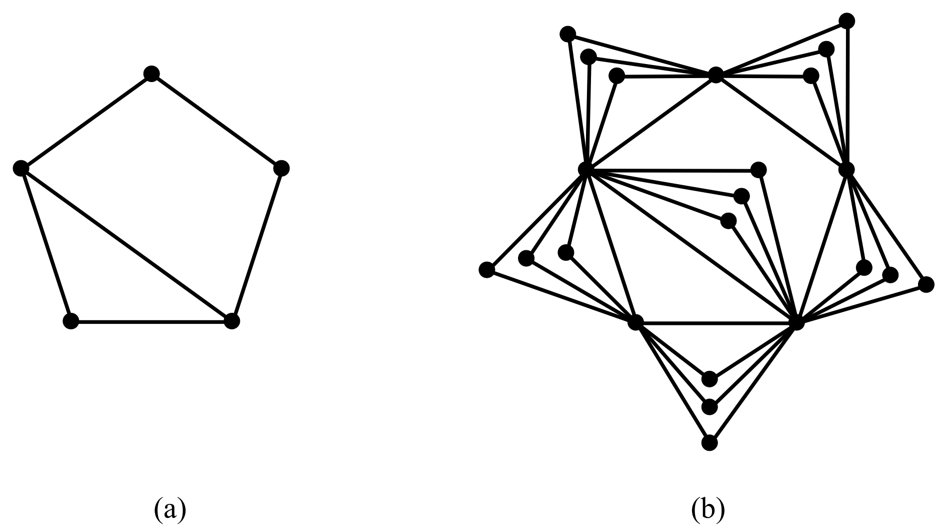


Figure 2. (a) The Graph G. (b) The k–th Semi Total Point Graph for k = 3.

Theorem 3. Let G be a connected $r-$regular graph with n vertices and m edges. Then

(i) $\mu_{R^k(G)}(\lambda) = (\lambda - 2)^{mk-n}(k+2-\lambda)^n \phi_G(\frac{\lambda^2 - \lambda(kr+r+2) + r(k+2)}{k+2-\lambda})$.

(ii) $\mu_{R^k(G)}(\lambda) = (\lambda - 2)^{mk-n}(\lambda - k - 2)^n \mu_G(\frac{\lambda^2 - \lambda(kr+2)}{\lambda - k - 2})$.

Proof. (i). Let $A(G)$ and $B(G)$ be the adjacency and incidence matrices of G, respectively, and I_n be a unit matrix of order n. By [5], the adjacency and distance matrices of $R^k(G)$ can be computed as follows:

$$A(R^k(G)) = \begin{pmatrix} 0_{mk} & \Gamma^t \\ \Gamma & A(G) \end{pmatrix} \quad ; \quad D(R^k(G)) = \begin{pmatrix} 2I_{mk} & 0 \\ 0 & ((k+1)r)I_n \end{pmatrix},$$

where $\Gamma = (\underbrace{B(G), B(G), \dots, B(G)}_{k \ times})$ and $\Gamma\Gamma^t = kA(G) + krI_n$. Then we have:

$$L(R^k(G)) = \begin{pmatrix} 2I_{mk} & -\Gamma^t \\ -\Gamma & (kr+r)I_n - A(G) \end{pmatrix}.$$

So,

$$\begin{aligned}
\mu_{R^k(G)}(\lambda) &= \det\begin{pmatrix} (\lambda-2)I_{mk} & \Gamma^t \\ \Gamma & (\lambda-kr-r)I_n + A(G) \end{pmatrix} \\
&= (\lambda-2)^{mk} \det\left((\lambda-kr-r)I_n + A(G) - \Gamma\frac{I_{mk}}{\lambda-2}\Gamma^t \right) \\
&= (\lambda-2)^{mk} \det\left((\lambda-kr-r)I_n + A(G) - \frac{kA(G)+krI_n}{\lambda-2} \right) \\
&= (\lambda-2)^{mk} \det\left(\frac{(\lambda-2)(\lambda-kr-r)I_n + (\lambda-2)A(G) - kA(G) - krI_n}{\lambda-2} \right) \\
&= (\lambda-2)^{mk-n} \det\Big(\big((\lambda-2)(\lambda-kr-r)-kr\big)I_n - A(G)(k+2-\lambda)\Big) \\
&= (\lambda-2)^{mk-n}(k+2-\lambda)^n \det\left(\frac{(\lambda-2)(\lambda-kr-r)-kr}{k+2-\lambda} I_n - A(G) \right). \qquad (1)
\end{aligned}$$

Thus,

$$\mu_{R^k(G)}(\lambda) = (\lambda-2)^{mk-n}(k+2-\lambda)^n \phi_G\left(\frac{\lambda^2-\lambda(kr+r+2)+r(k+2)}{k+2-\lambda}\right).$$

(ii). By considering $L(G) = D(G) - A(G)$ in (1), we have:

$$\begin{aligned}
\mu_{R^k(G)}(\lambda) &= (\lambda-2)^{mk-n}(\lambda-k-2)^n \det\left(\frac{\lambda^2-\lambda(kr+r+2)+r(k+2)}{\lambda-k-2} I_n + A(G)\right) \\
&= (\lambda-2)^{mk-n}(\lambda-k-2)^n \det\left(\frac{\lambda^2-\lambda(kr+2)}{\lambda-k-2} I_n - (rI_n - A(G))\right).
\end{aligned}$$

So, $\mu_{R^k(G)}(\lambda) = (\lambda-2)^{mk-n}(\lambda-k-2)^n \mu_G\left(\frac{\lambda^2-\lambda(kr+2)}{\lambda-k-2}\right)$, and the proof is completed. ■

3. THE KIRCHHOFF INDEX OF $R^k(G)$

In this section, we will compute the Kirchhoff index of $R^k(G)$, G is regular, by using the results obtained in the previous section. Gutman and Mohar [4] and Zhu [12] proved the following relationship between the Kirchhoff and the Laplacian eigenvalues of a graph:

Lemma 4. ([4, 12]). Let G be a connected graph with $n \geq 2$ vertices. Then

$$Kf(G) = n\sum_{i=1}^{n-1}\frac{1}{\mu_i}.$$

Let δ_i be the degree of vertex $v_i \in V(G)$. Zhou and Trinajstić [11] proved that:

Lemma 5. Let G be a connected graph with $n \geq 2$ vertices. Then

$$Kf(G) \geq -1 + (n-1)\sum_{v_i \in V(G)}\frac{1}{\delta_i}$$

with equality attained if and only if $G \cong K_n$ or $G \cong K_{t,n-t}$ for $1 \leq t \leq \left\lfloor \frac{n}{2} \right\rfloor$.

Gao, Luo and Liu in [3] obtained the Kirchhoff index of a graph G in terms of coefficients of the Laplacian polynomials as follows:

Lemma 6. [3]. Let G be a connected graph with $n \geq 2$ vertices and $\mu_G(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda$. Then

$$\frac{Kf(G)}{n} = -\frac{a_{n-2}}{a_{n-1}} (a_{n-2} = 1 \quad whenever \ \ n = 2).$$

Theorem 7. Let G be a connected $r-$regular graph with n vertices. Then

$$Kf(R^k(G)) = \frac{(kr+2)^2}{2(k+2)}Kf(G) + \frac{(n^2-n)(kr+2)}{2(k+2)} + \frac{n^2(k^2r^2-4)}{8} + \frac{n}{2}.$$

Proof. Suppose that $\mu_G(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-2}\lambda^2 + a_{n-1}\lambda.$ Then by Theorem 3 (ii),

$$\mu_{R^k(G)}(\lambda) = (\lambda-2)^{mk-n}(\lambda-k-2)^n \times \left[\left(\frac{\lambda^2-\lambda(kr+2)}{\lambda-k-2}\right)^n + \ldots \right.$$
$$\left. + a_{n-2}\left(\frac{\lambda^2-\lambda(kr+2)}{\lambda-k-2}\right)^2 + a_{n-1}\left(\frac{\lambda^2-\lambda(kr+2)}{\lambda-k-2}\right)\right]$$
$$= (\lambda-2)^{mk-n}\left[(\lambda^2-\lambda(kr+2))^n + \ldots + a_{n-2}(\lambda^2-\lambda(kr+2))^2(\lambda-k-2)^{n-2}\right.$$
$$\left. + a_{n-1}(\lambda^2-\lambda(kr+2))(\lambda-k-2)^{n-1}\right].$$

Suppose that C_μ^1 and C_μ^2 are the coefficients of λ and λ^2 in $\mu_{R^k(G)}$, respectively. Then,

$$C_\mu^1 = (-2)^{mk-n}a_{n-1}(-(kr+2))(-(k+2))^{n-1},$$
$$C_\mu^2 = (-2)^{mk-n}\left[a_{n-2}(kr+2)^2(-(k+2))^{n-2} + a_{n-1}(-(k+2))^{n-1}\right.$$
$$\left. + a_{n-1}(-(kr+2))(n-1)(-(k+2))^{n-2}\right]$$
$$+ (-2)^{mk-n-1}(mk-n)a_{n-1}(-(kr+2))(-(k+2))^{n-1}.$$

By Lemmas 4 and 6, we have:

$$\frac{Kf(R^k(G))}{n+mk} = -\frac{C_\mu^2}{C_\mu^1} = -\frac{a_{n-2}}{a_{n-1}}\cdot\frac{kr+2}{k+2} + \frac{1}{kr+2} + \frac{n-1}{k+2} + \frac{mk-n}{2}.$$

So,

$$Kf(R^k(G)) = -\frac{a_{n-2}}{a_{n-1}}\cdot\frac{(kr+2)(n+mk)}{(k+2)} + \frac{n+mk}{kr+2} + \frac{(n-1)(n+mk)}{k+2}$$
$$+ \frac{m^2k^2-n^2}{2}$$
$$= \frac{(kr+2)(n+mk)}{n(k+2)}Kf(G) + \frac{n+mk}{kr+2} + \frac{(n-1)(n+mk)}{k+2}$$
$$+ \frac{m^2k^2-n^2}{2},$$

Now by substituting $m = \frac{nr}{2}$ in the above equation the proof is completed. ■

In what follows, we give a lower bound for the Kirchhoff index of $R^k(G)$, when G is a connected regular graph.

Corollary 8. Let G be a r–regular graph with n vertices. Then,

$$Kf(R^k(G)) \geq \frac{(kr+2)^2(n^2-n-r)}{2r(k+2)} + \frac{(n^2-n)(kr+2)}{2(k+2)} + \frac{n^2(k^2r^2-4)}{8} + \frac{n}{2}$$

with equality attained if and only if $G \cong K_n$ or $G \cong K_{\frac{n}{2},\frac{n}{2}}$ and n is even.

Proof. By Lemma 5 and Theorem 7, we have:

$$\begin{aligned} Kf(R^k(G)) &\geq \frac{(kr+2)^2}{2(k+2)}\left(-1+\frac{n(n-1)}{r}\right) + \frac{(n^2-n)(kr+2)}{2(k+2)} + \frac{n^2(k^2r^2-4)}{8} + \frac{n}{2} \\ &= \frac{(kr+2)^2(n^2-n-r)}{2r(k+2)} + \frac{(n^2-n)(kr+2)}{2(k+2)} + \frac{n^2(k^2r^2-4)}{8} + \frac{n}{2}, \end{aligned}$$

proving the result. Clearly, this equality is attained if and only if $G \cong K_n$ or $G \cong K_{n/2,n/2}$ and n is even. ■

4. EXAMPLES

The aim of this section is to compute the Kirchhoff index of k–th semi total point special connected regular graphs.

Example 9. *The complete graph* K_n, $n \geq 2$. It is well known that K_n is $(n-1)$–regular and $Kf(K_n) = n-1$. Hence,

$$\begin{aligned} Kf(R^k(K_n)) &= \frac{(k(n-1)+2)^2}{2(k+2)}Kf(K_n) + \frac{(n^2-n)(k(n-1)+2)}{2(k+2)} \\ &\quad + \frac{n^2(k^2(n-1)^2-4)}{8} + \frac{n}{2} \\ &= \frac{k^2(n-1)^3 + k(n-1)^2(n+4) + 2(n-1)(n+2)}{2(k+2)} + \frac{k^2(n^2-n)^2 - 4n(n-1)}{8}. \end{aligned}$$

Example 10. *The complete bipartite graph* $K_{n,n}$. It is well known that $K_{n,n}$ is n–regular graph with $2n$ vertices. By [3], $Kf(K_{n,n}) = 4n-3$, and so

$$\begin{aligned} Kf(R^k(K_{n,n})) &= \frac{(kn+2)^2}{2(k+2)}Kf(K_{n,n}) + \frac{((2n)^2-(2n))(kn+2)}{2(k+2)} + \frac{(2n)^2(k^2n^2-4)}{8} + \frac{2n}{2} \\ &= \frac{(kn+2)((kn+2)(4n-3)+4n^2-2n)}{2(k+2)} + \frac{4n^2(k^2n^2-4)+8n}{8}. \end{aligned}$$

Example 11. *The cycle* C_n. By [8] $Kf(C_n) = \frac{n^3-n}{12}$ and so,

$$Kf(R^k(C_n)) = \frac{(2k+2)^2}{2(k+2)}Kf(C_n) + \frac{(n^2-n)(2k+2)}{2(k+2)} + \frac{n^2(4k^2-4)}{8} + \frac{n}{2}$$
$$= \frac{(k+1)^2(n^3-n)}{6(k+2)} + \frac{(k+1)(n^2-n)}{k+2} + \frac{n^2(k^2-1)+n}{2}.$$

Example 12. *The hypercube* Q_n. In [7], Liu et al. proved that Q_n is n–regular graph with 2^n vertices and $Kf(Q_n) = 2^n \sum_{i=1}^{n} \frac{C_n^i}{2i}$, where $2i$ with multiplicities C_n^i, $1 \le i \le n$, are the eigenvalues of the Laplacian matrix of the hypercube. Here, $C_n^i, 1 \le i \le n$, denotes the binomial coefficients. Hence,

$$Kf(R^k(Q_n)) = \frac{(kn+2)^2}{2(k+2)}Kf(Q_n) + \frac{(2^n(2^n-1))(kn+2)}{2(k+2)} + \frac{(2^n)^2(k^2n^2-4)}{8} + \frac{2^n}{2}$$
$$= 2^{n-1}\frac{(kn+2)^2}{(k+2)}\sum_{i=1}^{n}\frac{C_n^i}{2i} + \frac{2^{n-1}(2^n-1)(kn+2)}{k+2} + \frac{2^{2n}(k^2n^2-4)+2^{n+2}}{8}.$$

Example 13. *The cocktail–party graph* $CP(n)$. The cocktail–party graph $CP(n)$ is an (2n–2)–regular graph with 2n vertices and $Kf(CP(n)) = \frac{n^2+(n-1)^2}{n-1}$. This shows that,

$$Kf(R^k(CP(n))) = \frac{((2n-2)k+2)^2}{2(k+2)}Kf(CP(n)) + \frac{(2n(2n-1))((2n-2)k+2)}{2(k+2)}$$
$$+ \frac{(2n)^2(k^2(2n-2)^2-4)}{8} + \frac{2n}{2}$$
$$= \frac{((2n-2)k+2)^2}{2(k+2)} \cdot \frac{n^2+(n-1)^2}{n-1} + \frac{(2n(2n-1))((2n-2)k+2)}{2(k+2)}$$
$$+ 2n^2(k^2(n-1)^2-1)+n,$$

which completes our argument.

ACKNOWLEDGMENT. I am very pleased from the language editor of IJMC for several corrections on my paper.

REFERENCES

1. D. Bonchev, A. T. Balaban, X. Liu and D. J. Klein, Molecular cyclicity and centricity of polycyclic graphs, I: cyclicity based on resistances or reciprocal distances, *Int. J. Quantum Chem.* **50** (1994) 1–20.
2. F. R. Gantmacher, Theory of Matrices I, *Chelsea, New York*, (1960).

3. X. Gao, Y. F. Luo and W. W. Liu, Kirchhoff index in line, subdivision and total graphs of a regular graph, *Discrete Appl. Math*. **160** (2012) 560–565.
4. I. Gutman and B. Mohar, The quasi–Wiener and the Kirchhoff indices coincide, *J. Chem. Inf. Comput. Sci*. **36** (1996) 982–985.
5. S. R. Jog, S. P. Hande, I. Gutman and B. Bozkurt, Derived graphs of some graphs, *Kragujevac J. Math.* **36** (2012) 309–314.
6. D. J. Klein, M. Randić, Resistance distance, *J. Math. Chem.* **12** (1993) 81–95.
7. J. Liu, J. Cao, X.–F. Pan and A. Elaiw, The Kirchhoff index of hypercubes and related complex networks, *Discrete Dyn. Nat. Soc.* DOI:10.1155/2013/543189.
8. I. Lukovits, S. Nikolić and N. Trinajstić, Resistance distance in regular graphs, *Int. J. Quant. Chem.* **71** (1999) 217–225.
9. W. Wang, D. Yang and Y. Luo, The Laplacian polynomial and Kirchhoff index of graphs derived from regular graphs, *Discrete Appl. Math.* **161** (2013) 3063–3071.
10. H. Wiener, Structural determination of paraffin boiling points, *J. Am. Chem. Soc*. **69** (1945) 17–20.
11. B. Zhou and N. Trinajstić, A note on Kirchhoff index, *Chem. Phys. Lett.* **455** (2008) 120–123.
12. H.–Y. Zhu, D. J. Klein and I. Lukovits, Extensions of the Wiener number, *J. Chem. Inf. Comput. Sci.* **36** (1996) 420–428.