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Note on properties of First Zagreb Index of Graphs

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ABSTRACT

Let G be a graph. The first Zagreb $M_1(G)$ of graph G is defined as: $M_1(G) = \sum_{u\in V(G)} deg(u)^2$. In this paper, we prove that each even number except 4 and 8 is a first Zagreb index of a caterpillar. Also, we show that the fist Zagreb index cannot be an odd number. Moreover, we obtain the fist Zagreb index of some graph operations.

Keywords: Topological indices, the first and second Zagreb indices, tree, graph operation, strongly distance – balanced graph.

1. INTRODUCTION

Throughout this paper graph means simple connected graphs. Let G be a connected graph with vertex and edge sets $V(G)$ and $E(G)$, respectively. As usual, the degree of a vertex u of G is denoted by $deg(u)$ and it is defined as the number of edges incident with u. A topological index is a real number related to a graph. It must be a structural invariant, i.e., it preserves by every graph automorphisms. There are several topological indices have been defined and many of them have found applications as means to model chemical, pharmaceutical and other properties of molecules.

The Zagreb indices have been introduced more than thirty years ago by Gutman and Trinajestić [3]. The first Zagreb $M_1(G)$ of graph G is defined as: $M_1(G) = \sum_{u\in V(G)} deg(u)^2$, see [1,4,6]. A tree is an undirected graph in which any two vertices are connected by exactly one simple path. In other words, any connected graph without cycles is a tree. A caterpillar or caterpillar tree is a tree in which all the vertices of the caterpillar are within distance 1 of a central path. We denote the path graph and the cycle of order n by P_n and C_n, respectively.

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In what follows, four types of graphs resulting from edge subdivision will be introduced. Two of them, the subdivision graph and the total graph, belong to the folklore, while the other two were introduced in [1] and further investigated in [8]. For a connected graph *G*, define four related graphs as follows:

I. *S*(*G*) is the graph obtained by inserting an additional vertex in each edge of *G*. Equivalently, each edge of *G* is replaced by a path of length 2.

II. *R*(*G*) is obtained from *G* by adding a new vertex corresponding to each edge of *G*, then joining each new vertex to the end vertices of the corresponding edge. Another way to describe *R*(*G*) is to replace each edge of *G* by a triangle.

III. *Q*(*G*) is obtained from *G* by inserting a new vertex into each edge of *G*, then joining with edges those pairs of new vertices on adjacent edges of *G*.

IV. *T*(*G*) has as its vertices, the edges and vertices of *G*. Adjacency in *T*(*G*) is defined as adjacency or incidence for the corresponding elements of *G*.

2. MAIN RESULTS

A regular graph is a graph where each vertex has the same number of neighbors. A regular graph with vertices of degree *k* is called a *k*–regular graph or regular graph of degree *k*. If *G* has vertices $v_1, v_2, \ldots, v_n$, the sequence $(deg(v_1), deg(v_2), \ldots, deg(v_n))$ is called a degree sequence of *G*. We begin by the following simple proposition:

Proposition 1. If *G* is a graph, then $M_1(G)$ is not equal to 4 or 8.

Proof. If $M_1(G)$ is equal to 4, then *G* has degree sequence as (1,1,1,1) or (2) that is impossible and if $M_1(G)$ is equal to 8, then *G* has degree sequence as $(\underbrace{1,\ldots,1}_{8})$ or $(\underbrace{1,\ldots,1}_{4},2)$ or (2,2) that is a contradiction. So, $M_1(G)$ is not equal to 4 or 8. □

Proposition 2. Let *G* is a graph. Then $M_1(G)$ is an even number.

Proof. Let *G* is a graph and $v_1, v_2, \ldots, v_k$ are vertices of odd degree in it. It is a well-known fact that the number of vertices of odd degree in a graph is even. Thus *k* is an even number and so $deg^2(v_1) + deg^2(v_2) + \ldots + deg^2(v_k)$ is an even number, which completes the argument. □

By definition of first Zagreb index, it is not difficult to see that:

Lemma 3. Let T is a tree with more than 1 vertices and T' is obtained from T by adding a new vertex u and joining it to a pendent of T. Then $M_1(T')=M_1(T)+4$. C

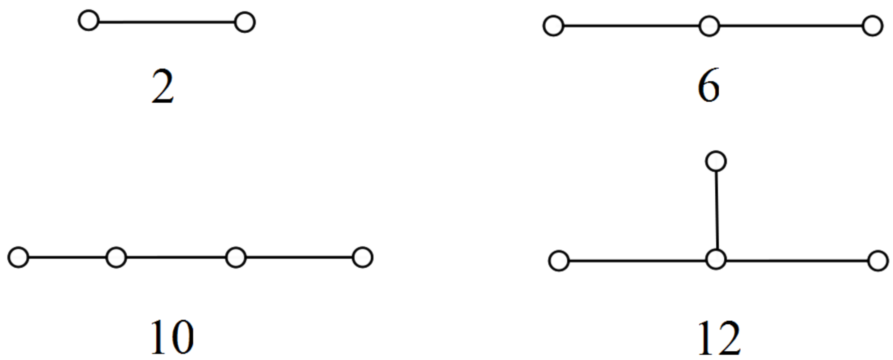

Figure 1. Some Tree Graphs with Their First Zagreb Indices.

Theorem 4. Each even number except 4 and 8 is a first Zagreb index of a caterpillar.

Proof. Use induction on n. By Figure 1, the result is valid for n = 2, 6, 10 or 12. Let n >12 and assume the theorem holds for n. Consider the number $m = n + 2$. Then by our assumption, there is a caterpillar T such that $M_1(T) = m - 4$. On the other hand, by lemma 3 there is a caterpillar T' that $M_1(T') = M_1(T) + 4$ and this completes the proof. C

The line graph $L(G)$ of a graph G is defined as follows: each vertex of $L(G)$ represents an edge of G, and any two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint in G, see [7].

Theorem 5. Let G is a connected graph. Then

1. $M_1(S(G))=M_1(G)+4|E(G)|$,
2. $M_1(R(G))=4M_1(G)+4|E(G)|$,
3. $M_1(T(G))=4M_1(G)+M_1(L(G))+8|E(L(G))|+4|E(G)|$,
4. $M_1(Q(G))=M_1(G)+ M_1(L(G))+8|E(L(G))|+4|E(G)|$.

Proof. The parts 1 and 2 is easy by definitions of $S(G)$ and $R(G)$, respectively. Now, we prove the part 3, to do this, suppose V_1 is the set of new vertices of $T(G)$ corresponding to edges of G. If x is a vertex of V_1 corresponding to edge uv of G, then $deg_{T(G)}(x)=deg_G(u)+deg_G(v)$. Thus, the sum of $deg^2_{T(G)}(x)$ over all vertices of V_1, is qual to:

$$M_1(L(G))+8|E(L(G))|+ 4|E(G)|.$$

On the other hand, if x is a vertex of $V(T(G))-V_1$, then $deg_{T(G)}(x)=2deg_G(x)$, which completes the proof of the part 3 and using similar arguments, we can prove the last section. □

The tensor product $G\otimes H$ of graphs G and H is a graph such that:

- The vertex set of $G\otimes H$ is the Cartesian product $V(G) \times V(H)$; and
- Vertices (u,u') and (v,v') are adjacent in $G\otimes H$ if and only if u' is adjacent with v' and u is adjacent with v [5].

Proposition 6. Let G and H be two graphs. Then $M_1(G\otimes H)=M_1(G)M_1(H)$.

Proof. It is clear that for each $(u,x)\in V(G\otimes H)$, $deg_{G\otimes H}((u,x))=deg_G(u)deg_H(x)$ and so the sum of $deg^2_{G\otimes H}(x)$ over all vertices of $V(G\otimes H)$, is qual to: $M_1(G)M_1(H)$. □

Suppose $G_1, \dots, G_n$ are connected rooted graphs with root vertices $r_1, \dots, r_n$, respectively. The generalized link $(G_1,\dots,G_n)^{(k_1,\dots,k_n)}_{(r_1,\dots,r_n)}$ is obtained by adding a new vertex x, then joining x to r_i by a path of length k_i, $i = 1,2, \dots, n$. Thus, we can write:

Proposition 7. Let $G_1,\dots,G_n$ are connected rooted graphs with root vertices $r_1,\dots,r_n$, respectively. Then $M_1((G_1,\dots,G_n)^{(k_1,\dots,k_n)}_{(r_1,\dots,r_n)}) = \sum_{i=1}^{n} M_1(G_i) + 2\sum_{i=1}^{n} deg_{G_i}(r_i)$ $+4\sum_{i=1}^{n} k_i + n(n-3)$, where $k_i>0$, $i=1,\dots, n$.

The second Zagreb $M_2(G)$ of graph G is defined as: $M_2(G) = \sum_{uv\in E(G)} deg(u)deg(v)$. So, it is clear that for some graphs, M_1 is larger than M_2 and for some graphs, M_2 is larger than M_1. But, we can say:

Proposition 8. Let G is a regular graph with $n>2$ vertices. Then $M_1(G)\leq M_2(G)$, with equality if and only if $G \cong C_n$.

Proof. Let G is a k–regular graph. Then for each uv of $E(G)$, we have $deg(u)deg(v)=r^2$ and so $M_2(G) = |E(G)|r^2$. Also, we obtain $M_1(G) = |V(G)|r^2$. On the other hand, since G is k–regular and $|V(G)| > 2$, so $|E(G)| \geq |V(G)|$, with equality if and only if $G\cong C_n$, which completes the proof. □

The graph X is said to be strongly distance–balanced if for any edge uv of X and any integer k, the number of vertices at distance k from u and at distance $k+1$ from v is equal to the number of vertices at distance *k+1* from u and at distance *k* from *v* [9]. The following proposition is directly obtained by definition of strongly distance–balanced graphs and Proposition 8:

Proposition 9. Let G is a strongly distance–balanced with $n>2$ vertices. Then $M_1(G) \leq M_2(G)$, with equality if and only if $G \cong C_n$. ◻

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