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Research Paper

Extremal Results and Inequalities for the Diminished Sombor Index of Graphs

Mohammad Hadi Akhbari[1⋆] and Fateme Movahedi[2]

[1] Department of Mathematics, Es.C., Islamic Azad University, Estahban, Iran
[2] Department of Mathematics, Faculty of Sciences, Golestan University, Gorgan, Iran

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Abstract

For any simple graph G having an edge set E, the Diminished Sombor index (DSO) is formulated as:

$$DSO(G) = \sum_{uv \in E} \frac{\sqrt{\zeta_u^2 + \zeta_v^2}}{\zeta_u + \zeta_v},$$

where ζ_u is the degree of the vertex u.

The present paper focuses on the DSO index of simple graphs, where precise bounds are derived based on fundamental structural parameters, including order, size, maximum degree, and minimum degree. We also establish sharp inequalities relating DSO to a collection of standard topological indices, namely the Randić, Geometric-Arithmetic, Zagreb, Sombor, and Harmonic indices, as well as inverse degree-related indices. For each bound, extremal graphs attaining equality are fully characterized. Additionally, we analyze the behavior of DSO for specific graph families, including trees and regular graphs.

1 Introduction

Let $G = (V, E)$ be a simple graph considered throughout this study; here, $|V|$ and $|E|$ correspond to its order and size. The degree of a vertex u, represented as ζ_u, corresponds to the cardinality of the set of vertices adjacent to it. We use the notations δ and Δ for the minimum and maximum degrees of G, respectively. Given two adjacent vertices u and v, we use uv to indicate the edge linking them. We use the standard graph notation from [1].

Topological indices are numerical invariants assigned to molecular graphs that serve as essential tools in mathematical chemistry. By providing a quantitative representation of molecular architecture, these descriptors facilitate the discovery of meaningful relationships between

*Corresponding author
E-mail addresses: hadi.akhbari@iau.ac.ir (M. H. Akhbari), f.movahedi@gu.ac.ir (F. Movahedi)
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structural attributes and diverse physicochemical characteristics. The capacity of these indices to predict properties without requiring costly laboratory synthesis makes them particularly valuable in computational screening workflows designed to identify promising drug leads or functional materials with desired attributes [2]. In what follows, we recall the definitions of several topological indices that will be needed in this paper.

- **First Zagreb index [3]:** $M_1(G) = \sum_{uv \in E} (\zeta_u + \zeta_v)$.
- **Second Zagreb index [4]:** $M_2(G) = \sum_{uv \in E} \zeta_u \zeta_v$.
- **Gutman-Milovanović index [5]:** For $\alpha, \beta \in \mathbb{R}$, $M_{\alpha,\beta}(G) = \sum_{uv \in E} (\zeta_u \zeta_v)^\alpha (\zeta_u + \zeta_v)^\beta$.
- **Randić index [6]:** $R(G) = \sum_{uv \in E} \frac{1}{\sqrt{\zeta_u \zeta_v}}$.
- **α-Randić index [7]:** For $\alpha \in \mathbb{R}$, $R_\alpha(G) = \sum_{uv \in E} (\zeta_u \zeta_v)^\alpha$.
- **Geometric-Arithmetic index [8]:** $GA(G) = \sum_{uv \in E} \frac{2\sqrt{\zeta_u \zeta_v}}{\zeta_u + \zeta_v}$.
- **Harmonic index [9]:** $H(G) = \sum_{uv \in E} \frac{2}{\zeta_u + \zeta_v}$.
- **Inverse degree index [9]:** $ID(G) = \sum_{uv \in E} \left(\frac{1}{\zeta_u^2} + \frac{1}{\zeta_v^2} \right)$.
- **Inverse sum indeg index [10]:** $ISI(G) = \sum_{uv \in E} \frac{\zeta_u \zeta_v}{\zeta_u + \zeta_v}$.
- **Sum-connectivity index [11]:** $\chi(G) = \sum_{uv \in E} \frac{1}{\sqrt{\zeta_u + \zeta_v}}$.
- **Sum-connectivity F-index [12]:** $SF(G) = \sum_{uv \in E} \frac{1}{\sqrt{\zeta_u^2 + \zeta_v^2}}$.
- **Symmetric division degree index [10]:** $SDD(G) = \sum_{uv \in E} \frac{\zeta_u^2 + \zeta_v^2}{\zeta_u \zeta_v}$.
- **Sombor index [13]:** $SO(G) = \sum_{uv \in E} \sqrt{\zeta_u^2 + \zeta_v^2}$.
- **Elliptic Sombor index [14]:** $ESO(G) = \sum_{uv \in E} (\zeta_u + \zeta_v) \sqrt{\zeta_u^2 + \zeta_v^2}$.

This index provides a new vertex-degree-based descriptor with geometric motivation, extending the family of Sombor-type indices and offering potential applications in chemical graph theory. For additional research on Sombor indices and the elliptic Sombor index, including their applications in chemical graph theory, see [15–20].
A recent modification of the SO was introduced in [21] as follows:

$$DSO(G) = \sum_{uv \in E} \frac{\sqrt{\zeta_u^2 + \zeta_v^2}}{\zeta_u + \zeta_v},$$

which is referred to as the diminished Sombor index in [22].

Movahedi et al. [22] investigated mathematical properties of the diminished Sombor index, identified extremal graphs, and provided numerical analyses alongside its prospective chemical applications, while a subsequent work by Movahedi [23] determined the unique tricyclic graph maximizing the diminished Sombor index for a given order and described its structural features. Das et al. [24] solved the minimization problem for the DSO index of tricyclic graphs. The study in [25] identifies and describes all molecular graphs with order n and cyclomatic number ℓ that achieve the smallest possible DSO index, assuming $n \geq 2(\ell - 1) \geq 4$. Movahedi [26] established sharp bounds for the DSO index using several classical indices and characterized the corresponding extremal graphs.

In this work, we focus on the study of the DSO index for simple graphs. We determine sharp bounds for *DSO* based on both structural graph invariants and various classic degree-based indices. Additionally, we study how the diminished Sombor index relates to other degree-based indices and derive new characterizations.

2 Preliminaries

Lemma 2.1. *([27]). Consider two sequences of positive real numbers, denoted by* $\mathbf{x} = (x_i)_{i=1}^n$, *and* $\mathbf{a} = (a_i)_{i=1}^n$. *Assuming* $r > 0$,

$$\frac{\left(\sum_{i=1}^n x_i\right)^{r+1}}{\left(\sum_{i=1}^n a_i\right)^r} \leq \sum_{i=1}^n \frac{x_i^{r+1}}{a_i^r}.$$

Equality holds if and only if either $r = 0$ *or* $\frac{x_1}{a_1} = \cdots = \frac{x_n}{a_n}$.

Lemma 2.2. *([28]). Suppose that* $\boldsymbol{\Theta} = (\Theta_i)_{i=1}^t$ *and* $\boldsymbol{\phi} = (\phi_i)_{i=1}^t$ *are two monotonically non-increasing sequences of non-negative numbers, with* $\Theta_1 > 0$ *and* $\phi_1 > 0$. *Furthermore, let* $\chi = (\chi_i)_{i=1}^t$ *where* $\chi_i \geq 0$. *Then,*

$$\left(\sum_{i=1}^t \chi_i \Theta_i^2\right)\left(\sum_{i=1}^t \chi_i \phi_i^2\right) \leq \max\left(\phi_1 \sum_{i=1}^t \chi_i \Theta_i,\ \Theta_1 \sum_{i=1}^t \chi_i \phi_i\right) \sum_{i=1}^t \chi_i \Theta_i \phi_i.$$

Lemma 2.3. *([29]). Let* $\mathbf{x} = (x_i)_{i=1}^n$ *and* $\mathbf{y} = (y_i)_{i=1}^n$, *be real sequences with* $\sum_{i=1}^n x_i = 1$ *and* $0 < \alpha \leq y_i \leq \beta < +\infty$. *Then,*

$$\sum_{i=1}^n x_i y_i + \alpha\beta \sum_{i=1}^n \frac{x_i}{y_i} \leq \alpha + \beta.$$

Lemma 2.4. *([30]). For any graph* G *of size* m,

$$GA(G) \geq \frac{2m\sqrt{\Delta}}{1+\Delta}.$$

Equality holds if and only if the endpoints of any arbitrary edge in G *possess degrees* 1 *and* Δ.

Lemma 2.5. *([31]). Let* G *be a graph with* n *vertices. Then,*

$$R(G) \leq \frac{n}{2}.$$

Equality holds if and only if G *is a disjoint union of regular graphs (not necessarily with equal degrees).*

Lemma 2.6. *([32]). Let* $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *be a convex function with* $x_i > 0$ *for* $1 \leq i \leq m$. *Then,*

$$\psi\left(\frac{x_1 + \cdots + x_m}{m}\right) \leq \frac{1}{m}\big(\psi(x_1) + \cdots + \psi(x_m)\big).$$

3 Main results

Theorem 3.1. *For a connected graph G,*

$$DSO(G) \leq \frac{1}{2\delta}\sqrt{SDD(G)M_2(G)}.$$

Equality holds if and only if G is regular.

Proof. For $uv \in E$, $2\delta \leq \zeta_u + \zeta_v \leq 2\Delta$. Thus,

$$\frac{1}{4\delta^2}SDD(G) = \frac{1}{4\delta^2}\left(\sum_{uv\in E}\frac{\zeta_u^2+\zeta_v^2}{\zeta_u\zeta_v}\right) \geq \sum_{uv\in E}\left[\left(\frac{\sqrt{\zeta_u^2+\zeta_v^2}}{\zeta_u+\zeta_v}\right)^2\frac{1}{\zeta_u\zeta_v}\right]. \tag{1}$$

By substituting $r=1$, $x_i = \frac{\sqrt{\zeta_u^2+\zeta_v^2}}{\zeta_u+\zeta_v}$, and $a_i = \zeta_u\zeta_v$ into Lemma 2.1 and using (1), we get

$$\frac{1}{4\delta^2}SDD(G) \geq \sum_{uv\in E}\frac{\left(\frac{\sqrt{\zeta_u^2+\zeta_v^2}}{\zeta_u+\zeta_v}\right)^2}{\zeta_u\zeta_v} \geq \frac{\left(\sum_{uv\in E}\frac{\sqrt{\zeta_u^2+\zeta_v^2}}{\zeta_u+\zeta_v}\right)^2}{\sum_{uv\in E}\zeta_u\zeta_v} = \frac{(DSO(G))^2}{M_2(G)}.$$

Therefore,

$$(DSO(G))^2 \leq \frac{1}{4\delta^2}SDD(G)M_2(G),$$

and consequently,

$$DSO(G) \leq \frac{1}{2\delta}\sqrt{SDD(G)M_2(G)}.$$

The bound is sharp if and only if the graph G satisfies the condition $\delta = \zeta_u = \zeta_v = \Delta$ for $uv \in E$. ■

Theorem 3.2. *For a connected graph G*

$$DSO(G) \geq \frac{\sqrt{2}}{2\Delta}\left(M_1(G) - 2ISI(G)\right).$$

Equality holds if and only if G is regular.

Proof. Applying Lemma 2.2 with the assignments $\Theta_i = \sqrt{\zeta_u^2+\zeta_v^2} = \phi_i$ and $\chi_i = \frac{1}{\zeta_u+\zeta_v}$, we get

$$\left(\sum_{uv\in E}\frac{\zeta_u^2+\zeta_v^2}{\zeta_u+\zeta_v}\right)\left(\sum_{uv\in E}\frac{\zeta_u^2+\zeta_v^2}{\zeta_u+\zeta_v}\right) \leq \max\left(\sqrt{2}\Delta\sum_{uv\in E}\frac{\sqrt{\zeta_u^2+\zeta_v^2}}{\zeta_u+\zeta_v}, \sqrt{2}\Delta\sum_{uv\in E}\frac{\sqrt{\zeta_u^2+\zeta_v^2}}{\zeta_u+\zeta_v}\right)\sum_{uv\in E}\frac{\zeta_u^2+\zeta_v^2}{\zeta_u+\zeta_v}.$$

Hence,

$$\sum_{uv\in E}\frac{\zeta_u^2+\zeta_v^2}{\zeta_u+\zeta_v} \leq \sqrt{2}\Delta\sum_{uv\in E}\frac{\sqrt{\zeta_u^2+\zeta_v^2}}{\zeta_u+\zeta_v}. \tag{2}$$

Consequently,

$$\begin{aligned}
DSO(G) &\geq \frac{1}{\sqrt{2}\Delta}\sum_{uv\in E}\frac{\zeta_u^2+\zeta_v^2}{\zeta_u+\zeta_v} \\
&= \frac{1}{\sqrt{2}\Delta}\left(\sum_{uv\in E}\frac{(\zeta_u+\zeta_v)^2}{\zeta_u+\zeta_v} - 2\sum_{uv\in E}\frac{\zeta_u\zeta_v}{\zeta_u+\zeta_v}\right) \\
&= \frac{1}{\sqrt{2}\Delta}\left(M_1(G) - 2ISI(G)\right).
\end{aligned}$$

The bound is sharp if and only if the graph G satisfies the condition $\zeta_u = \zeta_v$ for $uv \in E$. ■

Theorem 3.3. *Let G be a simple graph without a pendant vertex. Then,*

$$DSO(G) \geq \frac{\sqrt{2}}{2\Delta} ID(G).$$

Proof. Analogous to the proof of Theorem 3.2, we deduce from (2) that

$$\begin{aligned} DSO(G) &\geq \frac{1}{\sqrt{2}\Delta} \sum_{uv\in E} \frac{\zeta_u^2+\zeta_v^2}{\zeta_u+\zeta_v} \\ &\geq \frac{1}{\sqrt{2}\Delta} \sum_{uv\in E} \frac{\zeta_u^2+\zeta_v^2}{(\zeta_u\zeta_v)^2} \\ &= \frac{1}{\sqrt{2}\Delta} \sum_{uv\in E} \left(\frac{1}{\zeta_u^2}+\frac{1}{\zeta_v^2}\right) = \frac{\sqrt{2}}{2\Delta} ID(G). \end{aligned}$$

■

Corollary 3.4. *For a graph G of order n without a pendant vertex,*

$$DSO(G) \geq \frac{\sqrt{2}}{2}\left(\frac{n}{\Delta^2}\right).$$

Proof. Since $ID(G) \geq \frac{n}{\Delta}$, using Theorem 3.3, we get

$$DSO(G) \geq \frac{\sqrt{2}}{2\Delta} ID(G) \geq \frac{\sqrt{2}n}{2\Delta^2}.$$

■

Corollary 3.5. *Let G be a simple graph without a pendant vertex. Then,*

$$DSO(G) \geq \frac{\sqrt{2}}{\Delta}\left(\frac{n}{2m}\right)^2.$$

Proof. From Theorem 1 in [33], we have $ID(G) \geq \frac{n^2}{2m^2}$. Therefore, by applying Theorem 3.3, we get

$$DSO(G) \geq \frac{\sqrt{2}}{2\Delta} ID(G) \geq \frac{\sqrt{2}}{2\Delta}\left(\frac{n^2}{2m^2}\right) = \frac{\sqrt{2}n^2}{4m^2\Delta}.$$

■

Theorem 3.6. *For a graph G,*

$$DSO(G) \geq \frac{1}{2(\Delta+\delta)}\left(SO(G) + \delta\Delta\frac{(H(G))^2}{SF(G)}\right).$$

Equality holds if and only if every component of G is regular.

Proof. By substituting $x_i = \frac{\sqrt{\zeta_u^2+\zeta_v^2}}{(\zeta_u+\zeta_v)DSO(G)}$, $y_i = \zeta_u+\zeta_v$, $\alpha = 2\delta$, and $\beta = 2\Delta$ into Lemma 2.3,

$$\sum_{uv\in E} \sqrt{\zeta_u^2+\zeta_v^2} + 4\delta\Delta \sum_{uv\in E} \frac{\sqrt{\zeta_u^2+\zeta_v^2}}{(\zeta_u+\zeta_v)^2} \leq 2(\Delta+\delta)DSO(G). \tag{3}$$

By setting $x_i = \frac{1}{\zeta_u+\zeta_v}$, $a_i = \frac{1}{\sqrt{\zeta_u^2+\zeta_v^2}}$, and $r=1$ in Lemma 2.1, we have

$$\sum_{uv\in E} \frac{\sqrt{\zeta_u^2+\zeta_v^2}}{(\zeta_u+\zeta_v)^2} = \sum_{uv\in E} \frac{\left(\frac{1}{\zeta_u+\zeta_v}\right)^2}{\frac{1}{\sqrt{\zeta_u^2+\zeta_v^2}}} \geq \frac{\left(\frac{1}{2}\sum_{uv\in E}\frac{2}{\zeta_u+\zeta_v}\right)^2}{\sum_{uv\in E}\frac{1}{\sqrt{\zeta_u^2+\zeta_v^2}}} = \frac{\left(\frac{1}{2}H(G)\right)^2}{SF(G)}. \tag{4}$$

From (3) and (4), we get

$$SO(G) + 4\delta\Delta\left(\frac{\left(\frac{1}{2}H(G)\right)^2}{SF(G)}\right) \leq SO(G) + 4\delta\Delta\sum_{uv\in E}\frac{\sqrt{\zeta_u^2+\zeta_v^2}}{(\zeta_u+\zeta_v)^2} \leq 2(\Delta+\delta)DSO(G),$$

and this gives the result. We have equality if and only if each edge connects two vertices of the same degree. ■

Theorem 3.7. *For a connected graph G,*

$$DSO(G) \geq \frac{\sqrt{2}}{2}GA(G).$$

Equality holds if and only if G is a regular graph.

Proof. Using the geometric-quadratic mean inequality, it follows that

$$\sqrt{\zeta_u\zeta_v} \leq \sqrt{\frac{\zeta_u^2+\zeta_v^2}{2}}. \tag{5}$$

Therefore,

$$\sqrt{2}\left(\frac{\sqrt{\zeta_u\zeta_v}}{\zeta_u+\zeta_v}\right) \leq \frac{\sqrt{\zeta_u^2+\zeta_v^2}}{\zeta_u+\zeta_v}.$$

$$DSO(G) = \sum_{uv\in E}\frac{\sqrt{\zeta_u^2+\zeta_v^2}}{\zeta_u+\zeta_v} \geq \frac{\sqrt{2}}{2}\sum_{uv\in E}\frac{2\sqrt{\zeta_u\zeta_v}}{\zeta_u+\zeta_v} = \frac{\sqrt{2}}{2}GA(G).$$

Equality is achieved in (5) if and only if all vertex degrees are identical, which corresponds to the class of regular graphs. ■

Corollary 3.8. *For a simple graph G of size m,*

$$DSO(G) \geq \frac{m\sqrt{2\Delta}}{1+\Delta}.$$

Equality holds if and only if $G \simeq mK_2$.

Proof. Using Theorem 3.7 and Lemma 2.4, we get

$$DSO(G) \geq \frac{\sqrt{2}}{2}GA(G) \geq \frac{\sqrt{2}}{2}\left(\frac{2m\sqrt{\Delta}}{1+\Delta}\right) = \frac{m\sqrt{2\Delta}}{1+\Delta}.$$

According to Theorem 3.7, equality occurs in the first inequality if and only if the graph is regular. According to Lemma 2.4, equality in the second inequality holds if and only if every edge of G connects vertices of degrees 1 and Δ. Therefore, equality throughout holds if and only if each edge $uv \in E$ satisfies $\zeta_u = \zeta_v = 1$. Thus, $G \simeq mK_2$. ■

Corollary 3.9. *Let G be a triangle-free simple graph with n vertices and m edges. Then*

$$DSO(G) \geq 2\sqrt{2}\left(\frac{m}{n}\right)^2.$$

Proof. Applying Theorem 3.7 yields the lower bound $DSO(G) \geq \frac{\sqrt{2}}{2}GA(G)$. Furthermore, an application of the Cauchy–Schwarz (C–S) inequality leads to

$$GA(G) \geq m^2\left(\sum_{uv\in E}\frac{\zeta_u+\zeta_v}{2\sqrt{\zeta_u\zeta_v}}\right)^{-1} = \frac{m^2}{\sum_{uv\in E}\frac{\zeta_u+\zeta_v}{2\sqrt{\zeta_u\zeta_v}}}. \tag{6}$$

Using Lemma 2.5, the relation (6) and since for $uv \in E$, $\zeta_u+\zeta_v \leq n$, we get

$$DSO(G) \geq \frac{\sqrt{2}}{2}GA(G) \geq \frac{\sqrt{2}}{2}\left(\frac{m^2}{\sum_{uv\in E}\frac{\zeta_u+\zeta_v}{2\sqrt{\zeta_u\zeta_v}}}\right) \geq \frac{\sqrt{2}m^2}{n\sum_{uv\in E}\frac{1}{\sqrt{\zeta_u\zeta_v}}} \geq 2\sqrt{2}\frac{m^2}{n^2}.$$

Theorem 3.10. *For a simple graph G with $\delta \geq 1$,*

$$DSO(G) \leq \frac{1}{4\sqrt{\delta}}ESO(G).$$

Equality holds if and only if $G \simeq mK_2$.

Proof. By applying Lemma 2.6 to $\psi(x) = \frac{1}{x}$ and the geometric-quadratic inequality, we have

$$\frac{2}{\zeta_u+\zeta_v} \leq \frac{1}{2}\left(\frac{1}{\zeta_u}+\frac{1}{\zeta_v}\right) = \frac{1}{2}\left(\frac{\zeta_u+\zeta_v}{\zeta_u\zeta_v}\right) \leq \frac{\sqrt{2}}{2}\left(\frac{\zeta_u+\zeta_v}{\sqrt{\zeta_u+\zeta_v}}\right). \tag{7}$$

Since $\delta \leq \zeta_u \leq \Delta$ for any $u \in V$, using (7), we get

$$\begin{aligned} DSO(G) &= \sum_{uv\in E}\frac{\sqrt{\zeta_u^2+\zeta_v^2}}{\zeta_u+\zeta_v} \leq \frac{1}{2}\sum_{uv\in E}\left(\sqrt{\zeta_u^2+\zeta_v^2}\right)\left(\frac{2}{\zeta_u+\zeta_v}\right) \\ &\leq \frac{\sqrt{2}}{4}\sum_{uv\in E}\frac{\sqrt{\zeta_u^2+\zeta_v^2}(\zeta_u+\zeta_v)}{\sqrt{\zeta_u+\zeta_v}} \leq \frac{1}{4\sqrt{\delta}}\sum_{uv\in E}(\zeta_u+\zeta_v)\sqrt{\zeta_u^2+\zeta_v^2} \\ &= \frac{1}{4\sqrt{\delta}}ESO(G). \end{aligned}$$

Equality is attained in the preceding relations precisely when

$$\frac{\sqrt{\zeta_u^2+\zeta_v^2}}{\zeta_u+\zeta_v} = \frac{1}{4\sqrt{\delta}}(\zeta_u+\zeta_v)\sqrt{\zeta_u^2+\zeta_v^2}.$$

Consequently, this condition implies that equality occurs if and only if $\zeta_u = \zeta_v = 1$ for all $uv \in E$. For a vertex $u \in V$, $ecc_G(u) = \max\{\zeta_G(u,v)|v \in V\}$ in which $\zeta_G(u,v)$ is the length of

the shortest path connecting u and v. The radius $r(G)$ of G is given by $r(G) = \min\{ecc_G(u)|u \in V\}$.

Theorem 3.11. *For a connected graph G of order n,*

$$DSO(G) \geq \frac{SO(G)}{2(n-r(G))}.$$

Proof. Since for any $u \in V$, $\zeta_u \leq n - ecc_G(u)$, hence

$$\frac{2}{\zeta_u + \zeta_v} \geq \frac{2}{2n - 2r(G)} = \frac{1}{n - r(G)}.$$

$$\begin{aligned} DSO(G) &= \sum_{uv \in E} \frac{\sqrt{\zeta_u^2 + \zeta_v^2}}{\zeta_u + \zeta_v} = \frac{1}{2} \sum_{uv \in E} \left(\sqrt{\zeta_u^2 + \zeta_v^2}\right)\left(\frac{2}{\zeta_u + \zeta_v}\right) \\ &\geq \frac{1}{2} \sum_{uv \in E} \left(\sqrt{\zeta_u^2 + \zeta_v^2}\right)\left(\frac{1}{n - r(G)}\right) \\ &= \frac{1}{2(n - r(G))} \sum_{uv \in E} \sqrt{\zeta_u^2 + \zeta_v^2} = \frac{SO(G)}{2(n - r(G))}. \end{aligned}$$

Theorem 3.12. *For a simple graph G of size m,*

$$DSO(G) \leq \sqrt{m\,(m - 2M_{1,-2}(G))}.$$

Proof. From the C–S inequality, it follows that

$$\begin{aligned} (DSO(G))^2 &= \left(\sum_{uv \in E} \frac{\sqrt{\zeta_u^2 + \zeta_v^2}}{\zeta_u + \zeta_v}\right)^2 \leq m \sum_{uv \in E} \frac{\zeta_u^2 + \zeta_v^2}{(\zeta_u + \zeta_v)^2} \\ &= m\left(\sum_{uv \in E} \frac{(\zeta_u + \zeta_v)^2 - 2\zeta_u\zeta_v}{(\zeta_u + \zeta_v)^2}\right) = m\left(\sum_{uv \in E} 1 - 2\sum_{uv \in E} (\zeta_u\zeta_v)(\zeta_u + \zeta_v)^{-2}\right) \\ &= m\,(m - 2M_{1,-2}(G)), \end{aligned}$$

and this gives the result.

Theorem 3.13. *For a connected graph G of size m,*

$$DSO(G) \leq \frac{\sqrt{mSDD(G)}}{2}.$$

Equality holds if and only if G is a regular graph.

Proof. Based on the arithmetic-geometric mean inequality and since for any sequence of real numbers $(x_i)_{i=1}^n$, $\left(\sum_{i=1}^n x_i\right)^2 \leq n\sum_{i=1}^n x_i^2$, we get

$$\begin{aligned} (2DSO(G))^2 &= \left(2\sum_{uv \in E} \frac{\sqrt{\zeta_u^2 + \zeta_v^2}}{\zeta_u + \zeta_v}\right)^2 \leq \left(2\sum_{uv \in E} \frac{\sqrt{\zeta_u^2 + \zeta_v^2}}{2\sqrt{\zeta_u\zeta_v}}\right)^2 \\ &= \left(\sum_{uv \in E} \frac{\sqrt{\zeta_u^2 + \zeta_v^2}}{\sqrt{\zeta_u\zeta_v}}\right)^2 \leq m\sum_{uv \in E} \frac{\zeta_u^2 + \zeta_v^2}{\zeta_u\zeta_v} \\ &= mSDD(G). \end{aligned}$$

The bound is sharp if and only if the graph G satisfies the condition $\zeta_u = \zeta_v$ for $uv \in E$.

Theorem 3.14. *For a tree T of order n,*

$$DSO(T) \geq \frac{SO(T)}{n}.$$

Equality holds if and only if T is a star graph.

Proof. For any $uv \in E$, $\zeta_u + \zeta_v \leq n$. Therefore,

$$DSO(T) = \sum_{uv\in E} \frac{\sqrt{\zeta_u^2+\zeta_v^2}}{\zeta_u+\zeta_v} \geq \sum_{uv\in E} \frac{\sqrt{\zeta_u^2+\zeta_v^2}}{n} = \frac{SO(T)}{n}.$$

The bound is sharp if and only if $\zeta_u + \zeta_v = n$, which implies T is a star graph. ■

Theorem 3.15. *For a connected graph G,*

$$\frac{\sqrt{2}}{2}\delta H(G) \leq DSO(G) \leq \frac{\sqrt{2}}{2}\Delta H(G).$$

Equality holds if and only if G is a regular graph.

Proof. For any $u \in V$, $\delta \leq \zeta_u \leq \Delta$. Therefore, we get

$$\frac{\sqrt{2}}{2}\delta \sum_{uv\in E} \frac{2}{\zeta_u+\zeta_v} \leq \sum_{uv\in E} \frac{\sqrt{\zeta_u^2+\zeta_v^2}}{\zeta_u+\zeta_v} \leq \frac{\sqrt{2}}{2}\Delta \sum_{uv\in E} \frac{2}{\zeta_u+\zeta_v}.$$

Thus,

$$\frac{\sqrt{2}}{2}\delta H(G) \leq DSO(G) \leq \frac{\sqrt{2}}{2}\Delta H(G).$$

The bound is sharp if and only if the graph G satisfies the condition $\delta = \Delta$. ■

Theorem 3.16. *For a connected graph G with at least one edge,*

$$DSO(G) \geq \frac{(\chi(G))^2}{SF(G)}.$$

Equality holds if and only if G is a regular graph or a biregular graph.

Proof. From the C–S inequality, it follows that

$$\begin{aligned}
(\chi(G))^2 &= \left(\sum_{uv\in E} \frac{1}{\sqrt{\zeta_u+\zeta_v}}\right)^2 = \left(\sum_{uv\in E} \frac{\left(\zeta_u^2+\zeta_v^2\right)^{\frac{1}{4}}}{\sqrt{\zeta_u+\zeta_v}} \times \frac{1}{\left(\zeta_u^2+\zeta_v^2\right)^{\frac{1}{4}}}\right)^2 \\
&\leq \left(\sum_{uv\in E} \frac{\sqrt{\zeta_u^2+\zeta_v^2}}{\zeta_u+\zeta_v}\right)\left(\sum_{uv\in E} \frac{1}{\sqrt{\zeta_u^2+\zeta_v^2}}\right) \\
&= DSO(G)SF(G).
\end{aligned}$$

Therefore, we have

$$DSO(G) \geq \frac{(\chi(G))^2}{SF(G)}.$$

The bound is sharp if and only if $\frac{\left(\zeta_u^2+\zeta_v^2\right)^{\frac{1}{4}}}{\sqrt{\zeta_u+\zeta_v}} = \frac{c}{\left(\zeta_u^2+\zeta_v^2\right)^{\frac{1}{4}}}$. Therefore, $c^2 = \frac{\zeta_u^2+\zeta_v^2}{\zeta_u+\zeta_v}$. If $uv, vz \in E$, we have $\frac{\zeta_u^2+\zeta_v^2}{\zeta_u+\zeta_v} = \frac{\zeta_v^2+\zeta_z^2}{\zeta_v+\zeta_z}$ and consequently, $(\zeta_v+\zeta_z)(\zeta_u^2+\zeta_v^2) - (\zeta_u+\zeta_v)(\zeta_v^2+\zeta_z^2) = 0$. That is, G is a regular graph or a biregular graph.

4 Conclusion

In this paper, we established sharp bounds for the DSO index in terms of fundamental graph parameters and several classical topological indices, providing new characterizations and insights into the relationships between DSO and other degree-based indices. Our results contribute to the theoretical foundation of the DSO index and highlight its potential applications in mathematical chemistry and graph theory. For future research, it would be valuable to explore the behavior of the DSO index in specific families of graphs, investigate its extremal values under additional structural constraints, and analyze its predictive power in chemical graph theory for various molecular properties. Moreover, extending these results to weighted, directed, or dynamic graphs and studying algorithmic approaches for efficient computation of the DSO index in large-scale networks present promising directions for further study.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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