Iranian Journal of Mathematical chemistry

IJMC

DOI: 10.22052/IJMC.2026.258001.2093
Vol. 17, No. 2, 2026, pp. 217-232
Research Paper

An Appropriate Fractional Narayana Polynomials Neural Network Method for a Mathematical Model of the Lung Cancer

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Keywords:

Fractional Narayana polynomials neural network,
Lung cancer,
Cancer cells,
Immune cells,
Optimization algorithm

AMS Subject Classification (2020):

41A10; 65Z05; 92C42.

Article History:

Received: 21 November 2025
Accepted: 15 February 2026

Abstract

A mathematical model of lung cancer is used to analyze the dynamics of tumor growth and the interactions between cancer cells and immune cells. To obtain approximate solutions and improve understanding of the behavior of the state functions, a fractional Narayana polynomials neural network (FNPNN) with higher accuracy and better efficiency is proposed. For this purpose, we develop a method using a three-layer artificial neural network, which includes an input layer, a hidden layer, and an output layer. The fractional Narayana polynomials and $arcsinh(t)$ function are utilized as activation functions for the hidden and output layers of the network, respectively. The lung cancer model is reduced to the problem of solving a system of algebraic equations through the use of FNPNN and the Lagrange multipliers method. All computations are performed using Maple and MATLAB software. The convergence analysis is discussed. The efficiency and versatility of our suggested approach are confirmed by numerical modeling examples. The technique proposed in this work can be effortlessly applied to other scientific or engineering problems, providing the potential for substantial efficiency gains while keeping accuracy at an acceptable level.

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Academic Editor: Abbas Saadatmandi

Abbreviations

FNPNN: Fractional Narayana polynomials neural network
NNs: Neural networks
NPs: Narayana polynomials
NPNN: Narayana polynomials neural network

1 Introduction

Lung cancer ranks second in terms of diagnosis and is the primary cause of cancer-related mortality for both men and women globally [1–3]. Lung cancer cells have abnormalities in the regulatory systems (circuits) responsible for normal cell division and homeostasis. Normal lung cells become malignant through multiple stages of genetic and epigenetic changes. Eventually, this process leads to the development of invasive cancer through clonal expansion. A significant feature of cancer is the tumor microenvironment and the complex relationships between the various cell types it contains, as well as the signaling molecules they release [4]. Cancer stem cells make up a small and rare fraction of the entire cancer cell population (in most solid tumors, this fraction is less than 1%), and these cells have a high potential to generate tumors. It is believed that cancer stem cells are the primary drivers of all cancer characteristics, such as high tumorigenicity, a strong propensity for metastasis, immune system evasion, resistance to treatment, and cancer recurrence [5]. Metastasis, the spread of cancer cells from their original tumor to form new tumors in distant areas, consists of several steps. In lung cancer, metastasis is a key factor in predicting both patient recovery and the disease's overall progression. Lung cancer is often diagnosed at an advanced stage, after it has metastasized to other parts of the body [5]. For lung cancer, chemotherapy, surgery, and radiation therapy are the standard treatment approaches. Despite the fact that these treatments can prolong the survival period of patients with lung cancer, the 5-year overall survival rate remains unsatisfactory [6]. Recently, advances have been made in understanding cancer through the role of the immune system, as immune cells are vital to the development and treatment of cancer [6].

Polynomials play a significant role in different areas of mathematics and their applications, and they are employed in various numerical techniques to approximate and interpolate unknown functions in different functional equations where the solution to the given problem is represented as a linear combination of basis functions. Mathematical modeling of diverse diseases helps researchers to understand how diseases can spread through the body or population. The use of different polynomials can be seen in many mathematical models of diverse diseases as basis functions. For instance, Kolev et al. [7] investigated a mathematical model. This study proposes a mathematical model based on the kinetic theory of active particles to explain the response to treatment and tumor regression in lung cancer patients undergoing chemotherapy and radiotherapy. Mohandass et al. [8] proposed a convolutional NN using DenseNet 201 for early lung cancer detection, which is optimized using the Namib Beetle Optimization Algorithm on CT scans. Lee et al. [9] used a breath monitor system to detect lung tumors via convolutional NNs. Singh et al. [10] used digital pathology images to examine the manner in which convolutional NNs detect and diagnose colon and lung cancers. Aftab et al. [11] applied a graph NN model for learning Whole Slide Images representation of lung cancer. Shankara et al. [12] proposed a computer-aided model with convolution NN for identifying lung cancer. Othmani et al. [13] used deep NNs and Active Shape approach to detect lung cancer. Paikaray et al. [14] suggested an enhanced convolution NN for classifying different types of lung cancer, with its hyperparameters optimized using a gray wolf optimization technique. Agrawal et al. [15] used the Hermite wavelet operational matrix approach to solve fractional nonlinear can-

cer models. Karar and El-Brawany [16] presented a new controller model developed for the purpose of controlling hyperthermia thermal doses in the treatment of cancer. Karim et al. [17] employed the forward-backward sweep method to solve the time-fractional spatiotemporal SIR optimality system numerically. Gamal et al. [18] constructed the Chebyshev polynomial derivative-based spectral schemes for solving linear and nonlinear ordinary differential equations. Zaky et al. [19] constructed and analyzed a Legendre spectral-collocation method for the numerical solution of distributed-order fractional initial value problems. Abd-Elhameed et al. [20] presented a tau algorithm using seventh-kind Chebyshev polynomials for fractional delay differential equations. Abd-Elhameed et al. [21] selected the shifted Chebyshev polynomials of the first-kind as basis functions and employed the spectral tau method for obtaining the desired approximate solutions of fractional delay differential equations. Youssri et al. [22] developed a formula for fractional derivatives of the shifted Chebyshev polynomials of the third kind for solving a type of nonlinear fractional pantograph differential equations. Abd-Elhameed and Youssri [23] derived a formula for the high-order derivatives of Chebyshev polynomials of the fifth kind for obtaining the spectral solution of the convection-diffusion equation.

This research deals with applications of new polynomials, namely, fractional Narayana polynomials (FNPs) to numerically solve systems of differential equations obtained from a mathematical model of lung cancer. A system of differential equations, presented in [24], reveals the interactions between cancer cells and immune cells in lung tissues and cancer cells that have spread to different areas of the body as follows:

$$\begin{cases} \frac{dN(t)}{dt} = \lambda N(t)\left(1-\frac{N(t)}{k}\right) - \mu N(t)P(t) - \beta_1 N(t)I(t), \\ \frac{dI(t)}{dt} = \phi_1 I_0 + \phi_2 N^2(t) - \phi_3 I(t) - \beta_2 I(t)P(t), \\ \frac{dP(t)}{dt} = \gamma N(t)P(t) - \delta P(t) - \beta_3 I(t)P(t), \\ N(0) = N_0 > 0, \quad I(0) = I_0 > 0, \quad P(0) = P_0 > 0, \end{cases} \tag{1}$$

where $N(t)$ is the number of cancer cells in lung tissues at time t, $I(t)$ denotes the number of immune cells in lung tissues at time t, and $P(t)$ is the number of cancer cells that have spread to different areas of the body at time t, $t \in [0, T]$, and the appropriate initial conditions are $N_0 > 0$, $I_0 > 0$, $P_0 > 0$. A description of parameters in model (1) is provided in Table 1.

This study utilizes an optimization method that incorporates the NN, FNPs, and Lagrange multipliers to solve the nonlinear lung cancer model (1). Some of the primary benefits are listed below:

(i) The concept of FNPs, a generalization of Narayana polynomials, is presented for the first time in this study.
(ii) The convergence analysis is provided for the method.
(iii) An optimization algorithm is derived using the Lagrange multiplier method.
(iv) A few terms of the FNPs are utilized to attain highly accurate results.
(v) The considered problem is transformed into a system of algebraic equations, which can be solved using an appropriate numerical method.
(vi) The proposed approach is computer-based and easy to apply in any language that supports programming.
(vii) The method accuracy is shown by solving numerical examples.

This paper is divided into seven sections. After the Introduction, a description of Narayana polynomials is presented in Section 2. Section 3 provides a brief overview of the FNPs. Section 4 concerns the approximation method to solve the nonlinear mathematical model of lung cancer (1). Sections 5 and 6 examine the convergence of the suggested approach and the existence and uniqueness of solutions for the system (1), respectively. To verify the proposed approach,

Table 1: Definition of parameters in model (1).

Parameter	Description
λ	Growth rate of lung cancer cells
k	Carrying capacity of lung tissue
μ	Rate at which cancer cells spread from lung tissue to other body regions
β_1	Interaction between cancer cells and immune system cells
ϕ_1, ϕ_2, ϕ_3	Effects of growth factors
β_2	Interaction between immune cells and growth of cancer cells
γ	Rate at which cancer cells in lung tissue metastasize to other body regions
δ	Rate of death of cancer cells that have metastasized to other body regions
β_3	Interaction between blood vessels and cancer cells

numerical results are presented in Section 7. The conclusions and perspectives of this research are summarized in Section 8.

2 Narayana polynomials and function approximation

In this part of the paper, we discuss the definitions of NPs and the concept of function approximation using NPs.

Definition 2.1. Narayana numbers $\mathcal{N}_{i,j}$ are defined by the relations [25]:

$$\mathcal{N}_{0,j} = \mathcal{N}_{j,0} = \delta_{j0} \quad (j \in \mathbb{N}_0), \qquad \mathcal{N}_{i,j} = \frac{1}{i}\binom{i}{j}\binom{i}{j-1} \quad (i, j \in \mathbb{N}), \tag{2}$$

where δ_{ij} denotes the Kronecker delta function.

Definition 2.2. The analytical form of the NPs, $\mathbb{A}_n(\tau)$, is obtained by [25]:

$$\mathbb{A}_n(\tau) = \sum_{k=0}^{n} \mathcal{N}_{n+1,k+1}\tau^k = \sum_{k=0}^{n} \frac{1}{n+1}\binom{n+1}{k+1}\binom{n+1}{k}\tau^k, \qquad n \in \mathbb{N}_0. \tag{3}$$

The first NPs are given by $\mathbb{A}_0(\tau) = 1$, $\mathbb{A}_1(\tau) = 1 + \tau$, $\mathbb{A}_2(\tau) = 1 + 3\tau + \tau^2$, $\mathbb{A}_3(\tau) = 1 + 6\tau + 6\tau^2 + \tau^3$, and $\mathbb{A}_4(\tau) = 1 + 10\tau + 20\tau^2 + 10\tau^3 + \tau^4$. A function $f(\tau)$ can be approximated with the first $(n+1)$ NPs terms as:

$$f_n(\tau) \simeq \sum_{k=0}^{n} q_k \mathbb{A}_k(\tau) = Q^T R S_n(\tau), \tag{4}$$

where $Q^T = [q_0, q_1, \ldots, q_n]$, and

$$R = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ r_{1,0} & r_{1,1} & \cdots & r_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n,0} & r_{n,1} & \cdots & r_{n,n} \end{pmatrix}, \quad S_n(\tau) = [1, \tau, \tau^2, \ldots, \tau^n]^T, \tag{5}$$

and

$$r_{ij} = \begin{cases} \dfrac{1}{i+1}\binom{i+1}{j+1}\binom{i+1}{j}, & i \geq j, \\ 0, & i < j. \end{cases} \tag{6}$$

3 Fractional Narayana polynomials and neural network method

This section introduces the FNPs, function approximation, and the FNPNN implemented in this study.

Definition 3.1. The FNPs, $\mathbf{A}_m(t)$, are defined by the change of variable t^k into $t^{k+\alpha_k}$, $(k+\alpha_k > 0)$ on NPs as:

$$\mathbf{A}_m(t) = \sum_{k=0}^{m} \frac{1}{m+1}\binom{m+1}{k+1}\binom{m+1}{k} t^{k+\alpha_k}, \qquad m \in \mathbb{N}_0, \tag{7}$$

where α_k denotes control parameter. When $\alpha_k = 0$, FNPs and classical NPs coincide.

3.1 Function approximations

The functions $N(t)$, $I(t)$ and $P(t)$ in Equation (1) can be expressed in the following matrix form:

$$N(t) \simeq \mathbf{B}^T \mathbf{U} \mathbf{S}_{m_1}(t),\ I(t) \simeq \mathbf{C}^T \mathbf{V} \mathbf{S}_{m_2}(t),\ P(t) \simeq \mathbf{D}^T \mathbf{W} \mathbf{S}_{m_3}(t), \tag{8}$$

where

$$\mathbf{B}^T = [b_0, b_1, \ldots, b_{m_1}], \qquad \mathbf{C}^T = [c_0, c_1, \ldots, c_{m_2}], \qquad \mathbf{D}^T = [d_0, d_1, \ldots, d_{m_3}], \tag{9}$$

$$\mathbf{U} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ u_{1,0} & u_{1,1} & u_{1,2} & \cdots & u_{1,m_1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ u_{m_1,0} & u_{m_1,1} & u_{m_1,2} & \cdots & u_{m_1,m_1} \end{pmatrix}, \qquad \mathbf{V} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ v_{1,0} & v_{1,1} & v_{1,2} & \cdots & v_{1,m_2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ v_{m_2,0} & v_{m_2,1} & v_{m_2,2} & \cdots & v_{m_2,m_2} \end{pmatrix}, \tag{10}$$

$$\mathbf{W} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ w_{1,0} & w_{1,1} & w_{1,2} & \cdots & w_{1,m_3} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ w_{m_3,0} & w_{m_3,1} & w_{m_3,2} & \cdots & w_{m_3,m_3} \end{pmatrix}, \tag{11}$$

$$u_{ij} = \begin{cases} \dfrac{1}{i+1}\binom{i+1}{j+1}\binom{i+1}{j}, & i \geq j, \\ 0, & i < j. \end{cases} \qquad i = 1, 2, \ldots, m_1,\ \ j = 0, 1, \ldots, m_1, \tag{12}$$

$$v_{ij} = \begin{cases} \dfrac{1}{i+1}\binom{i+1}{j+1}\binom{i+1}{j}, & i \geq j, \\ 0, & i < j. \end{cases} \qquad i = 1, 2, \ldots, m_2,\ \ j = 0, 1, \ldots, m_2, \tag{13}$$

$$w_{ij} = \begin{cases} \dfrac{1}{i+1}\binom{i+1}{j+1}\binom{i+1}{j}, & i \geq j, \\ 0, & i < j. \end{cases} \qquad i = 1, 2, \ldots, m_3,\ \ j = 0, 1, \ldots, m_3, \tag{14}$$

and

$$\mathbf{S}_{m_i}(t) \triangleq [s_0^i(t), s_1^i(t), \ldots, s_{m_i}^i(t)]^T, \qquad i = 1, 2, 3, \tag{15}$$

$$s_j^i(t) = \begin{cases} 1, & j = 0, \\ t^{j+\zeta_j^i}, & j = 1, 2, \ldots, m_i, \end{cases} \qquad i = 1, 2, 3, \tag{16}$$

with ζ_j^i representing the control parameters.

The FNPs presented in this paper include control parameters ζ_j^i, which make these polynomials novel. As a result, the NPs are generalized into a new family of basis functions, namely the FNPs. This generalization has the advantage of reducing the approximation error and increasing the convergence of the method. Since our basis functions are chosen for the problem with exact solutions of the type $u(t) = t^{j+\zeta_j^i}, (j + \zeta_j^i \geq 0)$, we expect an accurate solution for this problem. The FNPs require only a few basis functions to achieve satisfactory results at a high level of accuracy.

3.2 Fractional Narayana polynomials neural network

In order to efficiently and accurately solve a mathematical model of lung cancer, a novel FNPNN method is proposed in this paper. The FNPNN method is structured with three layers that function as described below:
(i) The input t is provided by the network's input layer.
(ii) In the network's hidden layer, we select FNPs of varying degrees as activation functions.
(iii) The output layer of the network is a linear combination of FNPs with varying weights, and an activation function is applied to the input to produce the output.

Note that the activation function is the function that acts upon the input to get the output of the network. There are many types of activation functions such as sigmoid function, hyperbolic tangent function, ReLu function and others [26]. In this context, inverse hyperbolic sine functions are used as activation functions. Consequently, the following are the results of the FNPNNs, using input data t and parameters $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$:

$$\mathbf{N}(t, \mathbf{B}) = \operatorname{arcsinh}(N(t)), \qquad \mathbf{I}(t, \mathbf{C}) = \operatorname{arcsinh}(I(t)), \qquad \mathbf{P}(t, \mathbf{D}) = \operatorname{arcsinh}(P(t)), \tag{17}$$

where $N(t)$, $I(t)$ and $P(t)$ are represented as linear combinations:

$$N(t) = \sum_{i=0}^{m_1} b_i \mathbf{A}_i(t), \qquad I(t) = \sum_{j=0}^{m_2} c_j \mathbf{A}_j(t), \qquad P(t) = \sum_{k=0}^{m_3} d_k \mathbf{A}_k(t), \tag{18}$$

and $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ vectors are given by:

$$\mathbf{B}^T = [b_0, b_1, \ldots, b_{m_1}], \qquad \mathbf{C}^T = [c_0, c_1, \ldots, c_{m_2}], \qquad \mathbf{D}^T = [d_0, d_1, \ldots, d_{m_3}]. \tag{19}$$

4 Description of the approach

The focus of this section is to introduce the FNPNN approach for solving a mathematical model of lung cancer in Equation (1). We approximate $N(t)$, $I(t)$, and $P(t)$ by using the activation functions as:

$$N(t) \simeq \tilde{N}(t) = N_0 + t\mathbf{N}(t, \mathbf{B}), \qquad I(t) \simeq \tilde{I}(t) = I_0 + t\mathbf{I}(t, \mathbf{C}), \qquad P(t) \simeq \tilde{P}(t) = P_0 + t\mathbf{P}(t, \mathbf{D}). \tag{20}$$

The functions $N(t)$, $I(t)$ and $P(t)$ satisfy the initial conditions. By applying Equation (20) to the initial and boundary conditions of problem (1), we derive the following:

$$\Lambda_1 \triangleq \tilde{N}(0) - N_0 \simeq 0, \qquad \Lambda_2 \triangleq \tilde{I}(0) - I_0 \simeq 0, \qquad \Lambda_3 \triangleq \tilde{P}(0) - P_0 \simeq 0. \tag{21}$$

Using the approximations of (20) in (1), we have:

$$\begin{cases} \frac{d\tilde{N}(t)}{dt} - \lambda\tilde{N}(t)\left(1 - \frac{\tilde{N}(t)}{k}\right) + \mu\tilde{N}(t)\tilde{P}(t) + \beta_1\tilde{N}(t)\tilde{I}(t) \triangleq \mathbf{R}_1(t, \mathbf{B}, \mathbf{S}^1), \\ \frac{d\tilde{I}(t)}{dt} - \phi_1 I_0 - \phi_2\tilde{N}^2(t) + \phi_3\tilde{I}(t) + \beta_2\tilde{I}(t)\tilde{P}(t) \triangleq \mathbf{R}_2(t, \mathbf{C}, \mathbf{S}^2), \\ \frac{d\tilde{P}(t)}{dt} - \gamma\tilde{N}(t)\tilde{P}(t) + \delta\tilde{P}(t) + \beta_3\tilde{I}(t)\tilde{P}(t) \triangleq \mathbf{R}_3(t, \mathbf{D}, \mathbf{S}^3). \end{cases} \tag{22}$$

Here, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ denote unknown free coefficients. Additionally, $\mathbf{S}^i$, $(i = 1, 2, 3)$, are unknown control parameters, which are defined as follows:

$$\mathbf{S}^i = \left[\zeta_1^i,\ \zeta_2^i, \ldots, \zeta_{m_i}^i\right], \qquad i = 1, 2, 3. \tag{23}$$

The 2-norm of the residual functions is given by:

$$\mathfrak{R}\left(\mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{S}^1, \mathbf{S}^2, \mathbf{S}^3\right) = \int_0^T \left(\mathbf{R}_1^2(t, \mathbf{B}, \mathbf{S}^1) + \mathbf{R}_2^2(t, \mathbf{C}, \mathbf{S}^2) + \mathbf{R}_3^2(t, \mathbf{D}, \mathbf{S}^3)\right) dt. \tag{24}$$

It is required to evaluate the unknown vectors $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{S}^1$, $\mathbf{S}^2$, and $\mathbf{S}^3$ by employing the optimization problem as follows:

$$\min\ \mathfrak{R}\left(\mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{S}^1, \mathbf{S}^2, \mathbf{S}^3\right), \tag{25}$$

under the conditions in Equation (21) where $\mathfrak{R}$ is the objective function. To address this optimization problem, we assume the following:

$$\mathcal{J}^*[\mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{S}^1, \mathbf{S}^2, \mathbf{S}^3, \xi] = \mathfrak{R}\left(\mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{S}^1, \mathbf{S}^2, \mathbf{S}^3\right) + \xi\,\Lambda. \tag{26}$$

where $\Lambda = [\Lambda_1, \Lambda_2, \Lambda_3]^T$ is the known column vector of equality constraints and $\xi = [\xi_1, \xi_2, \xi_3]$ is the unknown Lagrange multipliers. By utilizing the Lagrange multiplier technique, we can derive the necessary and sufficient conditions for optimality, which are as follows:

$$\begin{cases} \dfrac{\partial\mathcal{J}^*}{\partial\xi} = 0, & \dfrac{\partial\mathcal{J}^*}{\partial\mathbf{B}} = 0, & \dfrac{\partial\mathcal{J}^*}{\partial\mathbf{C}} = 0, \\ \dfrac{\partial\mathcal{J}^*}{\partial\mathbf{D}} = 0, & \dfrac{\partial\mathcal{J}^*}{\partial\mathbf{S}^i} = 0, & i = 1, 2, 3. \end{cases} \tag{27}$$

By solving that system for the unknown vectors $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{S}^1$, $\mathbf{S}^2$, and $\mathbf{S}^3$, we attain the approximate optimal solution for Equation (1). In this study, the "fsolve" tool in Maple 18 is used to solve the algebraic system of equations derived in Equation (27). The procedure above outlines the application of the FNPNN method to solve the lung cancer mathematical model presented in Equation (1).

5 Convergence analysis

Theorem 5.1. *Consider* (e_n) *is a sequence in* X *such that* $Span\{\mathrm{e}_n | n \in \mathbb{N}\}$ *is dense in* X*;* $(X, \|\cdot\|)$ *is a normed linear space; and* $x_0 \in X$*. If* (x_n) *is the best approximation of* x_0 *in* $Span\{\mathrm{e}_1, \ldots, \mathrm{e}_n\}$*, then* (x_n) *converges to* x_0*.*

FNPNN Algorithm for solving a mathematical model of lung cancer in Equation (1)

Input: Initial conditions, number of basis functions (m_i, $i = 1, 2, 3$).

* Determine the unknown vectors $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ and matrices of NPs coefficients $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$ based on Equations (9)-(11).
* Determine the basis functions $s_j^i(t)$ by Equation (16).
* Determine the activation functions $\mathbf{N}(t, \mathbf{B})$, $\mathbf{I}(t, \mathbf{C})$ and $\mathbf{P}(t, \mathbf{D})$ using Equation (17).
* Calculate:
  - The initial conditions Λ_i, $i = 1, 2, 3$, using Equation (21).
  - The residual functions $\mathbf{R}_i$, $i = 1, 2, 3$, using Equation (22).
  - The 2-norm of the residual functions $\mathfrak{R}\left(\mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{S}^1, \mathbf{S}^2, \mathbf{S}^3\right)$ using Equation (24).
* Minimize the objective function $\mathfrak{R}\left(\mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{S}^1, \mathbf{S}^2, \mathbf{S}^3\right)$ under the constraints defined in Equation (21).
* Solve the nonlinear algebraic system using Equation (27).

Output: The approximate solutions are: $N(t) \simeq \tilde{N}(t) = N_0 + t\mathbf{N}(t, \mathbf{B})$, $I(t) \simeq \tilde{I}(t) = I_0 + t\mathbf{I}(t, \mathbf{C})$, $P(t) \simeq \tilde{P}(t) = P_0 + t\mathbf{P}(t, \mathbf{D})$.

Proof. Given that the closure of $Span\{\mathrm{e}_n | n \in \mathbb{N}\}$ is equal to X there exists a sequence (y_k) in $Span\{\mathrm{e}_n | n \in \mathbb{N}\}$ with $y_k \to x_0$. We suppose that $y_k \in Span\{\mathrm{e}_1, \ldots, \mathrm{e}_{n_k}\}$, in which (n_k) is an increasing sequence of natural numbers, without losing generality. Therefore, we can write

$$\|x_{n_k} - x_0\| \leq \|y_k - x_0\| \to 0, \quad (\text{as } k \to \infty). \tag{28}$$

It is also clear that the sequence $(\|x_n - x_0\|)_{n\in\mathbb{N}}$, consisting of non-negative real numbers, decreases. By applying (28), this sequence contains a convergent subsequence. As a result, (x_n) converges to x_0. ■

Remark 1. For each $T > 0$, we consider $X = L^2[0, T]$, equipped with $\|\cdot\|_2$, and $\mathrm{e}_n := \mathcal{B}_n$, the FNPs. Then, it implies from Theorem 5.1 that for each $x_0 \in L^2[0, T]$ the sequence (x_n) of best approximation of x_0 in $Span\{\mathcal{B}_1, \ldots, \mathcal{B}_n\}$, converges (in $\|\cdot\|_2$) to x_0.

Remark 2. Let $\tilde{\mathbf{N}}(t)$, $\tilde{\mathbf{I}}(t)$ and $\tilde{\mathbf{P}}(t)$ be the FNPNN approximate solutions of $\mathbf{N}(t)$, $\mathbf{I}(t)$ and $\mathbf{P}(t)$, respectively, we refer the reader to [27] for the error bound of approximations.

6 Existence and uniqueness of solutions for the system (1)

Lemma 6.1. *([28, 29]). Let $\Omega \subseteq \mathbb{R}^n$. Define the differential system*

$$\dot{x}(t) = f(t, x), \quad (t > 0), \tag{29}$$

with initial condition x_0, where $f : [0, \infty) \times \Omega \to \mathbb{R}^n$ satisfies locally Lipschitz condition with respect to x on $[0, \infty) \times \Omega$. Then this equation admits a unique solution $f(t, x)$.

Theorem 6.2. *For $0 < t < \infty$, the system of differential Equations (1) has a unique solution $\big(N(t), I(t), P(t)\big)$.*

Proof. Let $F : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ be defined for each $t \in [0, \infty)$ and $X = (N, I, P) \in \mathbb{R}^3$ by $F(t, X) = F(t, N, I, P) = \big(f_1(X), f_2(X), f_3(X)\big)$ where

$$\begin{aligned} f_1(X) &= \lambda N\left(1 - \tfrac{N}{k}\right) - \mu NP - \beta_1 NI, \\ f_2(X) &= \phi_1 I_0 + \phi_2 N^2 - \phi_3 I - \beta_2 IP, \\ f_3(X) &= \gamma NP - \delta P - \beta_3 IP. \end{aligned} \tag{30}$$

To prove that (1) admits a unique solution, by using Lemma 6.1, it is sufficient to show that F is locally Lipschitz in the region $[0, T] \times \mathcal{B}$ where M and T are both arbitrary positive real

numbers and $\mathcal{B} := \{X \in \mathbb{R}^3 : \ \|X\|_\infty \leq M\}$. Now, let $X = (N, I, P), \tilde{X} = (\tilde{N}, \tilde{I}, \tilde{P}) \in \mathcal{B}$. Then, it follows that

$$\begin{aligned}
\|F(X) - F(\tilde{X})\|_1 &= |f_1(X) - f_1(\tilde{X})| + |f_2(X) - f_2(\tilde{X})| + |f_3(X) - f_3(\tilde{X})| \\
&= |\lambda N(1 - \tfrac{N}{k}) - \mu NP - \beta_1 NI - (\lambda\tilde{N}(1 - \tfrac{\tilde{N}}{k}) - \mu\tilde{N}\tilde{P} - \beta_1\tilde{N}\tilde{I})| \\
&\quad + |\phi_1 I_0 + \phi_2 N^2 - \phi_3 I - \beta_2 IP - (\phi_1 I_0 + \phi_2\tilde{N}^2 - \phi_3\tilde{I} - \beta_2\tilde{I}\tilde{P})| \\
&\quad + |\gamma NP - \delta P - \beta_3 IP - (\gamma\tilde{N}\tilde{P} - \delta\tilde{P} - \beta_3\tilde{I}\tilde{P})| \\
&\leq |\lambda| \cdot |N - \tilde{N}| + \tfrac{|\lambda|M}{k}|N - \tilde{N}| + |\mu|M\big(|N - \tilde{N}| + |P - \tilde{P}|\big) \\
&\quad + |\beta_1|M\big(|N - \tilde{N}| + |I - \tilde{I}|\big) + \ |\phi_2|M \cdot |N - \tilde{N}| + |\phi_3| \cdot |I - \tilde{I}| \\
&\quad + |\beta_2|M\big(|I - \tilde{I}| + |P - \tilde{P}|\big) + \ |\gamma|M\big(|N - \tilde{N}| + |P - \tilde{P}|\big) \\
&\quad + |\delta| \cdot |P - \tilde{P}| + |\beta_3|M\big(|I - \tilde{I}| + |P - \tilde{P}|\big) \\
&= \Big((1 + \tfrac{M}{k})|\lambda| + (|\mu| + |\beta_1| + |\phi_2| + |\gamma|)M\Big)|N - \tilde{N}| \\
&\quad + \Big(|\phi_3| + (|\beta_1| + |\beta_2| + |\beta_3|)M\Big)|I - \tilde{I}| \\
&\quad + \Big(|\delta| + (|\mu| + |\gamma| + |\beta_2| + |\beta_3|)M\Big)|P - \tilde{P}| \\
&\leq \sigma(|N - \tilde{N}| + |I - \tilde{I}| + |P - \tilde{P}|) \\
&= \sigma\|X - \tilde{X}\|_1,
\end{aligned}$$

where $\sigma = \max\big\{(1 + \frac{M}{k})|\lambda| + (|\mu| + |\beta_1| + |\phi_2| + |\gamma|)M, \ \ |\phi_3| + (|\beta_1| + |\beta_2| + |\beta_3|)M, \ \ |\delta| + (|\mu| + |\gamma| + |\beta_2| + |\beta_3|)M\big\}$. Thus F is locally Lipschitz and, therefore, by using Lemma 6.1 the lung cancer model (1) admits a unique solution $\big(N(t), I(t), P(t)\big)$. ■

7 Solution method and approximate results

In this section, we validate and demonstrate the applicability of the proposed strategy for solving a mathematical model of lung cancer in Equation (1). All numerical calculations are performed using Maple 2024 and the CPU time (in seconds) is provided for an Intel Core i7 3.00GHz personal computer with 8 GB RAM. As the analytical solution for the given problem is unknown, we verify the reliability and accuracy of the technique by analyzing the residual error function. In order to draw the figures of approximations obtained for $N(t)$, $I(t)$ and $P(t)$, the following initial and parameter values are considered:

$$\begin{aligned}
&N_0 = 8, \ \ I_0 = 5, \ \ P_0 = 3, \ \ \lambda = 0.30, \ \ k = 10000, \ \ \mu = 0.01, \ \ \beta_1 = 0.01, \ \ \beta_2 = 0.01, \\
&\beta_3 = 0.04, \ \ \phi_1 = 0.03, \ \ \phi_2 = 0.04, \ \ \phi_3 = 0.01, \ \ \gamma = 0.07, \ \ \delta = 0.001.
\end{aligned} \tag{31}$$

The results obtained are depicted in Figures 1 to 3 with $m_1 = m_2 = m_3 = 2$ over a time period of 1 month (left side) and 5 months (right side). The runtime of the mentioned method for a time period of 5 months for several m_1, m_2 and m_3 values is summarized in Table 2.

Sensitivity analysis plays a central role in appraising the soundness and reliability of mathematical models across various disciplines. It helps in distinguishing which parameters have the

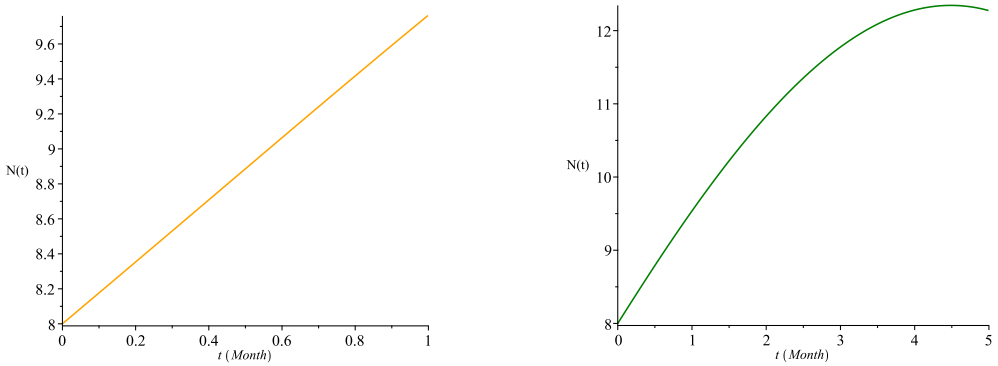

Figure 1: The FNPNN solutions of $N(t)$ with $m_1 = m_2 = m_3 = 2$ over a time period of 1 month (left side) and 5 months (right side).

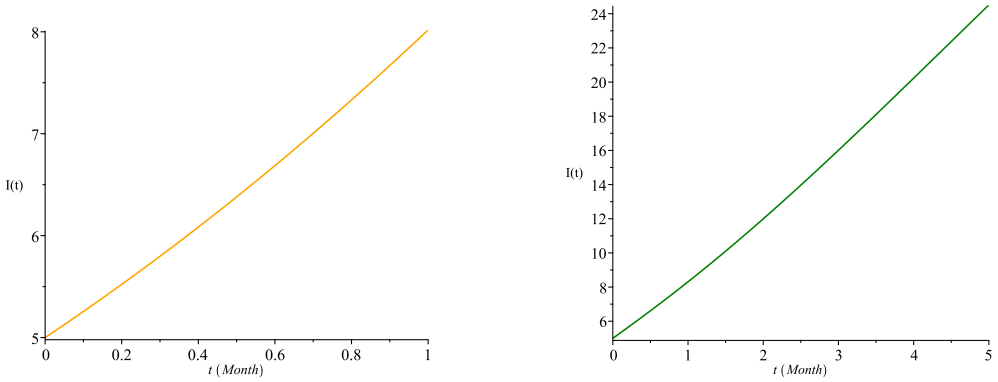

Figure 2: The FNPNN solutions of $I(t)$ with $m_1 = m_2 = m_3 = 2$ over a time period of 1 month (left side) and 5 months (right side).

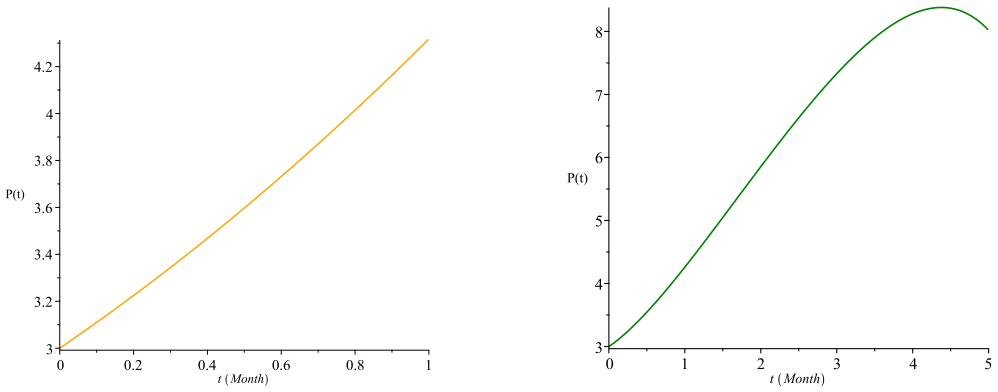

Figure 3: The FNPNN solutions of $P(t)$ with $m_1 = m_2 = m_3 = 2$ over a time period of 1 month (left side) and 5 months (right side).

Table 2: The runtime of the FNPNN method a time period of 5 months with several values of m_1, m_2 and m_3 (in seconds).

case	m_1	m_2	m_3	CPU times
1	1	1	1	5.17
2	2	1	2	9.76
3	2	2	2	12.27
4	3	3	2	15.43

most operative effect on outputs of the given model. By figuring out the sensitivity of the given model to different parameters, researchers can optimize the model structure or parameter values to achieve better performance. In this section, some scenarios will be designed to demonstrate the sensitivity of the suggested model to diverse parameters.

1. Sensitivity to λ: By choosing values of λ as $\lambda = 0.40$, model (1) is solved by the method stated in the previous subsection and figures of $N(t)$, $I(t)$ and $P(t)$ are depicted in Figure 4 with $m_1 = m_2 = m_3 = 2$ over a time period of 5 months.

2. Sensitivity to β_1: By choosing values of β_1 as $\beta_1 = 0.02$, model (1) is solved by the method stated in the previous subsection and figures of $N(t)$, $I(t)$ and $P(t)$ are depicted in Figure 5 with $m_1 = m_2 = m_3 = 2$ over a time period of 5 months.

3. Sensitivity to ϕ_2: By choosing values of ϕ_2 as $\phi_2 = 0.05$, model (1) is solved by the method stated in the previous subsection and figures of $N(t)$, $I(t)$ and $P(t)$ are depicted in Figure 6 with $m_1 = m_2 = m_3 = 2$ over a time period of 5 months.

The number of cancer cells in lung tissues increases over time and reaches a plateau after five months (Figure 1). The number of immune cells steadily increases (Figure 2). Cancer cells spreading to other organs and representing distant metastasis increase over time, but reach a plateau after five months (Figure 3). Figure 4 shows immune cells in the body ($I(t)$), lung cancer growth rate ($N(t)$), and metastatic cells or cells spreading to other body organs ($P(t)$) increase over time. However, the latter two reach a plateau after some time and then begin to decrease when $\lambda = 0.40$. Figures 5 and 6 show that $\beta_1 = 0.02$ and $\phi_2 = 0.05$ lead to the same results.

The mathematical lung cancer model (1) is also solved using the techniques discussed in Section 4, but with the NPNN approach. Table 3 lists the values obtained for the optimal residual function for various values of m_1, m_2, and m_3 using both NPNN and FNPNN over a time period of 5 months. The results presented in Table 3 indicate that the proposed method is more accurate than the NPNN-based results.

Table 3: The optimal residual function values for various values of m_i a time period of 5 months using NPNN and FNPNN, where $i = 1, 2, 3$.

case	m_1	m_2	m_3	Residual function (NPNN)	Residual function (FNPNN)
1	1	1	1	$5.4094E-02$	$3.2973E-05$
2	2	1	2	$6.1869E-03$	$6.3357E-07$
3	2	2	2	$3.3834E-03$	$1.8194E-08$
4	3	3	2	$9.3626E-05$	$3.1411E-11$

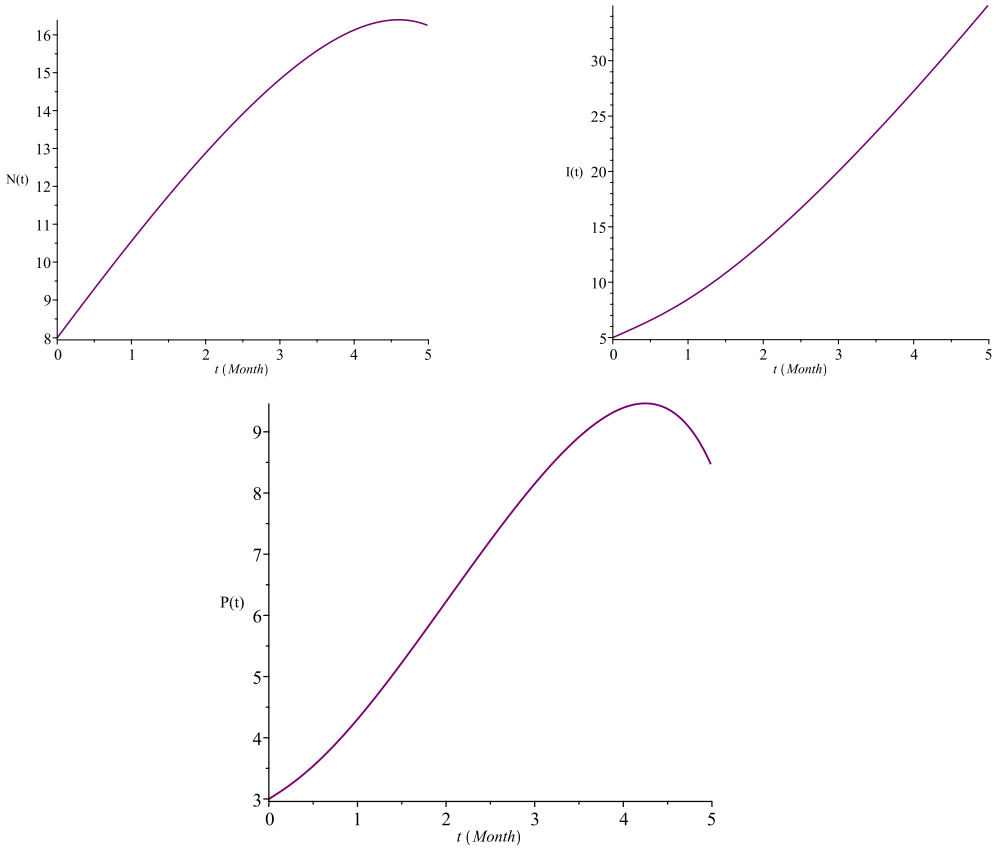

Figure 4: The FNPNN solutions of $N(t)$, $I(t)$ and $P(t)$ with $m_1 = m_2 = m_3 = 2$ and $\lambda = 0.40$ over a time period of 5 months.

8 Conclusion and future directions

In this paper, an effective algorithm based on the fractional Narayana polynomials neural network is applied for solving a mathematical model of lung cancer. We employed the fractional Narayana polynomials and $arcsinh(t)$ functions as the activation functions for both the hidden and output layers, respectively. The network is trained using the classical optimization approach. This approach effectively reduces the mathematical model of lung cancer to systems of algebraic equations. The suggested methodology is simulated numerically, and the computational results demonstrate the accuracy and efficiency of the FNPNN approach for the mentioned model. As the figures demonstrate, the suggested approach provides a stable solution and yields an approximate solution using a small number of basis functions. We recommend the following directions for future studies:
-The method presented here can be used in numerous applications, such as fractional variable-order models, fractional optimal control problems, fractional-order partial differential equations, fractional-order systems of partial differential equations, and various models in bioscience, etc.
– We can use fractional Bernoulli, fractional Chebyshev, fractional Legendre and fractional Laguerre polynomials instead of fractional Narayana polynomials.

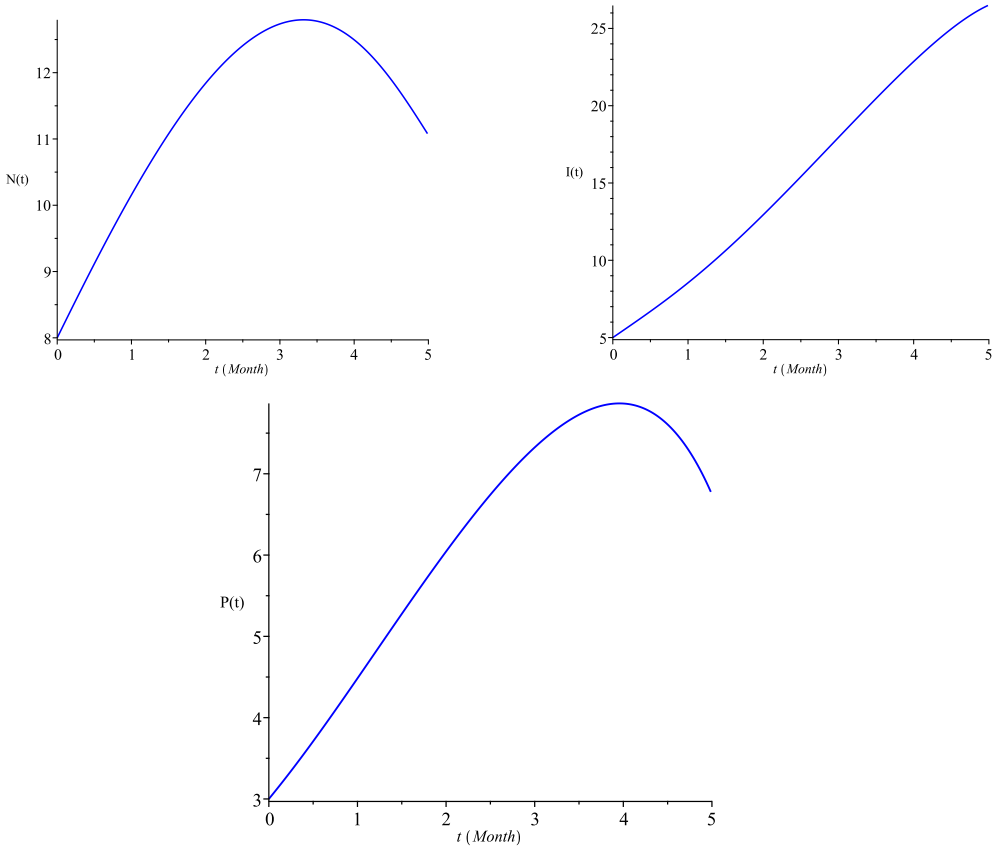

Figure 5: The graphs of the approximate solutions of $N(t)$, $I(t)$ and $P(t)$ with $m_1 = m_2 = m_3 = 2$ and $\beta_1 = 0.02$ over a time period of 5 months.

– The presented problem can be solved by using least squares-support vector regression.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

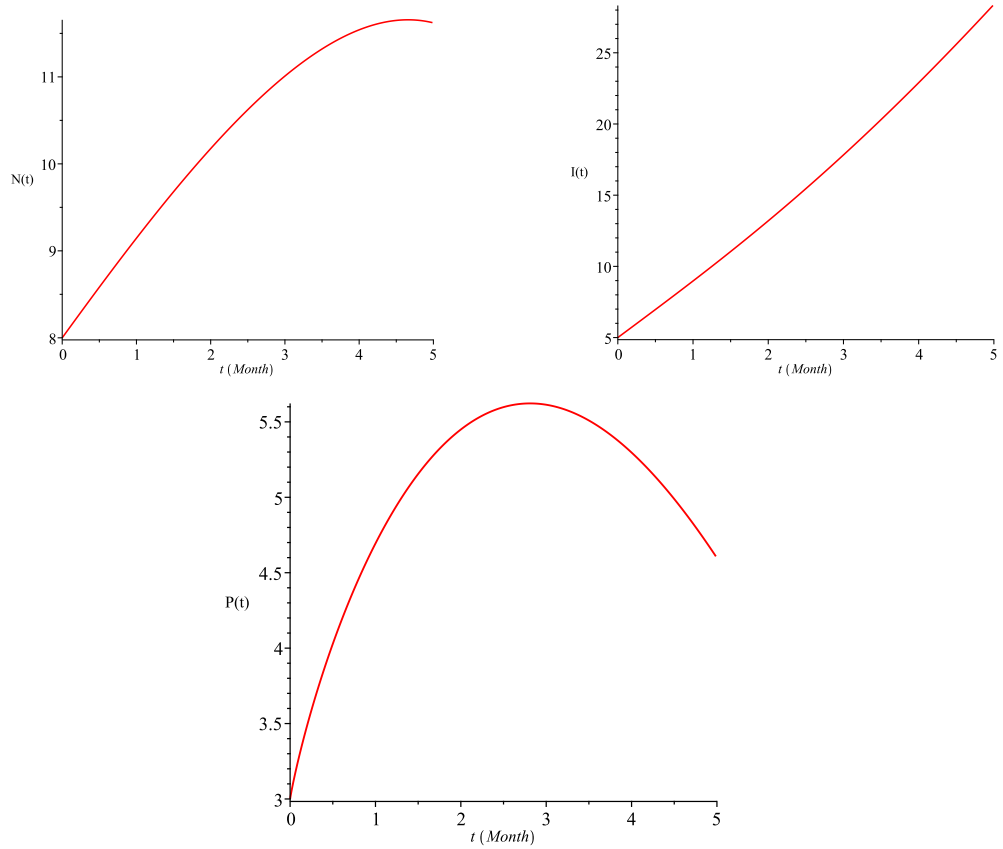

Figure 6: The FNPNN solutions of $N(t)$, $I(t)$ and $P(t)$ with $m_1 = m_2 = m_3 = 2$ and $\phi_2 = 0.05$ over a time period of 5 months.

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