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Research Paper

Topological Indices and Disjoint Path Cover Property of Graphs

Honggang Zhao[1] and Eminjan Sabir[1*]

[1]College of Mathematics and System Sciences, Xinjiang University, Urumqi, 830046, P. R. China

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Abstract

The disjoint path cover problem is closely related to the well-known Hamiltonian problem, which is a fundamental concept in graph theory. In the domains of bioinformatics and neuroinformatics, the existence of a disjoint path cover indicates the cascade effect within the signal transduction system and the reaction occurring in a metabolic pathway. One of the core subjects in exploring the disjoint path cover problem is to develop sufficient conditions. In this paper, we provide novel sufficient conditions, with respect to some significant topological indices including Harary index, first Zagreb index, forgotten topological index, reciprocal degree distance, eccentric connectivity index, eccentric distance sum, and connective eccentric index, for a connected graph to be disjoint path coverable.

1 Introduction

We focus solely on finite, connected, simple undirected graphs. *Simple graphs* are those that do not contain loops or multiple edges. For any notation and terminology not explained herein, we adhere to the conventions presented in [1]. For a graph $G = (V(G), E(G))$, the *order* $|V(G)|$ is denoted by n, and the *size* $|E(G)|$ is denoted by m. Let $V(G) = \{u_1, u_2, \ldots, u_n\}$ and its degree sequence be $d(u_1) \leq d(u_2) \leq \cdots \leq d(u_n)$ or simply as $d_1 \leq d_2 \leq \cdots \leq d_n$. We denote the *distance* between u_i and u_j as $d_G(u_i, u_j)$ and the *edge* joining them as u_iu_j. The *eccentricity* of a vertex u_i, denoted $\varepsilon_G(u_i)$, is defined as $\varepsilon_G(u_i) = \max\{d_G(u_i, u_j) \mid u_j \in V(G)\}$. The *distance sum* of u_i, denoted $D_G(u_i)$, is given by $D_G(u_i) = \sum_{u_j \in V(G)} d_G(u_i, u_j)$.

In 1993, both Ivanciuc et al. and Plavšić et al. independently proposed the Harary index as a molecular descriptor [2, 3]. The Harary index of G, symbolized by $H(G)$, was formulated as:

$$H(G) = \sum_{1 \leq i < j \leq n} \frac{1}{d_G(u_i, u_j)}.$$

*Corresponding author
E-mail addresses: zhg07214652@163.com (H. Zhao), eminjan20150513@163.com (E. Sabir)
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Gutman and Trinajstić estimated the total π-electron energy of conjugated systems in [4]. The formula they used contains two forms, which were later named the first Zagreb index $M_1(G)$ and the second Zagreb index $M_2(G)$. They were defined as:

$$M_1(G) = \sum_{u_i \in V(G)} d(u_i)^2 \quad \text{and} \quad M_2(G) = \sum_{u_i u_j \in E(G)} d(u_i)d(u_j).$$

Furtula and Gutman explored the branching of the carbon atom skeleton of the relevant underlying molecular graph and introduced a topological index named the *forgotten topological index*, denoted as $F(G)$ [5], which was formulated as:

$$F(G) = \sum_{u_i u_j \in E(G)} (d(u_i)^2 + d(u_j)^2) = \sum_{u_i \in V(G)} d(u_i)^3.$$

Alizadeh et al. and Li independently introduced a vertex-degree-weighted version of the Harary index called *reciprocal degree distance* or *additively weighted Harary index* [6, 7], which was denoted $RDD(G)$ for convenience and defined as:

$$RDD(G) = \sum_{1 \leq i < j \leq n} \frac{d(u_i) + d(u_j)}{d_G(u_i, u_j)}.$$

There are three topological indices that can forecast both physical and biological properties, and they possess great potential in the exploration of structure-activity/property relationships. The *eccentric connectivity index* (ECI) [8], denoted by $\xi^c(G)$, was defined as:

$$\xi^c(G) = \sum_{u_i \in V(G)} \varepsilon(u_i)d(u_i).$$

The *eccentric distance sum* (EDS) [9], denoted by $\xi^d(G)$, was defined as:

$$\xi^d(G) = \sum_{u_i \in V(G)} \varepsilon(u_i)D(u_i).$$

The *connective eccentric index* (CEI) [10], denoted by $\xi^{ce}(G)$, was defined as:

$$\xi^{ce}(G) = \sum_{u_i \in V(G)} \frac{d(u_i)}{\varepsilon(u_i)}.$$

For a graph G, a cycle C is referred to as a *Hamiltonian cycle* of G when all vertices of G are in C. A graph G is called *Hamiltonian* if it has a Hamiltonian cycle. A path P in G is designated as a *Hamiltonian path* of G when it contains all vertices of G. If a graph G contains a Hamiltonian path, it is called *traceable*. A graph G is called *Hamilton-connected* if for every pair of vertices in G, a Hamiltonian path connecting them exists. When G and H are two vertex-disjoint graphs, the *join* of G and H is represented by $G \vee H$.

For graph G, we define a *disjoint path cover* (DPC for short) as a set of paths, with the property that each vertex of G lies on precisely one of these paths. Given two disjoint vertex subsets $A = \{a_1, a_2, \ldots, a_t\}$ and $B = \{b_1, b_2, \ldots, b_t\}$ of a graph G, a *many-to-many t-disjoint path cover* (M-M t-DPC for short) of G is a DPC consisting of t paths and each of these paths connects a vertex a_i in A to a vertex $b_{\tau(i)}$ in B, where τ is a permutation of $\{1, 2, \ldots, t\}$. A graph G with at least $2t$ vertices is called a *many-to-many t-disjoint path coverable* (M-M t-DPCe for short) if, for any two disjoint vertex subsets $A = \{a_1, a_2, \ldots, a_t\}$ and $B = \{b_1, b_2, \ldots, b_t\}$, there exists a M-M t-DPC between them [11]. By modifying the subsets A and B, we can obtain two special cases of the M-M t-DPC: the *one-to-many t-disjoint path cover* (1-M t-DPC for short) with $A = \{a\}$ and $B = \{b_1, b_2, \ldots, b_t\}$ and the *one-to-one t-disjoint path cover* (1-1 t-DPC for short) with $A = \{a\}$ and $B = \{b\}$. A graph G with at least $t + 1$ vertices is *one-to-many t-disjoint path coverable* (1-M t-DPCe for short) if there exists a one-to-many t-disjoint path cover for any two disjoint vertex subsets $A = \{a\}$ and $B = \{b_1, b_2, \ldots, b_t\}$. A graph G with at least $t + 1$ vertices is *one-to-one t-disjoint path coverable* (1-1 t-DPCe for short) if there exists a one-to-one t-disjoint path cover for any two disjoint vertex subsets $A = \{a\}$ and $B = \{b\}$.

In the domains of bioinformatics and neuroinformatics, not only the existence of a DPC but also its structure indicates the cascade effect within the signal transduction system and the reaction occurring in a metabolic pathway [12]. Beyond theoretical interest, these path cover properties find profound applications in molecular systems where functional robustness depends on alternative reaction channels. Recent advances in topological analysis of molecular dynamics reveal that proteins like the villin headpiece subdomain utilize multiple disjoint conformational pathways for efficient folding, with each pathway acting as an independent route when others are obstructed [13]. This biological principle of redundancy directly corresponds to our M-M t-DPC framework, where t disjoint paths ensure system functionality despite local failures. In enzymatic mechanisms, graph transformation rules formalize how enzymes orchestrate complex multi-step reactions through precisely arranged atom rearrangements, with disjoint path structures preventing undesired cross-talk between reaction channels [14]. These principles extend to metabolic engineering, where artificial intelligence now designs robust biosynthetic pathways by intentionally creating multiple disjoint routes for critical transformations, which is a strategy that dramatically improves yield stability under fluctuating cellular conditions [15]. From a systems perspective, such multi-pathway architectures represent external dilation-dominated systems with surplus sink nodes that display correlated behavior across downstream agents [16]. Therefore, the threshold conditions we established for various DPC properties not only advance graph theory but also provide predictive tools for designing molecular systems with programmable robustness where the number of disjoint paths directly quantifies functional resilience against mutations, inhibitors, or environmental perturbations.

Besides, the DPC problem is strongly related to the well-known Hamiltonian problem, which is a fundamental one in graph theory. Actually, a Hamiltonian path joining a pair of vertices in a graph forms a M-M 1-DPC, 1-M 1-DPC, and 1-1 1-DPC of the graph. A graph of order $n \geq 3$ is 1-M 2-DPCe or M-M 1-DPCe if and only if it is Hamilton-connected. Moreover, a graph of order $n \geq 3$ is 1-1 2-DPCe if and only if it is Hamiltonian.

In 2013, Hua et al. gave a sufficient condition for a graph being traceable in terms of the Harary index [17]. Later, using similar approaches, Li proved the parallel results on the Hamiltonian and Hamilton-connected graphs [7]. For more research on Harary index and Hamiltonian properties, one can refer to [18–20]. It is natural to consider the sufficient conditions by using other topological indices. An presented a sufficient condition in terms of the first Zagreb index or the reciprocal degree distance for a graph to be Hamilton-connected [21]. An et al. also studied the first Zagreb index for k-connectivity, β-deficiency, and k-Hamiltonicity of graphs [22]. In 2023, Kori et al. gave a sufficient condition for a graph to be Hamiltonian in terms of $RDD(G)$ [23]. We point out that there is a counterexample to their result. Later, we will provide a few minor corrections of it. Zhu et al. presented sufficient conditions for Hamiltonian properties in graphs by leveraging the eccentric connectivity index, the eccentric distance sum, and the connective eccentricity index [24]. The k-leaf-connected graphs are a kind of generalization of Hamilton-connected graphs. In 2024, An et al. and Liu et al. independently presented sufficient conditions for k-leaf-connected graphs. Their work relied on multiple indices, such as the first Zagreb index, forgotten topological index, reciprocal degree distance, eccentric connectivity index, eccentric distance sum, and connective eccentric index [25, 26].

As mentioned earlier, the disjoint path coverable property of graphs is a natural extension of hamiltonicity. For disjoint path coverable graphs, there are several sufficient conditions with respect to Dirac-type (minimum degree), Ore-type (degree sum), Pósa-type and Bondy-type conditions [11, 27–29]. To our knowledge, for $t \geq 3$, there are no sufficient conditions in terms of topological indices for t-disjoint path coverable graphs. This motivates the current study.

In this paper, we use the above-mentioned topological indices to present several new sufficient conditions for a graph to be disjoint path coverable. In the next section, we give the necessary lemmas. Section 3 contains our main results and their proofs.

2 Preliminaries

In order to prove our main results, we need the following lemmas as our tools.

Lemma 2.1. *([29]). For an n-vertex graph G with $n \geq t+1 \geq 3$, let its degree sequence be $d_1 \leq d_2 \leq \cdots \leq d_n$. If there is no integer $t-1 \leq s \leq \frac{1}{2}(n+t-3)$ such that $d_{s-t+2} \leq s$ and $d_{n-s} \leq n-s+t-3$, then G is 1-1 t-DPCe.*

Lemma 2.2. *([29]). For an n-vertex graph G with $n \geq t+1 \geq 3$, let its degree sequence be $d_1 \leq d_2 \leq \cdots \leq d_n$. If there is no integer $t \leq s \leq \frac{1}{2}(n+t-2)$ such that $d_{s-t+1} \leq s$ and $d_{n-s} \leq n-s+t-2$, then G is 1-M t-DPCe.*

Lemma 2.3. *([29]). For an n-vertex graph G with $n \geq 2t \geq 2$, let its degree sequence be $d_1 \leq d_2 \leq \cdots \leq d_n$. If there is no integer $t+1 \leq s \leq \frac{1}{2}(n+t-1)$ such that $d_{s-t} \leq s$ and $d_{n-s} \leq n-s+t-1$, then G is M-M t-DPCe.*

Let K_n be a complete graph on n vertices. To facilitate our subsequent work, we present the following structure. For $1 \leq t \leq n-1$, we define:

$$Z_n^t = K_t \vee (K_{n-t-1} \cup K_1).$$

Actually, the graph Z_n^t is obtained from K_n by deleting $n-t-1$ edges that are incident to one common vertex in K_n. Obviously, the degree of this common vertex is t. Figure 1 illustrates this construction for $t=5$ and $n \geq 6$. The following results for Z_n^t can be readily verified, and all of them are optimal for t. **Remark 1.** (i): For $3 \leq t+1 \leq n$, Z_n^t is 1-1 t-DPCe.

(ii): For $3 \leq t+1 \leq n-1$, Z_n^{t+1} is 1-M t-DPCe.

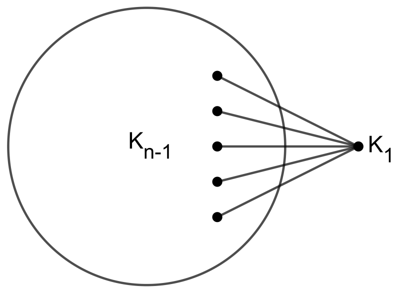
(iii): For $4 \leq 2t \leq n$, Z_n^{t+1} is M-M t-DPCe, and for $2 = 2t \leq n-2$, Z_n^{t+1} is not M-M t-DPCe.

3 Main results

3.1 Harary index

In this subsection, we will prove several theorems on the t-DPCe graphs based on Harary index, a significant topological descriptor in graph theory. For a graph G, there is a simple relationship between the Harary index, order, and degrees:

$$H(G) = \sum_{1 \leq i < j \leq n} \frac{1}{d_G(u_i, u_j)} = \frac{1}{2} \sum_{u_i \in V(G)} \left(\sum_{u_j \in V(G)} \frac{1}{d_G(u_i, u_j)} \right)$$



Z_n^5

Figure 1: Graph Z_n^5.

$$\leq \frac{1}{2}\sum_{u_i\in V(G)}\left(d(u_i)+\frac{1}{2}\left(n-1-d(u_i)\right)\right)=\frac{1}{2}\sum_{u_i\in V(G)}\left(\frac{n-1}{2}+\frac{1}{2}d(u_i)\right)$$
$$=\frac{1}{4}\left(n(n-1)+\sum_{u_i\in V(G)}d(u_i)\right).$$

Theorem 3.1. *For an n-vertex graph G with $n \geq 2t \geq 2$, if $H(G) \geq H(Z_n^{t+1})$, then G is M-M t-DPCe, unless $G \in \{Z_n^2, K_3 \vee (3K_1), K_4 \vee (3K_1)\}$.*

Proof. Suppose G is a graph satisfying the conditions above, yet G is not M-M t-DPCe. By Lemma 2.3, there must be an integer s with $t+1 \leq s \leq \frac{1}{2}(n+t-1)$ that satisfies both $d_{s-t} \leq s$ and $d_{n-s} \leq n-s+t-1$. Therefore

$$\begin{aligned}H(G) &\leq \frac{1}{4}\left(n(n-1)+\sum_{u_i\in V(G)}d(u_i)\right)\\ &\leq \frac{1}{4}\left(n(n-1)+(s-t)s+(n-2s+t)(n-s+t-1)+s(n-1)\right)\\ &=\frac{1}{4}(n(n-1)+(t+1)n+(n-t-2)(n-2)-(s-t-1)(2n+t-3s-4))\\ &=\frac{1}{4}(2(n^2-2n+t+2)-2(s-t-1)(n+t-2s-1)\\ &\quad-(s-t-1)(s-t-2)).\end{aligned}$$

Note that n, s, t are integers and $t+1 \leq s \leq \frac{1}{2}(n+t-1)$. This implies $H(G) \leq \frac{n^2-2n+t+2}{2}$ and $n \geq t+3$. By a simple calculation, we obtain $H(Z_n^{t+1}) = \frac{n^2-2n+t+2}{2}$. Then from $H(G) \geq H(Z_n^{t+1})$, it follows that $H(G) = \frac{n^2-2n+t+2}{2}$, $d_1 = \cdots = d_{s-t} = s$, $d_{s-t+1} = \cdots = d_{n-s} = n-s+t-1$, and $d_{n-s+1} = \cdots = d_n = n-1$, where $s = t+1$ or $s = t+2$ and $n = 2s-t+1$.

If $s = t+1$, then we have $d_1 = t+1$, $d_2 = \cdots = d_{n-t-1} = n-2$, and $d_{n-t} = \cdots = d_n = n-1$. It is inferred that $G_1 = Z_n^{t+1}$. If $s = t+2$ and $n = 2s-t+1$, then $n = t+5$. Note that $n \geq 2t \geq 2$. Then $d_1 = d_2 = t+2$, $d_3 = t+2$, and $d_4 = \cdots = d_{t+5} = t+4$, where $t \in \{1, 2, \ldots, 5\}$. Thus, $G_2 = K_{t+2} \vee (3K_1)$, $t \in \{1, 2, \ldots, 5\}$.

By **Remark 1** (iii), G_1 is M-M t-DPCe, unless $G_1 = Z_n^2$. Obviously, $K_{t+2} \vee (3K_1)$ is M-M t-DPCe for every $t \in \{3, 4, 5\}$, and $K_{t+2} \vee (3K_1)$ is not M-M t-DPCe for every $t \in \{1, 2\}$. ■

By a simple calculation, we obtain $H(Z_n^2) = \frac{n^2-2n+3}{2}$. Note that a graph G is M-M 1-DPCe if and only if it is Hamilton-connected. Then, the following result can be deduced from Theorem 3.1 when $t = 1$.

Corollary 3.2. *(**Li** [7]). For an n-vertex graph G, if $H(G) \geq \frac{n^2-2n+3}{2}$, then G is Hamilton-connected, unless $G = Z_n^2$, or $G = K_3 \vee (3K_1)$.*

Next, we will provide sufficient conditions for a graph to be 1-M t-DPCe. By replacing t with $l + 1$ in Lemma 2.2, we have the following lemma:

Lemma 3.3. *For an n-vertex graph G with $n \geq l + 2 \geq 3$, let its degree sequence be $d_1 \leq d_2 \leq \cdots \leq d_n$. If there is no integer $l+1 \leq s \leq \frac{1}{2}(n+l-1)$ such that $d_{s-l} \leq s$ and $d_{n-s} \leq n-s+l-1$, then G is 1-M $(l+1)$-DPCe.*

Note that Lemma 3.3 and Lemma 2.3 have similar forms. Thus, we can derive a result similar to Theorem 3.1 by applying Lemma 3.3. For completeness of the result, we need to consider whether $G_1 = Z_n^{l+1}$ and $G_2 = K_{l+2} \vee (3K_1)$ for $n \geq l + 2 \geq 3$ are 1-M $(l + 1)$-DPCe. By **Remark 1** (ii), Z_n^{l+1} is not 1-M $(l + 1)$-DPCe. We can observe that $K_{l+2} \vee (3K_1)$ is not 1-M $(l + 1)$-DPCe. It is worth noting that the result we derived here contains the parameter l. Thus, after making a replacement $l = t - 1$, we obtain the following theorem.

Theorem 3.4. *For an n-vertex graph G with $n \geq t + 1 \geq 3$, if $H(G) \geq H(Z_n^t)$, then G is 1-M t-DPCe, unless $G = H(Z_n^t)$, or $G = K_{t+1} \vee (3K_1)$.*

Since a graph G is 1-M 2-DPCe if and only if it is Hamilton-connected, Corollary 3.2 can also be directly obtained when $t = 2$ in Theorem 3.4.

At the end of this subsection, we obtain a sufficient condition for a graph to be 1-1 t-DPCe with a similar approach used in Theorem 3.4.

Theorem 3.5. *For an n-vertex graph G with $n \geq t + 1 \geq 3$, if $H(G) \geq H(Z_n^{t-1})$, then G is 1-1 t-DPCe, unless $G = Z_n^{t-1}$, or $G = K_t \vee (3K_1)$.*

Since a graph G is 1-1 2-DPCe if and only if it is Hamiltonian and $H(Z_n^1) = \frac{n^2-2n+2}{2}$, the following corollary can be deduced from Theorem 3.5 when $t = 2$.

Corollary 3.6. *(**Li** [7]). For an n-vertex graph G with $n \geq 3$, if $H(G) \geq \frac{n^2-2n+2}{2}$, then G is Hamiltonian, unless $G = Z_n^1$, or $G = K_2 \vee (3K_1)$.*

In the subsequent subsections, we will frequently use a method similar to the one in this subsection without further explanation.

3.2 First Zagreb index, reciprocal degree distance and forgotten topological index

The following several lemmas are crucial.

Lemma 3.7. *For an n-vertex graph G with $n \geq 2t \geq 2$ and $t \leq n - 5$, if $M_1(G) \geq M_1(Z_n^{t+1})$, then G is M-M t-DPCe, unless $G = Z_n^2$. Moreover, $M_1(G) = M_1(Z_n^{t+1})$ if and only if $G \cong Z_n^{t+1}$.*

Proof. Suppose G is a graph satisfying the conditions above, yet G is not M-M t-DPCe. By Lemma 2.3, there must be an integer s with $t+1 \leq s \leq \frac{1}{2}(n+t-1)$ that satisfies both $d_{s-t} \leq s$ and $d_{n-s} \leq n-s+t-1$. Therefore

$$\begin{aligned} M_1(G) &= \sum_{j=1}^{n} d_j^2 \\ &\leq (s-t)s^2 + (n-2s+t)(n-s+t-1)^2 + s(n-1)^2 \\ &= -s^3 + (5n+4t-4)s^2 + \left((n-1)^2 - 2(n+t-1)(2n+2t-1)\right) s \\ &\quad + (n+t)(n+t-1)^2. \end{aligned}$$

Consider the function:

$$f(x) = -x^3 + (5n+4t-4)x^2 + \left((n-1)^2 - 2(n+t-1)(2n+2t-1)\right) x,$$

where $t+1 \leq x \leq \frac{1}{2}(n+t-1)$. By differentiation, we derive:

$$f'(x) = -3x^2 + 2(5n+4t-4)x + \left((n-1)^2 - 2(n+t-1)(2n+2t-1)\right),$$

and $f''(x) = -6x + 2(5n+4t-4)$. Note that $t+1 \leq x \leq \frac{1}{2}(n+t-1)$ and $t \geq 1$. Then

$$f''(x) > 2(5n+4t-4) - 3(n+t) \geq 0.$$

It follows that $f(x)$ is a strictly convex function on $x \in [t+1, \frac{n+t-1}{2}]$, where n and t are integers. Thus, one can obtain $f(x) \leq \max\{f(t+1), f(\lfloor \frac{n+t-1}{2} \rfloor)\}$. Direct computation yields

$$f(\lfloor \tfrac{n+t-1}{2} \rfloor) - f(t+1) = \begin{cases} -\frac{1}{8}(t-n+4)(t-n+2)(t+3n-4), & \text{if } n+t \text{ is even,} \\ -\frac{1}{8}(t-n+5)(t-n+3)(t+3n-3), & \text{if } n+t \text{ is odd.} \end{cases}$$

Since $n \geq 2t \geq 2$ and $t \leq n-5$, $f(\lfloor \frac{n+t-1}{2} \rfloor) - f(t+1) \leq 0$. Then $f(x) \leq f(t+1)$. Note that

$$\begin{aligned} M_1(Z_n^{t+1}) &= (t+1)^2 + (t+1)(n-1)^2 + (n-t-2)(n-2)^2 \\ &= f(t+1) + (n+t)(n+t-1)^2. \end{aligned}$$

Then $M_1(G) \leq M_1(Z_n^{t+1})$. From $M_1(G) \geq M_1(Z_n^{t+1})$, we obtain $M_1(G) = M_1(Z_n^{t+1})$. Then all inequalities in the discussion above have to be equal. Thus, $s = t+1$, and we have $d_1 = t+1$, $d_2 = \cdots = d_{n-t-1} = n-2$ and $d_{n-t} = \cdots = d_n = n-1$. This implies $G \cong Z_n^{t+1}$, which is M-M t-DPCe except $t = 1$. Conversely, if $G \cong Z_n^{t+1}$, one can easily see that $M_1(G) = M_1(Z_n^{t+1})$. ■

Lemma 3.8. *For an n-vertex graph G with $n \geq 2t \geq 2$ and $t \leq n-5$, if $RDD(G) \geq RDD(Z_n^{t+1})$, then G is M-M t-DPCe, unless $G = Z_n^2$. Moreover, $RDD(G) = RDD(Z_n^{t+1})$ if and only if $G \cong Z_n^{t+1}$.*

Proof. Suppose G is a graph satisfying the conditions in Lemma 3.8, yet G is not M-M t-DPCe. By Lemma 2.3, there must be an integer s with $t+1 \leq s \leq \frac{1}{2}(n+t-1)$ that satisfies both $d_{s-t} \leq s$ and $d_{n-s} \leq n-s+t-1$. Therefore

$$\begin{aligned} RDD(G) &= \sum_{1 \leq i < j \leq n} \frac{d_i + d_j}{d_G(u_i, u_j)} = \sum_{u_j \in V(G)} d_j \left(\sum_{u_i \in V(G)} \frac{1}{d_G(u_i, u_j)} \right) \\ &\leq \sum_{u_j \in V(G)} d_j \left(d_j + \frac{1}{2}(n-1-d_j) \right) = \frac{1}{2} \left((n-1) \sum_{u_j \in V(G)} d_j + \sum_{u_j \in V(G)} d_j^2 \right) \end{aligned}$$

$$\begin{aligned}&\le \frac{1}{2}(n-1)((s-t)s+(n-2s+t)(n-s+t-1)+s(n-1))\\&\quad+\frac{1}{2}\left((s-t)s^2+(n-2s+t)(n-s+t-1)^2+s(n-1)^2\right)\\&=\frac{1}{2}(n+t)(n+t-1)(2n+t-2)+\frac{1}{2}(-s^3+(8n+4t-7)s^2\\&\quad-((n-1)(n+4t)+2(n+t-1)(2n+2t-1))s).\end{aligned}$$

Consider the function:

$$f(x)=-x^3+(8n+4t-7)x^2-((n-1)(n+4t)+2(n+t-1)(2n+2t-1))x,$$

where $t+1\le x\le \frac{1}{2}(n+t-1)$. By differentiation, we derive:

$$f'(x)=-3x^2+2(8n+4t-7)x-((n-1)(n+4t)+2(n+t-1)(2n+2t-1)),$$

and $f''(x)=-6x+2(8n+4t-7)$.
Note that $t+1\le x\le \frac{1}{2}(n+t-1)$ and $t\ge 1$. Then

$$f''(x)>2(8n+4t-7)-3(n+t)\ge 0.$$

It follows that $f(x)$ is a strictly convex function on $x\in[t+1,\frac{n+t-1}{2}]$, where n and t are integers. Thus, one can obtain $f(x)\le \max\{f(t+1),f(\lfloor\frac{n+t-1}{2}\rfloor)\}$. Direct computation yields

$$f(\lfloor\tfrac{n+t-1}{2}\rfloor)-f(t+1)=\begin{cases}-\frac{1}{8}(t-n+4)(t-n+2)(t+5n-6), & \textit{if } n+t \textit{ is even,}\\ -\frac{1}{8}(t-n+5)(t-n+3)(t+5n-5), & \textit{if } n+t \textit{ is odd.}\end{cases}$$

Since $n\ge 2t\ge 2$ and $t\le n-5$, $f(\lfloor\frac{n+t-1}{2}\rfloor)-f(t+1)\le 0$. Then $f(x)\le f(t+1)$. Note that

$$\begin{aligned}RDD(Z_n^{t+1})&=(t+n)(t+1)+\frac{1}{2}(t+n-1)(n-t-2)+(n-1)(n-1+n-2)\\&\quad+(n-1)(n-1)(n-2)+(n-2)(n-t-2)(n-t-3)\\&=f(t+1)+\frac{1}{2}(n+t)(n+t-1)(2n+t-2).\end{aligned}$$

Thus $RDD(G)\le RDD(Z_n^{t+1})$. From $RDD(G)\ge RDD(Z_n^{t+1})$, we have $RDD(G)=RDD(Z_n^{t+1})$. Then, all inequalities in the discussion above have to be equal. Thus, $s=t+1$, and we have $d_1=t+1$, $d_2=\cdots=d_{n-t-1}=n-2$ and $d_{n-t}=\cdots=d_n=n-1$. This implies $G\cong Z_n^{t+1}$, which is M-M t-DPCe except $t=1$. Conversely, if $G\cong Z_n^{t+1}$, one can easily see that $RDD(G)=RDD(Z_n^{t+1})$. ■

Lemma 3.9. *For an n-vertex graph G with $n\ge 2t\ge 2$ and $t\le n-5$, if $F(G)\ge F(Z_n^{t+1})$, then G is M-M t-DPCe, unless $G=Z_n^2$. Moreover, $F(G)=F(Z_n^{t+1})$ if and only if $G\cong Z_n^{t+1}$.*

Proof. Suppose G is a graph satisfying the conditions in Lemma 3.9, yet G is not M-M t-DPCe. By Lemma 2.3, there must be an integer s with $t+1\le s\le \frac{1}{2}(n+t-1)$ that satisfies both $d_{s-t}\le s$ and $d_{n-s}\le n-s+t-1$. Therefore

$$\begin{aligned}F(G)&=\sum_{u_j\in V(G)} d_j^3\\&\le (s-t)s^3+(n-2s+t)(n-s+t-1)^3+s(n-1)^3\\&=3s^4-(7n+8t-6)s^3+3(n+t-1)(3n+3t-2)s^2\end{aligned}$$

$$+\left((n-1)^3-(n+t-1)^2(5n+5t-2)\right)s+(n+t)(n+t-1)^3.$$

Consider the function defined as follows:

$$\begin{aligned}f(x)=&3x^4-(7n+8t-6)x^3+3(n+t-1)(3n+3t-2)x^2\\&+\left((n-1)^3-(n+t-1)^2(5n+5t-2)\right)x,\end{aligned}$$

where $t+1\le x\le \frac{1}{2}(n+t-1)$. By differentiation, we derive:

$$\begin{aligned}f'(x)=&12x^3-3(7n+8t-6)x^2+6(n+t-1)(3n+3t-2)x\\&+\left((n-1)^3-(n+t-1)^2(5n+5t-2)\right),\end{aligned}$$

and $f''(x)=36x^2-6(7n+8t-6)x+6(n+t-1)(3n+3t-2)$. Note that $x=\frac{7n+8t-6}{12}$ is the axis of symmetry of the function $f''(x)$. By direct calculation, we get

$$\begin{aligned}f''\left(\frac{7n+8t-6}{12}\right)&=36\left(\frac{7n+8t-6}{12}\right)^2-6(7n+8t-6)\left(\frac{7n+8t-6}{12}\right)\\&\quad+6(n+t-1)(3n+3t-2)\\&=\frac{1}{4}\left(24(n+t-1)(3n+3t-2)-(7n+8t-6)^2\right)\\&=\frac{1}{4}\left((8n+8t-8)(9n+9t-6)-(7n+8t-6)^2\right).\end{aligned}$$

Since $n\ge 6$ and $t\ge 1$, $f''(\frac{7n+8t-6}{12})>0$. Thus $f''(x)>0$. It follows that $f(x)$ is a strictly convex function on $x\in[t+1,\frac{n+t-1}{2}]$, where n and t are integers. Thus, one can obtain $f(x)\le \max\{f(t+1),f(\lfloor\frac{n+t-1}{2}\rfloor)\}$. Direct computation yields

$$f(\lfloor\tfrac{n+t-1}{2}\rfloor)-f(t+1)=\begin{cases}-\frac{1}{16}((t-n+2)(t-n+4)(t^2+(4n+2)t\\\quad+7n^2-26n+28), & \text{if } n+t \text{ is even},\\-\frac{1}{16}(t-n+3)\big((t+1)^3+(3n+6)(t+1)^2+3(n^2-4n\\\quad+4)(t+1)+(-7n^3+54n^2-108n+64)\big), & \text{if } n+t \text{ is odd}.\end{cases}$$

Now, we consider the function

$$\begin{aligned}g(t)=&(t+1)^3+(3n+6)(t+1)^2+3(n^2-4n+4)(t+1)\\&+(-7n^3+54n^2-108n+64),\end{aligned}$$

with $t\le n-5$ and $n\ge 2t\ge 2$. We obtain:

$$g'(t)=3(t+1)^2+6(n+2)(t+1)+3(n-2)^2.$$

Then $g'(t)\ge 0$. Thus, $g(t)$ is a strictly increasing function on $t\ge 1$. Since $n+t$ is odd and $1\le t\le n-5$, $t\le n-6$. Then $g(t)\le g(n-6)=-12n^2+54n+29\le 0$.

Obviously, $t^2+(4n+2)t+7n^2-26n+28>0$. Combining the above facts and $t\le n-5$, we have $f(\lfloor\frac{n+t-1}{2}\rfloor)-f(t+1)\le 0$. Then $f(x)\le f(t+1)$. Note that

$$\begin{aligned}F(Z_n^{t+1})&=(t+1)^3+(t+1)(n-1)^3+(n-t-2)(n-2)^3\\&=f(t+1)+(n+t)(n+t-1)^3.\end{aligned}$$

Then $F(G)\le F(Z_n^{t+1})$. From $F(G)\ge F(Z_n^{t+1})$, we have $F(G)=F(Z_n^{t+1})$. Then, all inequalities in the discussion above have to be equal. Thus, $s=t+1$, and we have $d_1=t+1$, $d_2=\cdots=d_{n-t-1}=n-2$ and $d_{n-t}=\cdots=d_n=n-1$. This implies $G\cong Z_n^{t+1}$, which is M-M t-DPCe except $t=1$. Conversely, if $G\cong Z_n^{t+1}$, one can easily see that $F(G)=F(Z_n^{t+1})$. ■

Let $\rho(G)$ denote either the first Zagreb index $M_1(G)$, reciprocal degree distance $RDD(G)$ or forgotten topological index $F(G)$. Combining Lemmas 3.7 to 3.9, we directly obtain the following theorem.

Theorem 3.10. *For an n-vertex graph G with $n \geq 2t \geq 2$ and $t \leq n-5$, if $\rho(G) \geq \rho(Z_n^{t+1})$, then G is M-M t-DPCe, unless $G = Z_n^2$. Moreover, $\rho(G) = \rho(Z_n^{t+1})$ if and only if $G \cong Z_n^{t+1}$.*

The following theorems can also be obtained in a similar manner with Theorems 3.4 and 3.5.

Theorem 3.11. *For an n-vertex graph G with $n-3 \geq t+1 \geq 3$, if $\rho(G) > \rho(Z_n^t)$ then G is 1-M t-DPCe. Moreover, $\rho(G) = \rho(Z_n^t)$ if and only if $G \cong Z_n^t$.*

Theorem 3.12. *For an n-vertex graph G with $n-2 \geq t+1 \geq 3$, if $\rho(G) > \rho(Z_n^{t-1})$, then G is 1-1 t-DPCe. Moreover, $\rho(G) = \rho(Z_n^{t-1})$ if and only if $G \cong Z_n^{t-1}$.*

Let $t = 1$ in Theorem 3.10 or $t = 2$ in Theorem 3.11, the following results are straightforward.

Corollary 3.13. *(**An** [21]). For an n-vertex graph G with $n \geq 6$, if $M_1(G) > n^3-5n^2+12n-6$, then G is Hamilton-connected. Moreover, $M_1(G) = n^3-5n^2+12n-6$ if and only if $G \cong Z_n^2$.*

Corollary 3.14. *(**An** [21]). For an n-vertex graph G with $n \geq 6$, if $RDD(G) > n^3 - \frac{9}{2}n^2 + \frac{19}{2}n - 6$, then G is Hamilton-connected. Moreover, $RDD(G) = n^3 - \frac{9}{2}n^2 + \frac{19}{2}n - 6$ if and only if $G \cong Z_n^2$.*

Corollary 3.15. *For an n-vertex graph G with $n \geq 6$, if $F(G) > F(Z_n^2) = n^4 - 7n^3 + 24n^2 - 38n + 30$, then G is Hamilton-connected. Moreover, $F(G) = F(Z_n^2)$ if and only if $G \cong Z_n^2$.*

Let $t = 2$ in Theorem 3.12, we can also obtain the following corollaries.

Corollary 3.16. *(**An et al.** [22]). For an n-vertex graph G with $n \geq 5$, if $M_1(G) > n^3 - 5n^2 + 10n - 6$, then G is Hamiltonian. Moreover, $M_1(G) = n^3 - 5n^2 + 10n - 6$ if and only if $G \cong Z_n^1$.*

Corollary 3.17. *For an n-vertex graph G with $n \geq 5$, if $RDD(G) > RDD(Z_n^1) = n^3 - \frac{9}{2}n^2 + \frac{17}{2}n - 5$, then G is Hamiltonian. Moreover, $RDD(G) = RDD(Z_n^1)$ if and only if $G \cong Z_n^1$.*

Corollary 3.18. *For an n-vertex graph G with $n \geq 5$, if $F(G) > F(Z_n^1) = n^4 - 7n^3 + 21n^2 - 29n + 16$, then G is Hamiltonian. Moreover, $F(G) = F(Z_n^1)$ if and only if $G \cong Z_n^1$.*

Corollary 3.17 is inconsistent with a result in [23]. By calculation, $RDD(Z_n^1)$ satisfies their condition, while Z_n^1 is not Hamiltonian. Thus, we present a corrected version of their result just by adding the minimum degree condition.

Corollary 3.19. *(**Kori et al.** [23]). For an n-vertex graph G with $n \geq 6$ with $\delta(G) \geq 2$, if $RDD(G) \geq n^3 - 7n^2 + 24n - 20$, then G is either Hamiltonian or is $K_2 \vee (2K_1 \cup K_{n-4})$.*

3.3 Eccentric connectivity index, eccentric distance sum, and connective eccentric index

Lemma 3.20. *For an n-vertex graph G with $n \geq 2t \geq 2$ and $t \leq n-5$, if*

$$\xi^c(G) \geq n^3 - 3n^2 - \frac{4m^2}{n} + 2n(t+2),$$

then G is M-M t-DPCe.

Proof. Suppose G is a graph satisfying the conditions in Lemma 3.20, yet G is not M-M t-DPCe. By Lemma 2.3, there must be an integer s with $t+1 \leq s \leq \frac{1}{2}(n+t-1)$ that satisfies both $d_{s-t} \leq s$ and $d_{n-s} \leq n-s+t-1$. Since $\varepsilon(u_i) \leq n-d_j$, therefore

$$\begin{aligned}\xi^c(G) &= \sum_{u_j\in V(G)} \varepsilon(u_j)d_j \leq \sum_{u_j\in V(G)} (n-d_j)\, d_j \\ &\leq n \sum_{u_j\in V(G)} d_j - \frac{1}{n}(\sum_{u_j\in V(G)} d_j)^2 = n \sum_{u_j\in V(G)} d_j - \frac{4m^2}{n} \\ &\leq n((s-t)s + (n-2s+t)(n-s+t-1) + s(n-1)) - \frac{4m^2}{n} \\ &= n^3 - 3n^2 + 2n - \frac{4m^2}{n} + n(3s^2 + (1-4t-2n)s - 3(t+1) \\ &\quad + (t+1)^2 + 2(t+1)n).\end{aligned}$$

Consider the function:

$$f(x) = 3x^2 + (1-4t-2n)x - 3(t+1) + (t+1)^2 + 2(t+1)n,$$

where $t+1 \leq s \leq \frac{1}{2}(n+t-1)$. By differentiation, we derive:
$f'(x) = 6x + 1 - 4t - 2n$, and $f''(x) = 6$.
It follows that $f(x)$ is a strictly convex function on $x \in [t+1, \frac{n+t-1}{2}]$, where n and t are integers. Thus, one can obtain $f(x) \leq \max\ \{f(t+1), f(\lfloor \frac{n+t-1}{2} \rfloor)\}$. Direct computation yields

$$f(\lfloor \tfrac{n+t-1}{2} \rfloor) - f(t+1) = \begin{cases} -\frac{1}{4}(t-n+2)(t-n+4), & \textit{if } n+t \textit{ is even,} \\ -\frac{1}{4}(t-n+3)(t-n+5), & \textit{if } n+t \textit{ is odd.} \end{cases}$$

Since $n \geq 2t \geq 2$ and $t \leq n-5$, $f(\lfloor \frac{n+t-1}{2} \rfloor) - f(t+1) \leq 0$. Then $f(x) \leq f(t+1)$. Thus $\xi^c(G) \leq n^3 - 3n^2 - \frac{4m^2}{n} + nf(t+1) = n^3 - 3n^2 - \frac{4m^2}{n} + 2n(t+2)$.

If $\xi^c(G) = n^3 - 3n^2 - \frac{4m^2}{n} + 2n(t+2)$, then all inequalities in the discussion above have to be equal. Thus, $s = t+1$, and we have $d_1 = t+1$, $d_2 = \cdots = d_{n-t-1} = n-2$ and $d_{n-t} = \cdots = d_n = n-1$. This infer that $G = Z_n^{t+1}$. Note that $\sum_{u_j\in V(G)} d_j^2 = \frac{1}{n}(\sum_{u_j\in V(G)} d_j)^2$ implies G is a regular graph. Since Z_n^{t+1} is not regular, the equality cannot hold simultaneously. Thus, $\xi^c(G) < n^3 - 3n^2 - \frac{4m^2}{n} + 2n(t+2)$, which is a contradiction. ∎

Lemma 3.21. *For an n-vertex graph G with $n \geq 2t \geq 2$ and $t \leq n-5$, if*

$$\xi^d(G) \leq \frac{1}{n(n-1)}(n^2 + n - 2t - 4)^2,$$

then G is M-M t-DPCe.

Proof. Suppose G is a graph satisfying the conditions in Lemma 3.21, yet G is not M-M t-DPCe. By Lemma 2.3, there must be an integer s with $t+1 \leq s \leq \frac{1}{2}(n+t-1)$ that satisfies both $d_{s-t} \leq s$ and $d_{n-s} \leq n-s+t-1$.

By the definition of the eccentric distance sum $\xi^d(G)$, the distance sum $D(u_j)$, we have $(n-1)\varepsilon(u_j) \geq D(u_j) \geq d_j + 2(n-1-d_j)$, $\sum_{u_j\in V(G)} d_j = 2m \leq n(n-1) < 2n(n-1)$, and

$$\xi^d(G) = \sum_{u_j\in V(G)} \varepsilon(u_j)D(u_j) \geq \sum_{u_j\in V(G)} \frac{(D(u_j))^2}{n-1} \geq \frac{1}{n-1} \sum_{u_j\in V(G)} (2(n-1) - d_j)^2$$

$$\geq \frac{1}{n(n-1)}\left(\sum_{u_j\in V(G)}(2(n-1)-d_j)\right)^2 = \frac{1}{n(n-1)}\left(2n(n-1)-\sum_{u_j\in V(G)}d_j\right)^2$$
$$\geq \frac{1}{n(n-1)}(2n(n-1)-(s-t)s-(n-2s+t)(n-s+t-1)-s(n-1))^2$$
$$= \frac{1}{n(n-1)}\left(-3s^2-(1-4t-2n)s-(t+1)^2-(t+1)(2n-3)+n^2+n-2\right)^2.$$

Consider the function defined as follows:

$$f(x) = -3x^2-(1-4t-2n)x-(t+1)^2-(t+1)(2n-3)+n^2+n-2,$$

where $t+1 \leq x \leq \frac{1}{2}(n+t-1)$. By differentiation, we derive:
$f'(x) = -6x-1+4t+2n$, and $f''(x) = -6$.
It follows that $f(x)$ is a strictly concave function on $x \in [t+1, \frac{n+t-1}{2}]$, where n and t are integers. Thus, one can obtain $f(x) \geq \min\{f(t+1), f(\lfloor\frac{n+t-1}{2}\rfloor)\}$. Direct computation yields

$$f(\lfloor\tfrac{n+t-1}{2}\rfloor)-f(t+1) = \begin{cases}\frac{1}{4}(t-n+2)(t-n+4), & \textit{if } n+t \textit{ is even,}\\ \frac{1}{4}(t-n+3)(t-n+5), & \textit{if } n+t \textit{ is odd.}\end{cases}$$

Since $n \geq 2t \geq 2$ and $t \leq n-5$, $f(\lfloor\frac{n+t-1}{2}\rfloor)-f(t+1) \geq 0$. Then $f(x) \geq f(t+1)$. Thus $\xi^d(G) \geq \frac{1}{n(n-1)}(f(t+1))^2 = \frac{1}{n(n-1)}(n^2+n-2t-4)^2$.

If $\xi^d(G) = \frac{1}{n(n-1)}(n^2+n-2t-4)^2$, then all inequalities in the discussion above have to be equal. Thus, $s = t+1$, and we have $d_1 = t+1$, $d_2 = \cdots = d_{n-t-1} = n-2$ and $d_{n-t} = \cdots = d_n = n-1$. This infer that $G = Z_n^{t+1}$. Note that $\frac{1}{n-1}\sum_{u_j\in V(G)}(2(n-1)-d_j)^2 = \frac{1}{n(n-1)}(\sum_{u_j\in V(G)}(2(n-1)-d_j))^2$ implies G is a regular graph. Since Z_n^{t+1} is not regular, the equality cannot hold simultaneously. Thus, $\xi^d(G) > \frac{1}{n(n-1)}(n^2+n-2t-4)^2$, which is a contradiction. ■

Lemma 3.22. *For an n-vertex graph G with $n \geq 2t \geq 2$ and $t \leq n-5$, if*

$$\xi^{ce}(G) \geq (n-1)(\frac{3t+2}{n}+n-3),$$

then G is M-M t-DPCe.

Proof. Suppose G is a graph satisfying the conditions in Lemma 3.22, yet G is not M-M t-DPCe. By Lemma 2.3, there must be an integer s with $t+1 \leq s \leq \frac{1}{2}(n+t-1)$ that satisfies both $d_{s-t} \leq s$ and $d_{n-s} \leq n-s+t-1$.

By the definition of the eccentric distance sum $\xi^d(G)$, the distance sum $D(u_j)$, we have $(n-1)\varepsilon(u_j) \geq D(u_j) \geq d_j + 2(n-1-d_j)$, and

$$\xi^{ce}(G) = \sum_{u_j\in V(G)}\frac{d_j}{\varepsilon(u_j)} \leq \sum_{u_j\in V(G)}\frac{n-1}{D(u_j)}d_j \leq (n-1)\sum_{u_j\in V(G)}\frac{d_j}{2(n-1)-d_j}$$
$$\leq (n-1)\left(\frac{s(s-t)}{2n-2-s}+\frac{(n-2s+t)(n-s+t-1)}{2n-2-(n-s+t-1)}+\frac{s(n-1)}{2n-2-(n-1)}\right)$$
$$\left(\text{because the function}\frac{x}{2(n-1)-x}\text{is strictly increasing on } x\in[1,n-1]\right)$$
$$= (n-1)\left(\frac{s(s-t)}{2n-2-s}+\frac{(n-s+t-1)(n-2s+t)}{n+s-t-1}+s\right)$$

$$\leq (n-1)\left(\frac{s(s-t)}{n+s-t-1}+n-s+t-\frac{2(s-t)(n-2s+t)}{n+s-t-1}\right)$$
$$\left(\text{because } \frac{s(s-t)}{2n-2-s}\leq\frac{s(s-t)}{n+s-t-1}\text{by the fact } 2\leq t+1\leq x\leq\frac{1}{2}(n+t-1)\right)$$
$$=(n-1)\left(\frac{4s^2+(2-5t-3n)s+(t+1)^2+(3n-4)(t+1)-2n+2}{n+s-t-1}+n-1\right).$$

Consider the function defined as follows:

$$f(x)=\frac{4x^2+(2-5t-3n)x+(t+1)^2+(3n-4)(t+1)-2n+2}{x+n-t-1},$$

where $t+1\leq x\leq \frac{1}{2}(n+t-1)$. By differentiation, we derive:

$$f'(x)=\frac{8x-3n-5t+2}{x+n-t-1}-\frac{4x^2+(2-5t-3n)x+(t+1)^2+(3n-4)(t+1)-2n+2}{(x+n-t-1)^2},$$

and

$$f''(x)=\frac{2(n-1)(7n-3t-5)}{(x+n-t-1)^3}.$$

Since $n\geq 2t\geq 2$ and $t\leq n-5$, $f''(x)>0$. It follows that $f(x)$ is a strictly convex function on $x\in[t+1,\frac{n+t-1}{2}]$, where n and t are integers. Thus, one can obtain $f(x)\leq \max\{f(t+1), f(\lfloor\frac{n+t-1}{2}\rfloor)\}$. Direct computation yields

$$f(\lfloor\tfrac{n+t-1}{2}\rfloor)-f(t+1)=\begin{cases}\frac{(n-4-t)((n-3)(n-t-1)+2)}{n(t+4-3n)}, & \textit{if } n+t \textit{ is even},\\ \frac{(n-3-t)(5+3t-6n-tn+n^2)}{n(t+3-3n)}, & \textit{if } n+t \textit{ is odd}.\end{cases}$$

Since $n\geq 2t\geq 2$ and $t\leq n-5$, $f(\lfloor\frac{n+t-1}{2}\rfloor)-f(t+1)\leq 0$. Then $f(x)\leq f(t+1)$. Thus, $\xi^{ce}(G)\leq (n-1)(f(t+1)+n-1)=(n-1)\left(\frac{3t+5}{n}+n-3\right)$.

If $\xi^{ce}(G)=(n-1)(\frac{3t+5}{n}+n-3)$, then all inequalities in the discussion above have to be equal. Thus, $s=t+1$, and we have $d_1=t+1$, $d_2=\cdots=d_{n-t-1}=n-2$ and $d_{n-t}=\cdots=d_n=n-1$. Note that $(n-1)\varepsilon(u_1)=D(u_1)=d_j+2(n-1-d_1)$ implies $\varepsilon(u_1)$ is not an integer. Then the equality cannot hold. Thus, $\xi^{ce}(G)<(n-1)(\frac{3t+5}{n}+n-3)$, which is a contradiction. ■

Combining Lemmas 3.20 to 3.22, we directly obtain the following theorem.

Theorem 3.23. *For an n-vertex graph G with $n\geq 2t\geq 6$ and $t\leq n-5$, if one of the following conditions holds,*
(i) $\xi^{c}(G)\geq n^3-3n^2-\frac{4m^2}{n}+2n(t+2)$,
(ii) $\xi^{d}(G)\leq \frac{1}{n(n-1)}(n^2+n-2t-4)^2$,
(iii) $\xi^{ce}(G)\geq (n-1)\left(\frac{3t+5}{n}+n-3\right)$,
then G is M-M t-DPCe.

The following theorems can also be obtained by a similar method.

Theorem 3.24. *For an n-vertex graph G with $n-3\geq t+1\geq 3$, if one of the following conditions holds,*
(i) $\xi^{c}(G)\geq n^3-3n^2-\frac{4m^2}{n}+2n(t+1)$,
(ii) $\xi^{d}(G)\leq \frac{1}{n(n-1)}(n^2+n-2t-2)^2$,
(iii) $\xi^{ce}(G)\geq (n-1)\left(\frac{3t+2}{n}+n-3\right)$,
then G is 1-M t-DPCe.

Theorem 3.25. *For an n-vertex graph G with $n-2 \geq t+1 \geq 3$, if one of the following conditions holds,*

(i) $\xi^c(G) \geq n^3 - 3n^2 - \frac{4m^2}{n} + 2nt$,

(ii) $\xi^d(G) \leq \frac{1}{n(n-1)}(n^2+n-2t)^2$,

(iii) $\xi^{ce}(G) \geq (n-1)\left(\frac{3t-1}{n} + n - 3\right)$,

then G is 1-1 t-DPCe.

Let $t = 2$ in Theorem 3.25, we obtain the following result.

Corollary 3.26. *(**Zhu et al.** [24]). For an n-vertex graph G with $n \geq 6$ and size m,*

(i) If $\xi^c(G) \geq n^3 - 3n^2 + 4n - \frac{4m^2}{n} > 0$, *then G is Hamiltonian.*

(ii) If $\xi^d(G) \leq \frac{1}{n(n-1)}(n^2+n-4)^2$, *then G is Hamiltonian.*

(iii) If $\xi^{ce}(G) \geq (n-1)\frac{n^2-3n+5}{n}$, *then G is Hamiltonian.*

Let $t = 1$ in Theorem 3.23 or $t = 2$ in Theorem 3.24. The Zhu's another result is straightforward.

Corollary 3.27. *(**Zhu et al.** [24]). For an n-vertex graph G with $n \geq 7$ and size m,*

(i) If $\xi^c(G) \geq n^3 - 3n^2 + 6n - \frac{4m^2}{n} > 0$, *then G is Hamilton-connected.*

(ii) If $\xi^d(G) \leq \frac{1}{n(n-1)}(n^2+n-6)^2$, *then G is Hamilton-connected.*

(iii) If $\xi^{ce}(G) \geq (n-1)\frac{n^2-3n+8}{n}$, *then G is Hamilton-connected.*

4 Concluding remarks

In this paper, novel sufficient conditions for a connected graph to be DPCe are presented, based on topological indices such as Harary index, first Zagreb index, forgotten topological index, reciprocal degree distance, eccentric connectivity index, eccentric distance sum, and connective eccentric index. Since most of the topological indices we used were transformed into quantities related to the degrees of the graph, we employed Bondy-type degree sequence conditions. Following this paper, many more indices, such as the Sombor index, the difference of Zagreb index, and the hyper-Zagreb index, are worthy of study.

Author Contributions. Honggang Zhao conceived of the presented idea, verified the analytical methods, developed the theory, and performed the computations. Eminjan Sabir provided the original research direction and supervised the findings of this work. All authors discussed the results and contributed to the final manuscript.

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