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Research Paper

Counterexamples to a Conjecture on the Mostar Index of a Graph and its Line Graph

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Abstract

The Mostar index is a distance-based topological index defined as a quantitative measure of distance balancedness and peripherality in graphs. Recently, in "An inequality for the Mostar index of line graphs of trees" [Commun. Comb. Optim. (2005), doi: 10.22049/cco.2025.30201.2356], the following conjecture regarding the Mostar index of graphs and their line graphs was proposed:
Let G be a simple connected graph on n vertices. Then

$$Mo(L_G) \leq Mo(G).$$

where, L_G denotes the line graph of the graph G. In this paper, counterexamples are presented to disprove the conjecture proposed in [1]. We also prove that the conjecture is not true for an infinite family of graphs with fixed cyclomatic number c, where $1 \leq c \leq 3$.

1 Introduction

Topological indices are numerical quantities associated with graphs that provide significant information about the structural properties of chemical compounds. The Mostar index proposed by Došlić [2] is one of the most significant distance-based topological indices that have been used to study the distance balancedness of graphs [3]. Let $G = (V, E)$ be a connected simple graph, where V denotes the set of vertices, and E denotes the set of edges of the graph G. Then the Mostar index of G is defined as:

$$Mo(G) = \sum_{e=uv \in E} |n_u(e|G) - n_v(e|G)|,$$

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where $n_u(e|G)$ is the number of vertices of G closer to u than v, and $n_v(e|G)$ is the number of vertices of G closer to v than u.

Various studies on different versions and properties of the Mostar index in different classes of graphs have been conducted recently [4–16].
A notable study by Sardar et al. [1] investigates the Mostar index of graphs and their corresponding line graphs. The authors established that for any tree, the Mostar index of its line graph is strictly less than that of the tree itself. Extending this observation, they generalized the result to connected graphs and proposed the following conjecture.

Conjecture 1.1. *Let G be a simple connected graph on n vertices. Then $Mo(L_G) \leq Mo(G)$.*

In this paper, we provide a counterexample to disprove Conjecture 1.1. Additionally, we propose several open problems related to the conjecture. It is also shown that for each cyclomatic number c, where $1 \leq c \leq 3$, there exist infinitely many graphs for which Conjecture 1.1 does not hold. Recall that for a tree, the cyclomatic number $c = 0$; graphs with $c = 1, 2$, and 3 are referred to as unicyclic, bicyclic, and tricyclic graphs, respectively.

2 Main results

In this section, we construct a counterexample to disprove Conjecture 1.1. Throughout, the contribution of an edge $e = uv$ to the Mostar index of a graph G is denoted by $\phi(e \mid G) = |n_u(e \mid G) - n_v(e \mid G)|$. We further prove that for each class of graphs with cyclomatic number c, where $1 \leq c \leq 3$, there exist infinitely many graphs for which Conjecture 1.1 does not hold.

Counterexample 2.1. *There exist connected graphs G for which Conjecture 1.1 does not hold.*

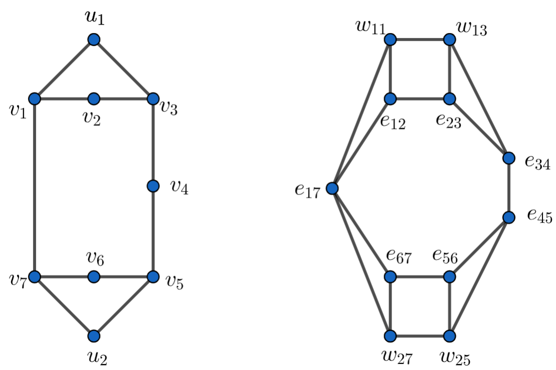

Figure 1: The graphs G and its associated L_G described in Counterexample 2.1 and Construction 2.4.

Consider the graph G shown in Figure 1. Let w_{ij} denote the edge connecting the vertices u_i and v_j in G, $i \leq j$. Similarly, let e_{ij} denote the edge connecting the vertex v_i and v_j in G, $i \leq j$. From direct computation,

$$\phi(e_{12}|G) = \phi(e_{67}|G) = \phi(w_{11}|G) = \phi(w_{27}|G) = 2,$$
$$\phi(e_{23}|G) = \phi(e_{56}|G) = \phi(w_{13}|G) = \phi(w_{25}|G) = 1,$$

$$\phi(e_{17}|G) = \phi(e_{34}|G) = \phi(e_{45}|G) = 0.$$

Therefore, $Mo(G) = 12$. Now, for the associated line graph L_G, from direct computation,

$$\begin{aligned}
&\phi(e_{12}e_{23}|L_G) = \phi(e_{56}e_{67}|L_G) = \phi(w_{11}w_{13}|L_G) = \phi(w_{25}w_{27}|L_G) = 1,\\
&\phi(w_{11}e_{12}|L_G) = \phi(w_{13}e_{23}|L_G) = \phi(w_{25}e_{56}|L_G) = \phi(w_{27}e_{67}|L_G) = 0,\\
&\phi(e_{23}e_{34}|L_G) = \phi(w_{13}e_{34}|L_G) = \phi(e_{45}e_{56}|L_G) = \phi(e_{45}w_{25}|L_G) = 1,\\
&\phi(w_{11}e_{17}|L_G) = \phi(e_{12}e_{17}|L_G) = \phi(e_{17}e_{67}|L_G) = \phi(e_{17}w_{27}|L_G) = 2,\\
&\phi(e_{34}e_{45}|L_G) = 0.
\end{aligned}$$

Thus, $Mo(L_G) = 16 > Mo(G) = 12$. Therefore Conjecture 1.1 is not true.

Now, we prove that there exist infinitely many graphs for which Conjecture 1.1 is not true. From this point onwards, the edges $v_iv_j, u_iu_j, u_iv_j, i \le j$ are denoted by e_{ij}, f_{ij} and w_{ij} respectively. We provide the following constructions to establish the claim.

Construction 2.2. *Let $C_{r,m}$ denote a unicyclic graph with cycle $C_r = v_0v_1 \dots v_rv_0$ together with a path $P_m = u_0u_1u_2 \dots u_m$ such that the pendant vertex u_0 of P_m is identified with v_0 of C_r(see, Figure 2).*

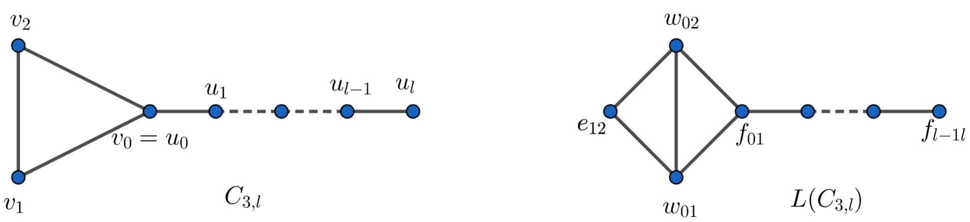

Figure 2: The graphs $C_{3,l}$ and its associated $L_{C_{3,l}}$ described in Theorem 2.5.

Construction 2.3. *Let $\Theta_{2,2,n-3}$ be a bicyclic graph obtained by taking two vertices v_0 and v_2 and connecting these two vertices with three different paths of length $2, 2, n-3$ respectively. The vertices are labeled as in Figure 3.*

Construction 2.4. *Consider the graph G shown in Figure 1. Let G_1 be the graph obtained by subdividing the edges v_4v_5 and v_1v_7 of G. Denote the newly inserted vertex between v_4 and v_5 in G_1 by v_4^1, and the vertex inserted between v_1 and v_7 by v_1^1.*

Next, let G_2 be the graph obtained by subdividing the edges $v_4v_4^1$ and $v_1v_1^1$ in G_1. Denote the new vertices inserted on these edges by v_4^2 and v_1^2, respectively.

Continuing in this manner, for $i = 3, 4, \dots,$ let G_i be the graph obtained from G_{i-1} by subdividing the edges $v_4v_4^{i-1}$ and $v_1v_1^{i-1}$ exactly once. Let the newly inserted vertices on these edges be denoted by v_4^i and v_1^i, respectively (see Figure 4).

Let w_{ij} denote the edge connecting the vertices u_i and v_j in G, for $i \le j$. Similarly, let e_{ij} denote the edge connecting the vertices v_i and v_j in G, for $i \le j$.

Now, in the graph G_i, let e'_{11}, e'_{54}, e'_{71}, and e'_{44} denote the edges $v_1v_1^i$, $v_5v_4^i$, $v_7v_1^1$, and $v_4v_4^1$, respectively. Also, for $j = 1, 2, \dots, i-1$, let e^j_{11} and e^j_{44} denote the edges $v_1^jv_1^{j+1}$ and $v_4^jv_4^{j+1}$, respectively, in G_i.

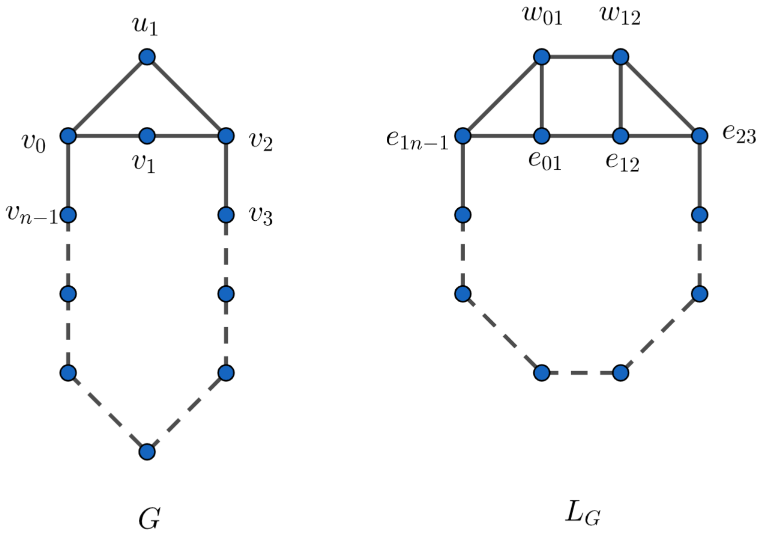

Figure 3: The graphs G and L_G described in Construction 2.3.

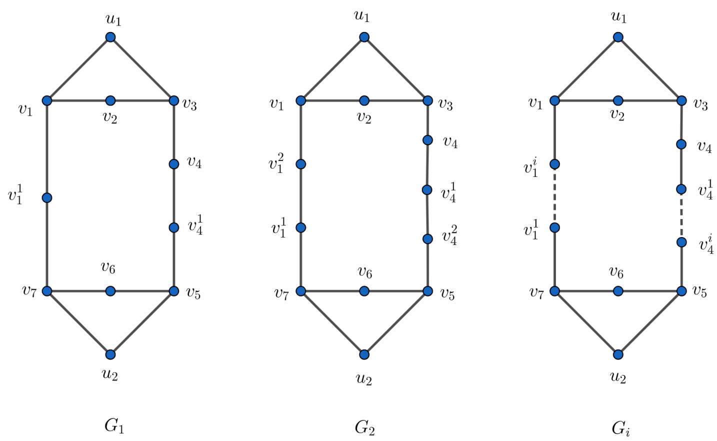

Figure 4: The graphs G_i described in Construction 2.4.

Theorem 2.5. *For each fixed positive integer $1 \leq c \leq 3$, there exist infinitely many graphs G with cyclomatic number c such that*

$$Mo(L_G) > Mo(G).$$

Proof. We consider the following cases,
Case I: c = 1 : Consider the graphs $C_{3,n-3}$ described in Construction 2.2 where $n \geq 6$. For the edges $f_{ii+1}, i = 1, \ldots n-4$ in G and for the edges $f_{i-1i}f_{ii+1}, i = 1, \ldots n-4$ in L_G, we have

$$\phi(f_{ii+1}|G) = \phi(f_{i-1i}f_{ii+1}|L_G).$$

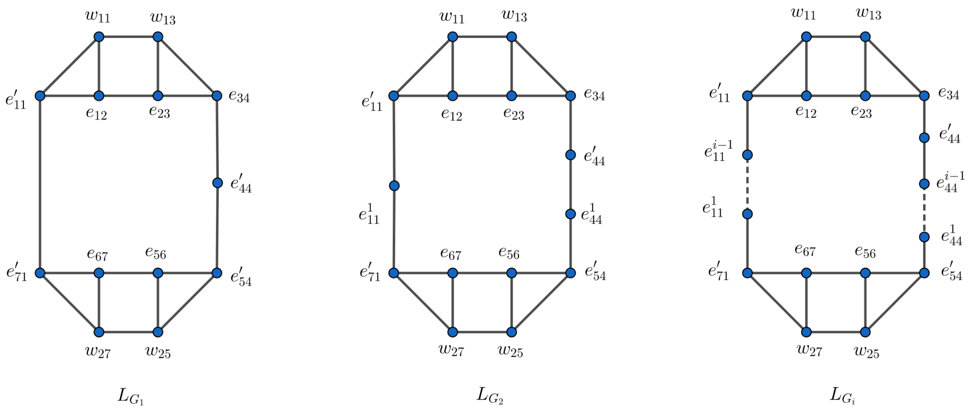

Figure 5: The line graphs L_{G_i} of graphs G_i described in Construction 2.4.

Also,

$$\phi(w_{01}|G) = \phi(w_{02}|G) = \phi(e_{12}w_{01}|L_G) = \phi(e_{12}w_{02}|L_G) = n-3, \phi(f_{01}|G) = n-6,$$
$$\phi(e_{12}|G) = \phi(w_{01}w_{02}|L_G) = 0, \phi(w_{01}f_{01}|L_G) = \phi(w_{02}f_{01}|L_G) = n-5.$$

Therefore,

$$\begin{aligned} Mo(L_G) - Mo(G) &= \phi(w_{01}f_{01}|L_G) + \phi(w_{02}f_{01}|L_G) - \phi(f_{01}|G) \\ &= 2n - 10 - (n-6) = n - 4 > 0, \end{aligned}$$

as $n \geq 6$. Therefore, for all the class of graphs $C_{3,n-3}$ with $n \geq 6$, $Mo(L_G) > Mo(G)$.

Case II: c = 2 : Consider the graphs G in Construction 2.3 with $n \geq 14$. When n is even, the edge $e_{\frac{n}{2}\frac{n}{2}+1}$ has contribution $\phi(e_{\frac{n}{2}\frac{n}{2}+1} \mid G) = 0$, and each of the remaining n edges e of G has contribution $\phi(e \mid G) = 1$. When n is odd, all the edges e of G have contribution $\phi(e \mid G) = 1$. Therefore,

$$Mo(G) = \begin{cases} n, & \text{if } n \text{ is even}, \\ n+1, & \text{if } n \text{ is odd}. \end{cases}$$

Now, in the case of L_G, from direct computation,

$$\phi(e_{1n-1}w_{01}|L_G) = \phi(e_{1n-1}e_{01}|L_G) = \phi(e_{01}e_{12}|L_G) = \phi(w_{01}w_{12}|L_G) = 0,$$
$$\phi(w_{01}e_{01}|L_G) = \phi(w_{12}e_{12}|L_G) = \phi(e_{23}w_{12}|L_G) = \phi(e_{23}e_{12}|L_G) = 0.$$

Now, for the remaining $n-5$ edges of L_G, when n is odd, the middle edge e of the path e_{23}—e_{1n-1} of length $n-5$ in L_G has contribution $\phi(e \mid L_G) = 0$, and each of the remaining edges e in this path has contribution $\phi(e \mid L_G) = 2$. Therefore, $Mo(L_G) = 2(n-6) = 2n-12$.

When n is even, the two middle edges of the path e_{23}—e_{1n-1} of length $n-5$ in L_G each have contribution $\phi(e \mid L_G) = 1$, and all the remaining edges e in this path have contribution $\phi(e \mid L_G) = 2$. Therefore, $Mo(L_G) = 2(n-7) + 2 = 2n - 12$.

Thus,

$$Mo(L_G) - Mo(G) = \begin{cases} n-12, & \text{if } n \text{ is even}. \\ n-13, & \text{if } n \text{ is odd}. \end{cases}$$

Now, when $n \geq 14$, $Mo(L_G) - Mo(G) > 0$.
Case III: c = 3 : Consider the graphs $G_i, i = 1, 2, \dots$ constructed as in Construction 2.4. From direct computation,

$$\phi(e_{12}|G_i) = \phi(e_{67}|G_i) = \phi(w_{11}|G_i) = \phi(w_{27}|G_i) = 2,$$
$$\phi(e_{23}|G_i) = \phi(e_{56}|G_i) = \phi(w_{13}|G_i) = \phi(w_{25}|G_i) = 1,$$
$$\phi(e_{34}|G_i) = \phi(e'_{11}|G_i) = \phi(e'_{71}|G_i) = \phi(e'_{54}|G_i) = \phi(e'_{44}|G_i) = 0,$$
$$\phi(e^j_{11}|G_i) = \phi(e^j_{44}|G_i) = 0, j = 1, 2, \dots i - 1.$$

Therefore, $Mo(G_i) = 12$ for all $i = 1, 2, 3, \dots$. Now, for the associated line graph $L_{G_i}, i = 1, 2, \dots$, from direct computation,

$$\phi(e_{12}e_{23}|L_{G_i}) = \phi(e_{56}e_{67}|L_{G_i}) = \phi(w_{11}w_{13}|L_{G_i}) = \phi(w_{25}w_{27}|L_{G_i}) = 1,$$
$$\phi(e_{23}e_{34}|L_G) = \phi(w_{13}e_{34}|L_G) = \phi(e'_{54}e_{56}|L_G) = \phi(e'_{54}w_{25}|L_G) = 1,$$
$$\phi(w_{11}e'_{11}|L_{G_i}) = \phi(e_{12}e'_{11}|L_{G_i}) = \phi(e'_{71}e_{67}|L_{G_i}) = \phi(e'_{71}w_{27}|L_{G_i}) = 2.$$

For all the remaining edges $e = uv$ connecting the e_{34}—e'_{54} path and the e'_{71}—e'_{11} path in L_{G_i}, the contribution $\phi(e|L_{G_i}) = 0$. Therefore, $Mo(L_{G_i}) = 16 > Mo(G_i) = 12$ for all $i = 1, 2, \dots$. Thus, there exist infinitely many tricyclic graphs for which Conjecture 1.1 does not hold. ■

3 Conclusion

In this study, we have demonstrated that Conjecture 1.1 does not hold in general for graphs with cyclomatic number $c = 1, 2, 3$. However, there exist several classes of graphs for which the conjecture may still be valid. The study concludes with the following open problems related to Conjecture 1.1. As shown by Sardar et al. [1], the inequality holds for all trees T; thus, it remains to characterize non-tree graphs G for which it holds.

Problem 3.1. *Characterize graphs G (beyond trees) for which*

$$Mo(L_G) < Mo(G).$$

Problem 3.2. *Characterize the graphs G for which*

$$Mo(L_G) = Mo(G).$$

In this paper, we have constructed unicyclic, bicyclic, and tricyclic graphs to disprove Conjecture 1.1. In general, there may be infinitely many graphs for which Conjecture 1.1 does not hold for fixed c-cyclic graphs, for each $c \geq 4$.

Problem 3.3. *Prove that for each fixed $c \geq 4$, there exist infinitely many $c-$cyclic graphs G with*

$$Mo(L_G) > Mo(G).$$

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Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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