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Research Paper

Robust Numerical Approach for Solving Robin Boundary Value Problems

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Abstract

In this work, we introduce and develop spectral collocation techniques for solving second-order differential equations (SODEs) arising in chemical processes such as catalytic reactions, diffusion-reaction systems, and thermal conduction in reactive media, where Robin boundary conditions naturally emerge due to combined flux and concentration constraints. The proposed approach can be roughly represented as a truncated series of modified shifted fourth-kind Chebyshev polynomials (4KCPs). The unknown expansion coefficients are determined using the spectral collocation method. Collocation nodes were the shifted 4KCPs roots. The resulting nonlinear algebraic system is solved efficiently using Newton's method. We present a theorem that shows the truncation error rapidly converges with respect to the number of retained modes. The method's applicability and effectiveness are demonstrated using some numerical examples.

1 Introduction

Chebyshev polynomials (CPs), a well-known class of orthogonal polynomials, are used in many areas of mathematics and engineering. CPs are frequently employed in approximation theory for the accurate estimation and representation of functions. Their exceptional ability to minimize any major errors makes them suitable for providing very accurate estimates. In signal processing, CPs are used to create filters with specific frequency responses. CPs also occur in other fields, including mechanics, mathematical physics, and control theory, see [1, 2]. Different types

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of CPs are used in numerical analysis to approximate solutions of various differential equations (DEs); see, for instance, seventh-kind CPs are used as basis functions in [3] to solve fractional delay differential equations, in [4] the authors utilized first-kind CPs as basis functions to solve the time fractional nonlinear Burgers' equation. For more studies see, [5–7].

Spectral methods have garnered a lot of interest due to their exceptional accuracy and efficiency in solving differential equations [8–10]. These methods are divided into three general categories: the collocation, tau, and Galerkin methods. The best choice depends on the type of difficulty. Although collocation methods are particularly helpful for nonlinear problems or equations with complex coefficients [11–13], the Galerkin method supports theoretical analysis and provides efficient error estimates [14, 15]. On the other hand, the tau method [16–18] performs well in situations with complicated or nonlinear boundary conditions, when the Galerkin approach would not be suitable, and collocation [19, 20] can become computationally expensive. For more studies, see [21, 22].

Explicit 4KCP-based collocation techniques have shown amazing efficiency in solving many problems [23–25]. Recent developments in collocation methods have further increased their applicability. The flexibility and stability of spectral collocation techniques that utilize orthogonal polynomials have also been proven by their successful application to a range of models. For instance, Fadugba et al. [26] solved the second-order Fredholm integro-differential equations using CPs. Mulimani and Srinivasa [27] solved the Benjamin-Bona-Mahony equation using ultraspherical polynomials. These developments demonstrate the growing importance of collocation techniques in numerical analysis and their ability to deal with increasingly complex mathematical models.

Because of the wide variety of phenomena they may explain, non-linear equations [28–30] are essential in many areas of mathematics, physics, and engineering. Compared to linear equations, these equations have more challenging solutions. Nonlinear equations are found in many engineering fields, such as electrical circuits, control systems, and structural analysis. Numerical methods are required since the majority of nonlinear problems do not have analytical solutions. There are many attempts to solve these problems analytically and numerically; for example, the authors of [31] employed a modified cubic B-spline for solving nonlinear Fisher equation. Furthermore, the authors of [32] treated the nonlinear Fokas–Lenells equation in optical fiber using the variational method. For more studies, see [33–39].

The advantages of our proposed technique can be summarized as follows:

- By choosing modified sets of shifted 4KCPs as basis functions, a few retained modes produce highly accurate approximations.
- The approach requires fewer computations to achieve the desired precision.
- Our technique can treat both linear and nonlinear equations.

We point out here that the novelty of our contribution to this paper can be listed as follows:

- Some relations and formulas of the modified sets of shifted 4KCPs are presented in simple forms.
- The employment of these basis functions in the numerical treatment of linear and nonlinear SODEs is new.

The organization of this paper is as follows. Section 2 presents the properties of 4KCPs. Section 3 details the collocation algorithm and choice of basis functions to solve SODEs. The error analysis is studied in Section 4. Section 5 provides numerical results and comparisons with existing methods. Finally, Section 6 concludes the paper with remarks on the significance of the findings.

2 Overview of 4KCPs and their shifted one

The 4KCPs $\mathrm{W}_i(x)$ in the interval $[-1,1]$ are special Jacobi polynomials $P_i^{r,s}(x)$, $r,s>-1$, that can be defined as [40, 41]:

$$\mathrm{W}_i(x)=\frac{2^{2\,i}}{\binom{2\,i}{i}}\,P_i^{\left(\frac{1}{2},-\frac{1}{2}\right)}(x). \tag{1}$$

The orthogonality relation of $\mathrm{W}_i(x)$ is given by

$$\int_{-1}^{1}\mathrm{W}_i(x)\,\mathrm{W}_j(x)\,\sqrt{\frac{1-x}{1+x}}dx=\pi\,\delta_{i,j}, \tag{2}$$

where $\delta_{i,j}$ is the well-known Kronecker delta.
The recurrence relation of these polynomials is

$$\mathrm{W}_i(x)=2\,x\,\mathrm{W}_{i-1}(x)-\mathrm{W}_{i-2}(x),\quad \mathrm{W}_0(x)=1,\quad \mathrm{W}_1(x)=2\,x+1,\quad i=2,3,\dots \tag{3}$$

Theorem 2.1. *([42]). For all $n\geq q$, the q-th derivative D_x^q of $\mathrm{W}_n(x)$ is*

$$D_x^q\,\mathrm{W}_n(x)=\sum_{i=0}^{n-q}\mu_{n,i,q}\,\mathrm{W}_i(x), \tag{4}$$

where

$$\mu_{n,i,q}=\frac{2^q\,(-1)^{i+n+q}}{(q-1)!}\begin{cases}\frac{\left(\frac{1}{2}(-i+n+q-2)\right)!\left(\frac{1}{2}(i+n+q)\right)!}{\left(\frac{1}{2}(-i+n-q)\right)!\left(\frac{1}{2}(i+n-q)\right)!}, & \text{if } (n-i-q)\text{ even},\\ \frac{\left(\frac{1}{2}(-i+n+q-1)\right)!\left(\frac{1}{2}(i+n+q-1)\right)!}{\left(\frac{1}{2}(-i+n-q-1)\right)!\left(\frac{1}{2}(i+n-q+1)\right)!}, & \text{if } (n-i-q)\text{ odd},\\ 0, & \text{otherwise}.\end{cases} \tag{5}$$

The shifted FKCPs $\mathrm{W}_i^{A,B}(x)$ on $[A,B]$ are described as:

$$\mathrm{W}_i^{A,B}(x)=\mathrm{W}_i\left(\frac{2\,x-(A+B)}{B-A}\right). \tag{6}$$

Corollary 2.2. *Based on the orthogonality relation (2), one has*

$$\int_A^B\mathrm{W}_i^{A,B}(x)\,\mathrm{W}_j^{A,B}(x)\,\omega_1(x)\,dx=\frac{\pi\,(B-A)}{2}\,\delta_{i,j}, \tag{7}$$

where $\omega_1(x)=\sqrt{\frac{x-B}{A-x}}$.

Proof. Based on the definition of $\mathrm{W}_i^{A,B}(x)=\mathrm{W}_i\left(\frac{2\,x-(A+B)}{B-A}\right)$ and after replacing x with $\frac{2\,x-(A+B)}{B-A}$ in the orthogonality relation (2) we get the desired result. ■

Lemma 2.3. *The power form of $\mathrm{W}_j^{A,B}(x)$ can be represented as*

$$\mathrm{W}_j^{A,B}(x)=\sum_{r=0}^{j}\mu_{r,j}\,x^r, \tag{8}$$

where

$$\mu_{r,j}=\frac{4^r\,(-1)^{j-r}\,\Gamma(j+r+1)}{\Gamma(2\,r+1)\,(B-A)^r\,\Gamma(j-r+1)}\;{}_2F_1\left(r-j,j+r+1;r+\frac{1}{2};\frac{A}{A-B}\right). \tag{9}$$

Proof. The proof of this lemma depends on using the following power form of $\mathrm{W}_j^{0,1}(x)$ defined in [42]

$$\mathrm{W}_j^{0,1}(x) = \sum_{r=0}^{j} \frac{2^{2\,r}\,(-1)^{j-r}\,\Gamma(j+r+1)}{\Gamma(2\,r+1)\,\Gamma(j-r+1)}\, x^r. \tag{10}$$

Now, replace $x \to \frac{x-A}{B-A}$ in the previous equation, one has

$$\mathrm{W}_j^{A,B}(x) = \sum_{r=0}^{j} \frac{2^{2\,r}\,(-1)^{j-r}\,\Gamma(j+r+1)}{\Gamma(2\,r+1)\,\Gamma(j-r+1)} \left(\frac{x-A}{B-A}\right)^r, \tag{11}$$

which can be written after using the relation $(x-A)^r = \sum_{n=0}^{r} x^n (-A)^{r-n} \binom{r}{n}$, as

$$\mathrm{W}_j^{A,B}(x) = \sum_{r=0}^{j} \sum_{n=0}^{r} \frac{4^r\,(-1)^{j-r}\,(B-A)^{-r}\,(-A)^{r-n}\binom{r}{n}\,\Gamma(j+r+1)}{\Gamma(2\,r+1)\,\Gamma(j-r+1)}\, x^n. \tag{12}$$

Therefore, the desired result may be obtained after expanding, rearranging and collecting similar terms. ∎

Corollary 2.4. *For all $n \geq q$, the following formula is valid*

$$D_x^q\,\mathrm{W}_n^{A,B}(x) = \sum_{i=0}^{n-q} \zeta_{n,i,q}\,\mathrm{W}_i^{A,B}(x), \tag{13}$$

where $\zeta_{n,i,q} = \frac{2^q}{(B-A)^q}\,\mu_{n,i,q}$ and $\mu_{n,i,q}$ is defined in (5).

Proof. It is possible to derive the proof of this corollary by substituting $\left(\frac{2x-(A+B)}{B-A}\right)$ instead of x in Theorem 2.1. ∎

3 Treatment for the SODEs

This section is devoted to analyzing a collocation approach to solve the SODEs [43, 44]:

$$\eta''(x) = g(x, \eta, \eta'), \quad A \leq x \leq B, \tag{14}$$

subject to Robin boundary conditions

$$\alpha_1\,\eta(A) + \beta_1\,\eta'(A) = \gamma_1, \tag{15}$$
$$\alpha_2\,\eta(B) + \beta_2\,\eta'(B) = \gamma_2, \tag{16}$$

where α_1, α_2, β_1, β_2, γ_1, and γ_2 are constants.

To proceed in our proposed collocation method, the following transformation

$$\chi(x) = \eta(x) - \hat{\eta}(x), \tag{17}$$

where

$$\hat{\eta}(x) = \frac{\gamma_2\,(\alpha_1 A + \beta_1) - \gamma_1\,(\beta_2 + \alpha_2 B)}{\alpha_2\,(\alpha_1(A-B) + \beta_1) - \alpha_1\beta_2} + \frac{\alpha_2\gamma_1 - \alpha_1\gamma_2}{\alpha_2\,(\alpha_1(A-B) + \beta_1) - \alpha_1\beta_2}\, x, \tag{18}$$

is used to convert the SODEs (14) governed by the conditions (15) and (16) into the following modified equation:

$$\chi''(x) = f(x, \chi, \chi'), \quad A \leq x \leq B, \tag{19}$$

where

$$f(x,\chi,\chi') = g(x,\chi+\hat{\eta},\chi'+\hat{\eta}'), \tag{20}$$

subject to Robin boundary conditions

$$\alpha_1\,\chi(A) + \beta_1\,\chi'(A) = 0, \tag{21}$$

$$\alpha_2\,\chi(B) + \beta_2\,\chi'(B) = 0. \tag{22}$$

Finally, we can solve the modified equation (19) governed by the homogeneous conditions (21) and (22) instead of (14), which is governed by (15) and (16).

Remark 1. To deduce the transformation (17)-(18). Firstly, assume Equation (17) where

$$\hat{\eta}(x) = y_0\,x + y_1, \tag{23}$$

inserting Equations (17) and (23) into Equations (15) and (16), placing $\chi(A)=\chi(B)=\chi'(A)=\chi'(B)=0$ and solving the two equations resulting to get $\hat{\eta}(x)$ in (18).

3.1 Trial functions

Consider the following trial functions

$$\mathcal{Z}_k(x) = \left(b_k + a_k\,x + x^2\right)\,\mathrm{W}_k^{A,B}(x), \tag{24}$$

where

$$a_k = \frac{1}{\phi_k}\,\left[2\beta_1\left(2\beta_2 k\tau\left(A+3B+k^2\mu+k\mu\right) - 3\alpha_2\lambda\left(A+k^2\mu+k\mu\right)\right)\right. \tag{25}$$

$$\left. +\alpha_1\lambda\left(2\beta_2\left(Ak\tau + B\left(k^2+k+3\right)\right) - 3\alpha_2\lambda\mu\right)\right], \tag{26}$$

$$b_k = \frac{1}{\phi_k}\,\left[\beta_1\left(2\beta_2\left(3A^2\left(k^2+\tau\right) - 2AB\left(k^2+k+3\right)\left(k^2+\tau\right) + B^2\left(k^2+k+3\right)\right)\right)\right. \tag{27}$$

$$\left. +3\alpha_2\beta_1 B(B-A)\left(B-2A\left(k^2+\tau\right)\right) + \alpha_1 A\lambda\left(\beta_2\left(3A-2B\left(k^2+k+3\right)\right) + 3\alpha_2 B\lambda\right)\right], \tag{28}$$

$$\phi_k = \beta_1\left(3\alpha_2\lambda(2k\tau+1) - 4\beta_2 k\left(k^2+k+2\right)\tau\right) + \alpha_1\lambda\left(3\alpha_2\lambda - \beta_2(2k\tau+3)\right), \tag{29}$$

and $\mu = A+B$, $\lambda = A-B$, $\tau = k+1$.

Lemma 3.1. *The first and second derivatives of $\mathcal{Z}_k(x)$ are*

$$\begin{aligned}
\frac{d\,\mathcal{Z}_k(x)}{d\,x} &= \left(a_k\,x + b_k + x^2\right)\sum_{i=0}^{n-1}\zeta_{n,i,1}\,\mathrm{W}_i^{A,B}(x) + (a_k+2x)\,\mathrm{W}_k^{A,B}(x),\\
\frac{d^2\,\mathcal{Z}_k(x)}{d\,x^2} &= 2\,\mathrm{W}_k^{A,B}(x) + 2\,(2\,x+a_k)\sum_{i=0}^{n-1}\zeta_{n,i,1}\,\mathrm{W}_i^{A,B}(x) + (x^2+a_k\,x+b_k)\sum_{i=0}^{n-2}\zeta_{n,i,2}\,\mathrm{W}_i^{A,B}(x).
\end{aligned} \tag{30}$$

Proof. The first and second derivatives of $\mathcal{Z}_k(x)$ defined in (24) are

$$\begin{aligned}
\frac{d\,\mathcal{Z}_k(x)}{d\,x} &= \left(a_k\,x + b_k + x^2\right)\frac{d\,\mathrm{W}_k^{A,B}(x)}{d\,x} + (a_k+2x)\,\mathrm{W}_k^{A,B}(x),\\
\frac{d^2\,\mathcal{Z}_k(x)}{d\,x^2} &= 2\,\mathrm{W}_k^{A,B}(x) + 2\,(2\,x+a_k)\frac{d\,\mathrm{W}_k^{A,B}(x)}{d\,x} + (x^2+a_k\,x+b_k)\frac{d^2\,\mathrm{W}_k^{A,B}(x)}{d\,x^2}.
\end{aligned} \tag{31}$$

Now, the application of Corollary 2.4 after putting $q=1$, $q=2$ enables us to get the desired result. ∎

3.2 Collocation solution for the SODEs with homogeneous Robin boundary conditions

Assume that the SODEs (19), governed by the homogeneous Robin boundary conditions (21) and (22).
Now, consider

$$\begin{aligned}&\mathbf{P}_M([A,B]) = \operatorname{span}\{\mathcal{Z}_k(x) : k = 0,1,\dots,M\},\\ &\Delta_M([A,B]) = \{\chi_M(x) \in \mathbf{P}_M([A,B]) : \alpha_1\, \mathcal{Z}_i(A) + \beta_1\, \mathcal{Z}'_i(A) = \alpha_2\, \mathcal{Z}_i(B) + \beta_2\, \mathcal{Z}'_i(B) = 0\}.\end{aligned} \tag{32}$$

Then, any $\chi_M(x) \in \Delta_M([A,B])$ may be expressed as

$$\chi_M(x) = \sum_{k=0}^{M} c_k\, \mathcal{Z}_k(x). \tag{33}$$

Now, the residual $R(x)$ of Equation (19) may be calculated to give

$$R(x) = \chi''_M(x) - f(x, \chi_M, \chi'_M), \tag{34}$$

by virtue of Lemma 3.1, $R(x)$ can be rewritten as

$$\begin{aligned}R(x) &= \sum_{k=0}^{M} c_k \left[2\,\mathrm{W}_k^{A,B}(x) + 2\,(2\,x + a_k) \sum_{k=0}^{n-1} \zeta_{n,k,1}\, \mathrm{W}_k^{A,B}(x) + (x^2 + a_k\, x + b_k) \sum_{k=0}^{n-2} \zeta_{n,k,2}\, \mathrm{W}_k^{A,B}(x)\right]\\ &- f\left(x, \sum_{k=0}^{M} c_k \left[\left(b_k + a_k\, x + x^2\right)\, \mathrm{W}_k^{A,B}(x)\right], \sum_{k=0}^{M} c_k \left[\left(a_k\, x + b_k + x^2\right) \sum_{k=0}^{n-1} \zeta_{n,k,1}\, \mathrm{W}_k^{A,B}(x) + (a_k + 2x)\, \mathrm{W}_k^{A,B}(x)\right]\right).\end{aligned} \tag{35}$$

Now, the application of collocation method enables us to get the following $(M+1)$ algebraic system of equations in the unknown expansion coefficients c_k

$$R(x_i) = 0, \quad i = 1,2,\dots,M+1, \tag{36}$$

where $\{x_i : i = 1,2,\dots,M+1\}$ are the first $M+1$ distinct roots of $\mathrm{W}_{M+1}^{A,B}(x)$. And therefore, system in (36) can be solved using Newton's iterative methods.

3.3 Application of Equations (14)-(16) in chemistry

One well-known example of application of Equations (14)-(16) is reactive diffusion in a catalytic slab, where the concentration of a chemical species is represented by $\eta(x)$ and the nonlinear reaction kinetics (such as Langmuir-Hinshelwood or Arrhenius-type terms) are encoded by $g(x, \eta, \eta')$. When surface reactions and diffusive flow coexist at the boundaries, such as in the case of a partially absorbing or reactive wall, robin boundary conditions are created.

4 Error bound

In this section, our aim is to demonstrate that when M approaches infinity, $R(x)$ converges to zero. For the unknown function $\chi(x)$, we derive various error bounds and derivatives of this function by using similar steps as in [45].

Theorem 4.1. *Assume that* $\frac{d^i\,\chi(x)}{d\,X^i} \in \boldsymbol{C}([A,B])$, $i = 0,1,2,\dots,M+3$ *and let* $\chi_M(x)$ *be the proposed approximate solution belonging to* $\Delta_M([A,B])$, *and define*

$$\mathcal{L}_M = \sup_{x\in[A,B]} \left| \frac{d^{M+3}\,\chi(x)}{d\,x^{M+3}} \right|. \tag{37}$$

Consequently, the following estimate holds:

$$\|\chi(x) - \chi_M(x)\|_2 \leq \frac{\mathcal{L}_M\,(B-A)^{M+\frac{7}{2}}}{\sqrt{2\,M+7}\,(M+3)!}. \tag{38}$$

Proof. Consider the following Taylor expansion of $\chi(x)$ about the point $x = A$:

$$u_M(x) = \sum_{i=0}^{M+2} \left(\frac{d^i\,\chi(x)}{d\,x^i} \right)_{x=A} \frac{(x-A)^i}{i!}, \tag{39}$$

$$\chi(x) - u_M(x) = \frac{x^{M+3}}{(M+3)!} \left(\frac{d^{M+3}\,\chi(x)}{d\,x^{M+3}} \right)_{x=c}, \quad c \in [A,B]. \tag{40}$$

Because $\chi_M(x)$ is the best approximation solution of $\chi(x)$, we can apply the concept of best approximation [9]:

$$\begin{aligned} \|\chi(x) - \chi_M(x)\|_2^2 &\leq \|\chi(x) - u_M(x)\|_2^2 \\ &\leq \int_A^B \frac{{\mathcal{L}_M}^2\,x^{2\,(M+3)}}{((M+3)!)^2}\,d\,x \\ &= \frac{{\mathcal{L}_M}^2\,(B-A)^{2\,M+7}}{(2\,M+7)\,((M+3)!)^2}, \end{aligned} \tag{41}$$

and therefore, we have

$$\|\chi(x) - \chi_M(x)\|_2 \leq \frac{\mathcal{L}_M\,(B-A)^{M+\frac{7}{2}}}{\sqrt{2\,M+7}\,(M+3)!}. \tag{42}$$

■

Theorem 4.2. *Suppose that* $\chi(x)$, $\chi_M(x)$ *and* $\frac{d^i\,\chi(x)}{d\,X^i}$ *satisfy the condition of Theorem 4.1 and*

$$\mathrm{T}_{M,n} = \sup_{x\in[A,B]} \left| \frac{d^{M-n+3}\,\chi(x)}{d\,x^{M-n+3}} \right|. \tag{43}$$

The following estimate is therefore valid:

$$\left\| \frac{d^n\,(\,\chi(x) - \chi_M(x))}{d\,x^n} \right\|_2 \leq \frac{\mathrm{T}_{M,n}\,(B-A)^{M-n+\frac{7}{2}}}{\sqrt{2\,(M-n)+7}\,(M-n+3)!}, \quad n = 1,2. \tag{44}$$

Proof. Assuming that $\frac{d^n\,\bar{v}(x)}{d\,x^n}$ is the Taylor expansion of $\frac{d^n\,\chi(x)}{d\,x^n}$ about the point $x = A$, the residual between $\frac{d^n\,\chi(x)}{d\,x^n}$ and $\frac{d^n\,\bar{v}(x)}{d\,x^n}$ can be expressed as follows:

$$\frac{d^n\,(\,\chi(x) - \bar{v}(x))}{d\,x^n} = \frac{x^{M-n+3}}{(M-n+3)!} \left(\frac{d^{M-n+3}\,\chi(x)}{d\,x^{M-n+3}} \right)_{x=c}, \quad c \in [A,B]. \tag{45}$$

The concept of the best approximation states that if $\frac{d^n\,\chi_M(x)}{d\,x^n}$ is the best approximate solution of $\frac{d^n\,\chi(x)}{d\,x^n}$, we obtain

$$\left\|\frac{d^n\,(\,\chi(x)-\chi_M(x))}{d\,x^n}\right\|_2 \le \left\|\frac{d^n\,(\,\chi(x)-\bar{v}(x))}{d\,x^n}\right\|_2. \tag{46}$$

Now, by taking the same procedures as in Theorem 4.1, the desired result could be obtained. ■

Theorem 4.3. *Assuming f satisfies Lipschitz condition in its second and third arguments, then*

$$\|R(x)\|_2 \le \frac{\mathrm{T}_{M,2}\,(B-A)^{M+\frac{3}{2}}}{\sqrt{2\,M+3}\,(M+1)!} + \tau_1\,\frac{\mathcal{L}_M\,(B-A)^{M+\frac{7}{2}}}{\sqrt{2\,M+7}\,(M+3)!} + \tau_2\,\frac{\mathrm{T}_{M,1}\,(B-A)^{M+\frac{5}{2}}}{\sqrt{2\,M+5}\,(M+2)!}. \tag{47}$$

Proof. Subtracting Equation (34) from Equation (19), we get

$$R(x) = \chi''(x) - \chi''_M(x) - [f(x,\chi,\chi') - f(x,\chi_M,\chi'_M)], \tag{48}$$

Since, f satisfies Lipschitz condition in its second and third arguments, then there exist two Lipschitz constants τ_1 and τ_2 such that

$$|f(x,\chi,\chi') - f(x,\chi_M,\chi'_M)| \le \tau_1\,|\chi(x)-\chi_M(x)| + \tau_2\,|\chi'(x)-\chi'_M(x)|. \tag{49}$$

Now, taking $\|.\|_2$, for both sides of Equation (48), and using the previous inequality, we get

$$\|R(x)\|_2 \le \frac{\mathrm{T}_{M,2}\,(B-A)^{M+\frac{3}{2}}}{\sqrt{2\,M+3}\,(M+1)!} + \tau_1\,\frac{\mathcal{L}_M\,(B-A)^{M+\frac{7}{2}}}{\sqrt{2\,M+7}\,(M+3)!} + \tau_2\,\frac{\mathrm{T}_{M,1}\,(B-A)^{M+\frac{5}{2}}}{\sqrt{2\,M+5}\,(M+2)!}. \tag{50}$$

■

5 Illustrative examples

Example 5.1. ([46, 47]). Consider the following equation

$$\eta''(x) = \frac{1}{2}e^{-x}\left(\eta'(x)^2 + \eta(x)^2\right), \quad 0 \le x \le 1, \tag{51}$$

subject to

$$\eta(0) - \eta'(0) = 0, \tag{52}$$

$$\eta(1) + \eta'(1) = 2\,e, \tag{53}$$

where the exact solution of this problem is $\eta(x) = e^x$.
Table 1 shows a comparison of L_∞ errors between our method at $M = 11$ and the method in [46]. Table 2 presents a comparison of absolute errors between our method at $M = 11$ and the method in [47]. Figure 1 illustrates the absolute errors at different values of M.

Table 1: Comparison of L_∞ errors of Example 5.1.

Method in [46] at ($N = 32$, $h = 0.032$)	5.09370×10^{-13}
Our method at ($M = 11$)	5.96745×10^{-16}

Table 2: Comparison of absolute errors of Example 5.1.

	Method in [47] at $h = 0.01$				
x	2PDD4	2PDAM4	DAM4	Our method at $M = 11$	CPU time
0.1	5.32×10^{-12}	2.79×10^{-11}	8.54×10^{-12}	6.66134×10^{-16}	
0.2	5.87×10^{-12}	1.35×10^{-10}	2.23×10^{-11}	4.44089×10^{-16}	
0.3	6.69×10^{-12}	3.24×10^{-10}	4.57×10^{-11}	0	
0.4	7.76×10^{-12}	5.88×10^{-10}	7.80×10^{-11}	0	
0.5	9.08×10^{-12}	9.22×10^{-10}	1.19×10^{-10}	0	1.954
0.6	1.06×10^{-12}	1.32×10^{-9}	1.68×10^{-10}	4.44089×10^{-16}	
0.7	1.24×10^{-11}	1.79×10^{-9}	2.24×10^{-10}	4.44089×10^{-16}	
0.8	1.44×10^{-11}	2.32×10^{-9}	2.88×10^{-10}	4.44089×10^{-16}	
0.9	1.66×10^{-11}	2.91×10^{-9}	3.59×10^{-10}	8.88178×10^{-16}	

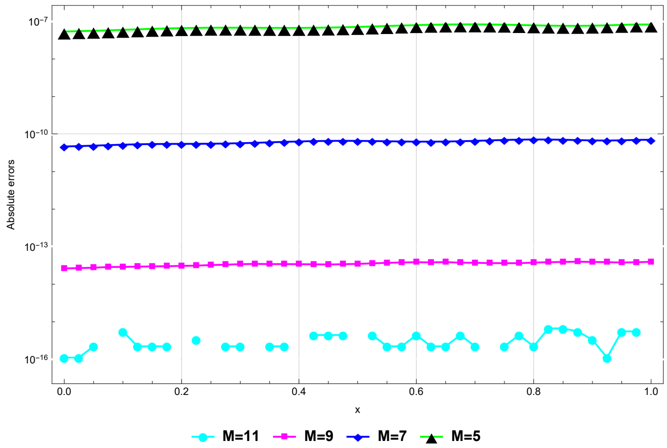

Figure 1: The absolute errors of Example 5.1 at different values of M.

Example 5.2. ([43, 46]). Consider the following equation

$$\eta''(x) = -e^{-2\,\eta(x)}, \quad 0 \leq x \leq 1, \tag{54}$$

subject to

$$-\eta(0) + \eta'(0) = 1, \tag{55}$$

$$\eta(1) + \eta'(1) = \frac{1}{2} + \log(2), \tag{56}$$

where the exact solution of this problem is $\eta(x) = \log(x + 1)$.
Table 3 shows a comparison of L_∞ errors between our method at $M = 20$ and the methods in [46] and [43]. Figure 2 illustrates the absolute errors at different values of M. Also, Figure 3 illustrates the maximum absolute errors at different values of M.

Example 5.3. ([46, 47]). Consider the following equation

$$\eta''(x) = \eta(x) - 2\,\cos(x), \quad \frac{\pi}{2} \leq x \leq \pi, \tag{57}$$

Table 3: Comparison of L_∞ errors of Example 5.2.

Method in [46] at ($N = 128$, $h = \frac{1}{64}$)	4.76696×10^{-12}
Method in [43] at ($N = 19$)	3.33067×10^{-16}
Our method at ($M = 20$)	2.22045×10^{-16}

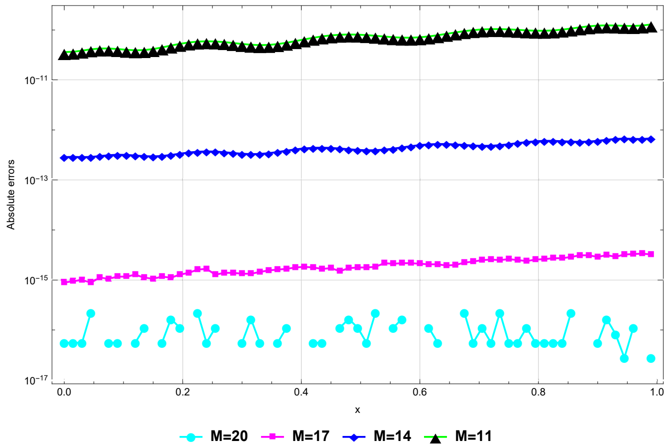

Figure 2: The absolute errors of Example 5.2 at different values of M.

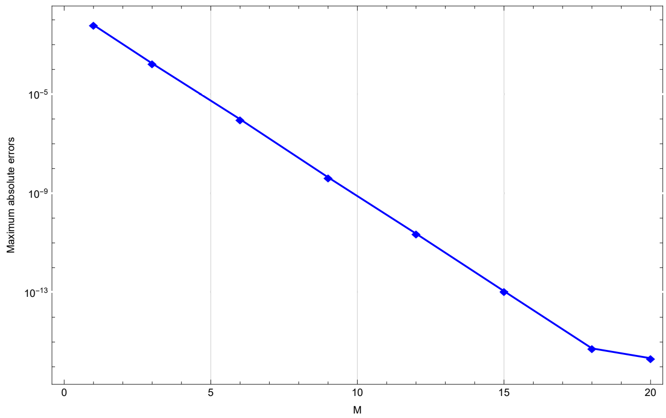

Figure 3: The maximum absolute errors of Example 5.2 at different values of M.

subject to

$$3\,\eta\left(\frac{\pi}{2}\right) + \eta'\left(\frac{\pi}{2}\right) = -1, \tag{58}$$

$$4\,\eta(\pi) + \eta'(\pi) = -4, \tag{59}$$

where the exact solution of this problem is $\eta(x) = \cos(x)$.
Table 4 shows a comparison of L_∞ errors between our method at $M = 12$ and the methods in [46] and [47]. Table 5 presents a comparison of absolute errors between our method at $M = 12$ and the method in [47]. Figure 4 illustrates the comparability of analytic and approximate solution at $M = 12$.

Table 4: Comparison of L_∞ errors of Example 5.3.

Method in [46] at ($N = 50$, $h = 0.032$)	5.65024×10^{-14}
Method in [47] at ($h = 0.01$)	2.47×10^{-10}
Our method at ($M = 12$)	6.127045×10^{-15}

Table 5: Comparison of absolute errors of Example 5.3.

	Method in [47] at $h = 0.01$				
x	2PDD4	2PDAM4	DAM4	Our method at $M = 12$	CPU time
1.5708	2.05×10^{-10}	2.05×10^{-10}	2.05×10^{-10}	5.38458×10^{-15}	
1.7279	1.70×10^{-10}	1.71×10^{-10}	1.71×10^{-10}	2.91434×10^{-15}	
1.8850	1.56×10^{-10}	1.63×10^{-10}	1.63×10^{-10}	1.43982×10^{-15}	
2.0420	1.49×10^{-10}	1.66×10^{-10}	1.69×10^{-10}	1.26982×10^{-15}	
2.1991	1.48×10^{-10}	1.78×10^{-10}	1.91×10^{-10}	1.249×10^{-15}	
2.3562	1.52×10^{-10}	2.00×10^{-10}	2.29×10^{-10}	3.33067×10^{-16}	6.689
2.5133	1.60×10^{-10}	2.31×10^{-10}	2.86×10^{-10}	3.33067×10^{-16}	
2.6704	1.73×10^{-10}	2.72×10^{-10}	3.65×10^{-10}	2.27596×10^{-15}	
2.8274	1.91×10^{-10}	3.25×10^{-10}	4.68×10^{-10}	3.30291×10^{-15}	
2.9845	2.16×10^{-10}	3.92×10^{-10}	6.03×10^{-10}	5.44009×10^{-15}	

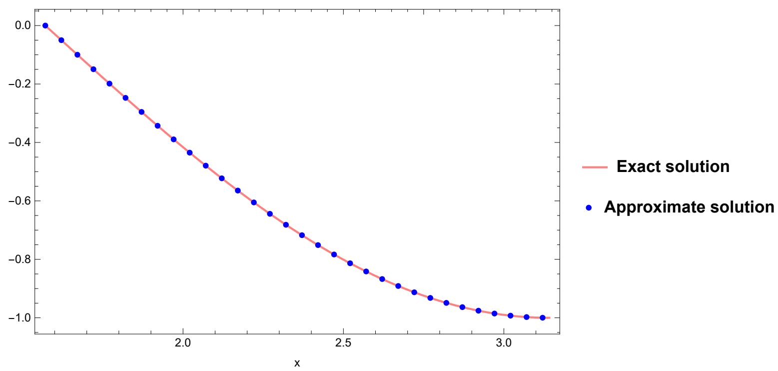

Figure 4: Comparability of analytic and approximate solution of Example 5.3 at $M = 12$.

Example 5.4. ([44, 48]). Consider the following equation

$$\eta''(x) + \frac{2\eta'(x)}{x} + \eta(x)^5 = 0, \quad 0 \le x \le 1, \tag{60}$$

subject to

$$\eta'(0) = 0, \quad \eta(1) = \frac{\sqrt{3}}{2}, \tag{61}$$

where the exact solution of this problem is $\eta(x) = \sqrt{\frac{3}{x^2+3}}$.

Table 6 shows a comparison of maximum absolute errors between our method at $M = 15$ and the methods in [44] and [48]. Table 7 presents the maximum absolute errors at different values of M. Figure 5 illustrates the L_∞ errors at different values of M.

Table 6: Comparison of maximum absolute errors of Example 5.4.

Method in [44] at ($h = \frac{1}{128}$)	5.5912×10^{-11}
Method in [48] at ($h = \frac{1}{128}$)	1.37828×10^{-10}
Our method at ($M = 15$)	2.23779×10^{-16}

Table 7: Maximum absolute errors of Example 5.4.

M	3	6	9	12	15
Error	1.23583×10^{-5}	3.35475×10^{-8}	8.83282×10^{-11}	1.60168×10^{-13}	2.23779×10^{-16}
CPU time	2.157	2.272	2.288	2.35	2.515

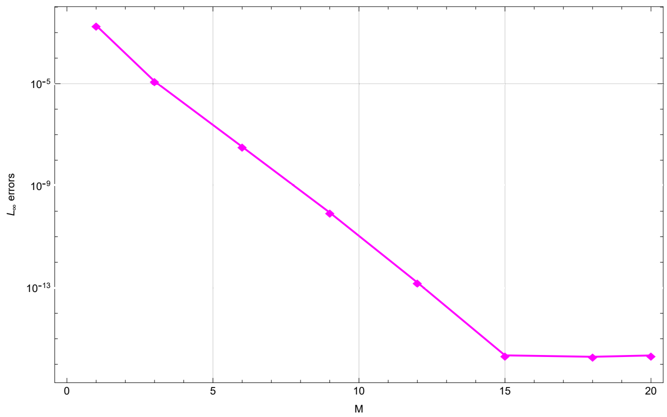

Figure 5: The L_∞ errors of Example 5.4 at different values of M.

Example 5.5. ([46, 49]). Consider the following equation

$$\eta''(x) = 0.5(\eta(x) + x + 1)^3, \quad 0 \le x \le 1, \tag{62}$$

subject to

$$-\eta(0) + \eta'(0) = -\frac{1}{2}, \tag{63}$$

$$\eta(1) + \eta'(1) = 1, \tag{64}$$

where the exact solution of this problem is $\eta(x) = \frac{2}{2-x} - x - 1$.
Table 8 shows a comparison of L_∞ errors between our method at $M = 21$ and the methods in [46] and [49]. Table 9 presents the different errors at different values of M. In addition, Table 10 shows the CPU time used in Table 9. Figure 6 illustrates the comparability of analytic and approximate solution at $M = 21$.

Table 8: Comparison of L_∞ errors of Example 5.5.

Method in [46] at ($N = 128$, $h = \frac{1}{128}$)	2.86215×10^{-13}
Method in [49] at ($h = 0.01$)	1.1374×10^{-10}
Our method at ($M = 21$)	2.22045×10^{-16}

Table 9: Errors of Example 5.5.

M	3	6	9	12	15	18	21
MAE	2.89×10^{-3}	1.94×10^{-5}	1.22×10^{-7}	7.65×10^{-10}	4.51×10^{-12}	2.65×10^{-14}	2.22×10^{-16}
L_∞	2.92×10^{-3}	1.95×10^{-5}	1.22×10^{-7}	7.65×10^{-10}	4.51×10^{-12}	2.71×10^{-14}	4.71×10^{-16}
L_2	1.84×10^{-3}	1.34×10^{-5}	8.33×10^{-8}	5.34×10^{-10}	3.10×10^{-12}	1.86×10^{-14}	1.79×10^{-16}

Table 10: CPU time used of Table 9.

M	3	6	9	12	15	18	21
MAE	1.218	1.406	1.828	1.953	1.534	1.532	1.829
L_∞	1.374	1.578	2.031	2.234	1.846	1.86	2.219
L_2	2.358	3.25	5.422	6.093	6.331	7.063	8.501

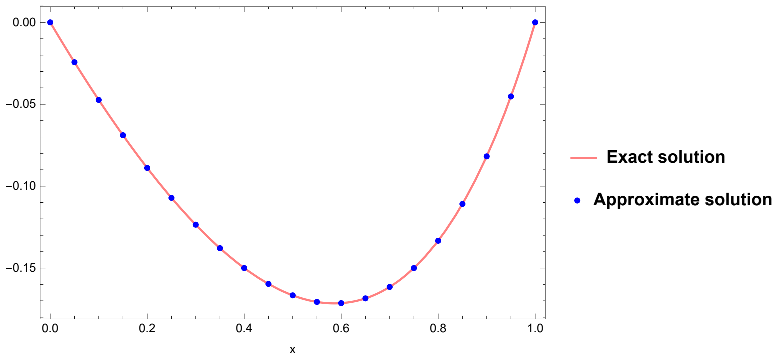

Figure 6: Comparability of analytic and approximate solution of Example 5.5 at $M = 21$.

Example 5.6. ([44]). Consider the following equation

$$\left(x^2 - 6x - 1\right)\eta'(x) + \left(-x^2 + 5x + 6\right)\eta(x) + \eta''(x) = e^x - (x-6)(x+1), \quad 0 \le x \le 1, \quad (65)$$

subject to

$$\eta(0) + \eta'(0) = 2, \quad (66)$$
$$2\,\eta(1) - \eta'(1) = 2, \quad (67)$$

where the exact solution of this problem is $\eta(x) = x\,e^x + 1$.
Table 11 shows a comparison of maximum absolute errors between our method at $M = 12$ and method in [44]. Table 12 presents the different errors at different values of M. In addition, Table 13 shows the CPU time used in Table 12. Figure 7 illustrates the absolute errors at different values of M.

Table 11: Comparison of maximum absolute errors of Example 5.6.

Method in [44] at $(h = \frac{1}{80})$	2.5888×10^{-11}
Our method at $(M = 12)$	8.88178×10^{-16}

Table 12: Errors of Example 5.6.

M	2	4	6	8	10	12
MAE	1.079×10^{-2}	5.059×10^{-6}	6.945×10^{-9}	7.601×10^{-12}	3.108×10^{-15}	8.881×10^{-16}
L_∞	1.079×10^{-2}	5.059×10^{-6}	6.945×10^{-9}	7.602×10^{-12}	7.105×10^{-15}	5.329×10^{-15}
L_2	52.48×10^{-3}	1.837×10^{-6}	2.289×10^{-9}	2.555×10^{-12}	6.887×10^{-16}	2.738×10^{-16}

Table 13: CPU time used of Table 12.

M	2	4	6	8	10	12
MAE	1.032	1.344	1.594	1.703	2.234	2.517
L_∞	1.173	1.531	1.75	1.922	2.5	2.72
L_2	1.61	2.266	2.625	3.078	3.953	4.064

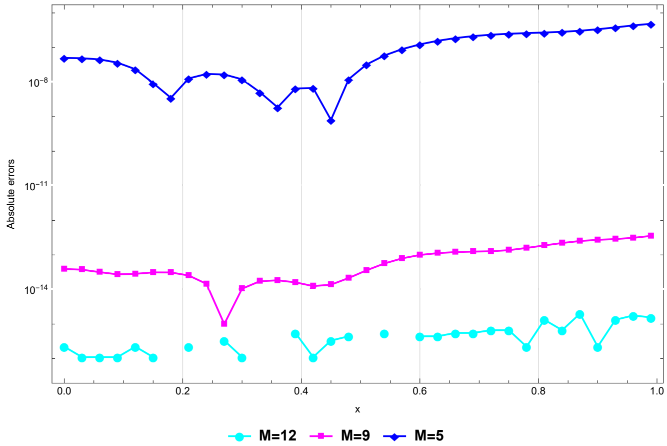

Figure 7: The absolute errors of Example 5.6 at different values of M.

6 Concluding remarks

In this study, we have demonstrated and examined a precise collocation solver for a specific linear and nonlinear SODEs. Some theoretical results concerned with modified sets of shifted 4KCPs were the keys to implementing our numerical algorithm for solving the linear and nonlinear SODEs. Additionally, we discussed the truncation error of the approximate modified sets of shifted 4KCPs solution that were given. To confirm the validity and correctness of the suggested technique, several numerical results and comparisons were shown. We do think that the proposed technique can be expanded to more generic models in other fields of physics, mathematics, and engineering. As an expected future work, we aim to employ the developed theoretical results in this paper along with suitable spectral methods to treat some other problems. All codes were written and debugged by *Mathematica* 11 on HP Z420 Workstation, Processor: Intel(R) Xeon(R) CPU E5-1620 v2 - 3.70GHz, 16 GB Ram DDR3, and 512 GB storage.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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