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Research Paper

On the Energy and Nullity of Non-Uniform Path and Cycle Semigraphs

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Abstract

Graph energy, originating in Hückel molecular orbital theory, remains central to mathematical chemistry. Motivated by heterogeneous linear and cyclic molecular structures, we study non-uniform path and cycle semigraphs, where original edges are subdivided by $n_i \geq 1$ middle vertices. We show the adjacency matrix decomposes into a symmetric tridiagonal core, whose spectrum comprises all non-zero eigenvalues, plus zero rows from middle vertices. For paths, a continuant recurrence for the characteristic polynomial and parity arguments yield spectral symmetry and precise nullity conditions. For cycles, a wraparound determinant formula characterizes when the spectrum is symmetric about zero and provides exact criteria for the presence and multiplicity of specific zero eigenvalues. Consequently, the energy of each semigraph equals the energy of its core matrix, yielding clean expressions for energy and nullity from the $\{n_i\}$ parameters. Uniform cases arise as immediate corollaries and are consistent with spectral invariants in chemically inspired models.

1 Introduction

In spectral graph theory, eigenvalues encode subtle structural information about networks. Two classical spectral invariants are the *graph energy* and the *nullity*. Graph energy defined as the sum of the absolute values of the adjacency eigenvalues, originated in Hückel molecular orbital theory as an approximation to the total π-electron energy of conjugated hydrocarbons [1]. It has since become a central descriptor in mathematical chemistry [2, 3]. The nullity of a

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graph, the multiplicity of the eigenvalue 0, is likewise important, with connections to molecular stability and non-bonding orbitals [4]. Motivated by heterogeneous conjugated chains and cyclic polymers, where local environments vary along a linear or cyclic backbone, we study the energy and nullity of certain non-uniform semigraphs.

The concept of a semigraph, introduced by Sampathkumar, extends the classical notion of a graph by allowing edges to be incident with more than two vertices [5]. While the spectral theory of ordinary graphs is well developed, the spectral analysis of semigraphs is comparatively nascent. Related developments include color energy for semigraphs [6] and Randić-type energies for uniform hypergraphs [7], underscoring that modeling choices (semigraphs vs. hypergraphs) lead to distinct spectral frameworks.

We focus on two families of non-uniform semigraphs: (i) the *Non-Uniform Path Semigraphs* $P_{k+1}^{n_1,n_2,\ldots,n_k}$ and (ii) the *Non-Uniform Cycle Semigraphs* $C_k^{n_1,n_2,\ldots,n_k}$. Each structure is formed by connecting k segments in a linear or cyclic fashion, where the ith segment has length $n_i \geq 1$. Non-uniformity refers to the fact that the n_i are allowed to differ, modeling positional heterogeneity along the chain or ring.

Our main objective is to characterize the nullity and to provide a tractable method for computing the energy of these semigraphs. The analysis rests on a key structural property of the adjacency matrix: rows corresponding to pure middle vertices are zero. By deleting the corresponding columns, we pass to a principal core matrix that collects exactly the non-zero eigenvalues of the semigraph. Consequently, the problem reduces to studying a symmetric tridiagonal core of size $(k+1) \times (k+1)$ in the path case and $k \times k$ in the cycle case.

For the path cores, we derive continuant-type recurrences for the characteristic polynomials and use parity to establish spectral symmetry and to obtain precise nullity conditions. For the cycle cores, we prove a wrap-around determinant identity, characterize when the spectrum is symmetric about zero, and give exact criteria for the presence with multiplicity two of the zero eigenvalue. Throughout, continuant techniques underlie polynomial recurrences [8]. A key consequence is that the energy of a non-uniform semigraph equals the energy of its core, so only the non-zero eigenvalues need be computed. Uniform cases appear as immediate corollaries.

This paper is structured as follows. We first analyze the non-uniform Path Semigraphs and find their adjacency matrices, establishing the recurrence relation for its characteristic polynomial corresponding to the core matrix, proving the symmetry of its spectrum, and deriving a formula for its nullity based on the nature of k. We then extend this analysis to the Non-Uniform Cycle Semigraphs, revealing a more intricate relationship for its characteristic polynomial, symmetry criteria, and a complete nullity formula, including the notable case $k \equiv 0 \pmod 4$. We conclude with remarks on energy computations and directions for future work (including Laplacian-based analogues [9]).

2 Preliminaries

Definition 2.1. Semigraph:([5]). G is a semigraph represented by a pair (V, E) where V is a non-empty set whose elements are called vertices of G and E is a set of m-tuples, called edges of semigraph G, of distinct vertices, for various m $(m \geq 2)$ satisfying the following conditions:

1. At most one vertex in common on any two edges.
2. Two edges $(u_1, u_2, \ldots, u_k)$ and $(w_1, w_2, \cdots, w_r)$ are considered to be equal if
    - $r = k$ and
    - either $u_j = w_j$ for $j = 1, 2, \ldots, k$ or $u_j = w_{r+1-j}$, for $j = 1, 2, \ldots, k$.

The vertices in a semigraph are divided mainly into three namely end vertices, middle vertices and middle-end vertices, depending upon their location in an edge. For the edge $E = (w_1, w_2, \ldots, w_n)$, w_1 and w_n are called the end vertices of E and $w_2, w_3, \cdots w_{n-1}$ are called the middle vertices of E. Also, a vertex which is a middle vertex for one edge and an end vertex of another edge is called an (m, e)-vertex. An (m, e)-vertex is drawn as a hollow circle with a small tangent drawn to it indicating it is an end vertex of the other edge and the middle vertex is drawn as a hollow circle.

Definition 2.2. Adjacent vertices:([10]). Two vertices in a semigraph are stated to be adjacent if they belong to the same edge.
We define the following types of vertices for a semigraph:

1. w_i is said to be a pure end vertex if it is an end vertex of every edge to which it belongs.
2. w_i is said to be a pure middle vertex if it is a middle vertex of every edge to which it belongs.
3. w_i is said to be a middle end vertex if it is middle vertex of at least one edge and end vertex of at least one other edge.

Definition 2.3. Adjacency matrix:([5]).
Let $G(V, E)$ be a semigraph with vertex set $V = \{1, 2, ..k\}$ and edge set $E = \{e_1, e_2, ...e_q\}$. Adjacency matrix of $G(V, E)$ is a $k \times k$ matrix $A = [a_{ij}]$ which is defined as follows:

1. For every edge $e_i = (i_1, i_2, ...i_s)$ of G with $i_1, i_2, ..i_s$ being vertices of V, $\forall i_r \in e_i$; $r = 1, 2, ...s$
   - $a_{i_1 i_r} = r - 1$,
   - $a_{i_s i_r} = s - r$.
2. All the remaining entries of adjacency matrix of G are zero.

Remark 1. (**Semigraphs vs. hypergraphs**): A hypergraph $H = (V, \mathcal{E})$ has edges that are unordered nonempty subsets of V; different edges may share more than one vertex, and adjacency is typically encoded via an incidence matrix or (for uniform hypergraphs) higher-order tensors. In a *semigraph*, each edge is an *ordered* m-tuple $(u_1, \ldots, u_m)$, any two edges share *at most one* vertex, and adjacency entries depend on the *positions* of incident vertices (see Def. 2.3). Consequently, the spectral objects and energy notions for semigraphs differ from those for hypergraphs. (see, for instance, Nandi–Gutman–Nath [6] on the energy of the *color matrix* for semigraphs, and Shirdel–Mortezaee–Alaameri [7] on the Randić matrix energy of uniform hypergraphs).

Definition 2.4. Spectrum of adjacency matrix of semigraph:([5]). If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues of a matrix M with multiplicities $m_1, m_2, \cdots, m_n$ then spectrum of adjacency matrix of semigraph M is $\begin{pmatrix} \lambda_1 & \lambda_2 & ... & \lambda_n \\ m_1 & m_2 & ... & m_n \end{pmatrix}$.

Definition 2.5. Nullity:([11]). The nullity of a usual graph $\eta(G)$ is the algebraic multiplicity of eigenvalue zero in the spectrum of adjacency matrix of the graph G.

Definition 2.6. (**Nullity of a semigraph**): The nullity of a semigraph $\eta(G)$ is the algebraic multiplicity of eigenvalue zero in the spectrum of adjacency matrix of the semigraph G. It is denoted by $\eta(G)$.

Definition 2.7. Energy of a semigraph:([12]). Let G be a semigraph with n vertices and its adjacency matrix be $A(G)$. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues of adjacency matrix then Energy of semigraph G is denoted by $E(G)$ and is defined by

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

3 Nullity and energy of non-uniform semigraphs

In this section we define two classes of Non-uniform semigraphs: (i) Non-uniform Path Semigraphs, $P_{k+1}^{n_1,n_2,\cdots,n_k}$,
(ii) Non-uniform Cycle Semigraphs, $C_k^{n_1,n_2,\cdots,n_k}$.
Then we find a recurrence formula to calculate the characteristic polynomial and mention some properties related to eigenvalues. Also, we find the nullity and energy of these two classes of semigraphs.

3.1 Non-uniform path semigraphs, $P_{k+1}^{n_1,n_2,\cdots,n_k}$

Definition 3.1. (**Non-uniform Path Semigraphs**): Non-uniform Path Semigraphs with $\sum_{i=1}^{k} n_i + 1$ vertices $v_{1,1}, v_{1,2}, \cdots, v_{1,n_1}, v_{2,1}, \cdots, v_{k+1,1}$ and k edges $e_1 = (v_{1,1}, v_{1,2}, \cdots, v_{2,1}), e_2 = (v_{2,1}, v_{2,2}, \cdots, v_{3,1}) \cdots, e_k = (v_{k,1}, v_{k,2}, \cdots, v_{k+1,1})$ are denoted by $P_{k+1}^{n_1,n_2,\cdots,n_k}$ and is given in Figure 1.

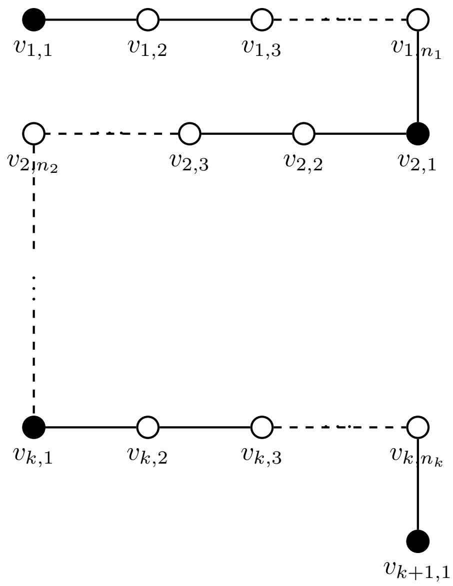

Figure 1: Non-uniform path semigraph, $P_{k+1}^{n_1,n_2,\cdots,n_k}$.

The adjacency matrix $A_{P_{k+1}^{n_1,n_2,\cdots,n_k}}$ of this semigraph is given by

$$
\begin{array}{c|ccccccccccccccccccc|}
 & v_{11} & v_{12} & \cdots & v_{1n_1} & v_{21} & v_{22} & \cdots & v_{2n_2} & v_{31} & \cdots & v_{3n_3} & \cdots & v_{k1} & v_{k2} & \cdots & v_{kn_k} & v_{(k+1)1} \\
\hline
v_{11} & 0 & 1 & \cdots & n_1-1 & n_1 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
v_{12} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
v_{13} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
v_{1n_1} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
v_{21} & n_1 & n_1-1 & \cdots & 1 & 0 & 1 & \cdots & n_2-1 & n_2 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
v_{22} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
v_{2n_2} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
v_{31} & 0 & 0 & \cdots & 0 & n_2 & n_2-1 & \cdots & 1 & 0 & \cdots & n_3-1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
v_{32} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
v_{3n_3} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
v_{41} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
v_{k1} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & n_k-1 & n_k \\
v_{k2} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
v_{k3} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
v_{kn_k} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
v_{(k+1)1} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & n_k & n_k-1 & \cdots & 1 & 0
\end{array}.
$$

Theorem 3.2. *Let A be a real symmetric tridiagonal matrix given by*

$$
A = A_{k+1} = \begin{array}{c}
 \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{k-1} \\ v_k \\ v_{k+1}
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} v_1 & v_2 & v_3 & \cdots & v_{k-1} & v_k & v_{k+1} \end{array} \\
\left(\begin{array}{ccccccc}
0 & n_1 & 0 & \cdots & 0 & 0 & 0 \\
n_1 & 0 & n_2 & \cdots & 0 & 0 & 0 \\
0 & n_2 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & n_{k-1} & 0 \\
0 & 0 & 0 & \cdots & n_{k-1} & 0 & n_k \\
0 & 0 & 0 & \cdots & 0 & n_k & 0
\end{array}\right)_{(k+1)\times(k+1)}
\end{array},
$$

where n_i's are non negative integers. Let $m = k+1$. Assume that $P_m(\lambda) = \det(A_m - \lambda I_m)$ be the characteristic polynomial of $A = A_m$. Then,

$$P_m(\lambda) = -\lambda P_{m-1}(\lambda) - n_{m-1}^2 P_{m-2}(\lambda) \quad \textit{for} \;\; m \geq 2,$$

with initial conditions $P_0(\lambda) = 1$ and $P_1(\lambda) = -\lambda$.

Proof. Take cofactor expansion along the last row of the matrix $A_m - \lambda I_m$,

$$
A - \lambda I_m = \begin{array}{c}
 \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{m-2} \\ v_{m-1} \\ v_m
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} v_1 & v_2 & v_3 & \cdots & v_{m-2} & v_{m-1} & v_m \end{array} \\
\left(\begin{array}{ccccccc}
-\lambda & n_1 & 0 & \cdots & 0 & 0 & 0 \\
n_1 & -\lambda & n_2 & \cdots & 0 & 0 & 0 \\
0 & n_2 & -\lambda & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -\lambda & n_{m-2} & 0 \\
0 & 0 & 0 & \cdots & n_{m-2} & -\lambda & n_{m-1} \\
0 & 0 & 0 & \cdots & 0 & n_{m-1} & -\lambda
\end{array}\right)_{m\times m}
\end{array}.
$$

The expansion gives

$$P_m(\lambda) = (-\lambda)\cdot \det(A_{m-1} - \lambda I_{m-1}) - n_{m-1} \cdot \det \begin{pmatrix} -\lambda & n_1 & \cdots & 0 & 0 \\ n_1 & -\lambda & \cdots & 0 & 0 \\ 0 & n_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\lambda & 0 \\ 0 & 0 & \cdots & n_{m-2} & n_{m-1} \end{pmatrix}$$
$$= (-\lambda)P_{m-1}(\lambda) - n_{m-1}\cdot (n_{m-1}\det(A_{m-2} - \lambda I_{m-2}))$$
$$= -\lambda P_{m-1}(\lambda) - n_{m-1}^2 P_{m-2}(\lambda).$$

The initial conditions follow directly. $P_0(\lambda) = \det(\text{Empty matrix}) = 1$ (by convention) and $P_1(\lambda) = \det(-\lambda) = -\lambda$. ■

Theorem 3.3. *Let A be the $(k+1)\times(k+1)$ real symmetric tridiagonal matrix given by*

$$A = \begin{matrix} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{k-1} \\ v_k \\ v_{k+1} \end{matrix} \begin{matrix} v_1 & v_2 & v_3 & \cdots & v_{k-1} & v_k & v_{k+1} \\ \end{matrix} \begin{pmatrix} 0 & n_1 & 0 & \cdots & 0 & 0 & 0 \\ n_1 & 0 & n_2 & \cdots & 0 & 0 & 0 \\ 0 & n_2 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n_{k-1} & 0 \\ 0 & 0 & 0 & \cdots & n_{k-1} & 0 & n_k \\ 0 & 0 & 0 & \cdots & 0 & n_k & 0 \end{pmatrix}_{(k+1)\times(k+1)},$$

where n_i's are non negative integers.Then all eigenvalues of A are real. Also the spectrum is symmetric with respect to the origin, i.e., if λ_j is an eigenvalue of A then $-\lambda_j$ is also an eigenvalue of A.

Proof. Since A is a real symmetric matrix then all its eigenvalues are real. Suppose $m = k+1$, $A = A_m$ and $P_m(\lambda)$ denote the characteristic polynomial of A_m.

First we are going to prove that $P_m(\lambda)$ is an even function of λ if m is even, and an odd function if m is odd. We use the method of induction to prove these cases.
For $m = 0$, $P_0(\lambda) = det(A_0 - \lambda I) = det(Empty\ matrix) = 1$ (By convention) is an even function.
For $m = 1$, $P_1(\lambda) = det(A_1 - \lambda I) = det(-\lambda) = -\lambda$ is an odd function.
So the result holds for $m = 0$ and $m = 1$.
Assume that the result holds for all integers up to $m-1$. We are going to show that result is true for m.
From Theorem 3.2 we get $P_m(\lambda) = -\lambda P_{m-1}(\lambda) - n_{m-1}^2 P_{m-2}(\lambda)$.
If m is even then $m-1$ is odd and $m-2$ is even. By the method of induction, P_{m-1} is odd and P_{m-2} is even. So,

$$P_m(-\lambda) = -(-\lambda)P_{m-1}(-\lambda) - n_{m-1}^2 P_{m-2}(-\lambda)$$
$$= \lambda(-P_{m-1}(\lambda)) - n_{m-1}^2 P_{m-2}(\lambda) = P_m(\lambda).$$

So, $P_m(\lambda)$ is an even function when m is even.
If m is odd, then $m-1$ is even and $m-2$ is odd. By the method of induction, P_{m-1} is an even function and P_{m-2} is an odd function.

$$P_m(-\lambda) = -(-\lambda)P_{m-1}(-\lambda) - n_{m-1}^2 P_{m-2}(-\lambda)$$
$$= \lambda(P_{m-1}(\lambda)) - n_{m-1}^2(-P_{m-2}(\lambda)) = -P_m(\lambda).$$

So, $P_m(\lambda)$ is an odd function when m is odd.
If λ_j is a root of characteristic polynomial of $A = A_m$, then $P_m(\lambda_j) = 0$.

- If m is even, then P_m is an even function. So, $P_m(-\lambda_j) = P_m(\lambda_j) = 0$.
- If m is odd, then P_m is an odd function. So, $P_m(-\lambda_j) = -P_m(\lambda_j) = -0 = 0$.

Therefore, in both cases, if λ_j is an eigenvalue of $P_m(\lambda)$ then $-\lambda_j$ is also an eigenvalue. So, the spectrum is symmetric with respect to the origin. This completes the proof. ■

Theorem 3.4. *Let A be the $(k+1)\times(k+1)$ matrix given in Theorem 3.2.*
The characteristic polynomial $P_{k+1}(\lambda) = \det(A_{k+1} - \lambda I)$ is given as follows:
***If** $k+1 = 2r$ **is even,** then $P_{k+1}(\lambda) = \sum_{j=0}^{r} M_j \lambda^{(k+1)-2j}$.*
***If** $k+1 = 2r+1$ **is odd,** then $P_{k+1}(\lambda) = -\lambda\left(\sum_{j=0}^{r} M_j \lambda^{k-2j}\right)$, where the coefficients M_j for $P_{k+1}(\lambda)$ are defined by:*

$$M_j = \sum_{\substack{1\le i_1<i_2<\cdots<i_j\le k \\ |i_p-i_q|\ge 2 \text{ for } p\ne q}} \left(\prod_{l=1}^{j}(-n_{i_l}^2)\right),$$

for $j = 1, 2, \ldots, r$ and $M_0 = 1$.

Proof. Suppose $m = k+1$. We will prove this result by the method of induction applied on k. Let $P_m(\lambda) = \det(A_m - \lambda I)$.
Consider $m = 1$ and $m = 2$.

- For $m = 1$: $A_1 = 0$. So, $P_1(\lambda) = \det(-\lambda) = -\lambda$. Also, for $m = 1$, we have $k = 0$. So, $P_1(\lambda) = -\lambda(\lambda^0) = -\lambda$.
- For $m = 2$: $A_2 = \begin{pmatrix} 0 & n_1 \\ n_1 & 0 \end{pmatrix}$. So, $P_2(\lambda) = \det\begin{pmatrix} -\lambda & n_1 \\ n_1 & -\lambda \end{pmatrix} = \lambda^2 - n_1^2$.
  Also, for $m = 2$, we have $k = 0$. Thus, $P_2(\lambda) = \lambda^2 + M_1$. Here, $M_1 = -n_1^2$. So, $P_2(\lambda) = \lambda^2 - n_1^2$.

Assume that the theorem holds for all integers up to $m-1$. We want to prove the result for m. From Theorem 3.2, we have

$$P_m(\lambda) = -\lambda P_{m-1}(\lambda) - n_{m-1}^2 P_{m-2}(\lambda).$$

We are going to split $k+1$ into two cases.
Case 1: $k+1$ **is even.** Let $k+1 = 2r$. So, $k = 2r-1$ is odd and $k-1 = 2(r-1)$ is even. By the inductive hypothesis:

$$P_k(\lambda) = -\lambda\left(\sum_{j=0}^{r-1} M_j^{(k)} \lambda^{k-1-2j}\right),$$
$$P_{k-1}(\lambda) = \sum_{j=0}^{r-1} M_j^{(k-1)} \lambda^{k-1-2j}.$$

Using the recurrence relation:

$$P_{k+1}(\lambda) = -\lambda\left[-\lambda\sum_{j=0}^{r-1} M_j^{(k)}\lambda^{k-1-2j}\right] - n_k^2\left[\sum_{j=0}^{r-1} M_j^{(k-1)}\lambda^{k-1-2j}\right]$$

$$= \lambda^2 \sum_{j=0}^{r-1} M_j^{(k)} \lambda^{k-1-2j} - n_k^2 \sum_{j=0}^{r-1} M_j^{(k-1)} \lambda^{k-1-2j}$$
$$= \sum_{j=0}^{r-1} M_j^{(k)} \lambda^{k+1-2j} - \sum_{j=0}^{r-1} n_k^2 M_j^{(k-1)} \lambda^{k-1-2j}.$$

Reindex the second sum with $l = j+1$ (so $j = l-1$), its general term becomes $n_k^2 M_{l-1}^{(k-1)} \lambda^{k-1-2(l-1)} = n_k^2 M_{l-1}^{(k-1)} \lambda^{k+1-2l}$.

$$P_{k+1}(\lambda) = M_0^{(k)} \lambda^{k+1} + \sum_{j=1}^{r-1} M_j^{(k)} \lambda^{k+1-2j}$$
$$- \left(\sum_{l=1}^{r-1} n_k^2 M_{l-1}^{(k-1)} \lambda^{k+1-2l} + n_k^2 M_{r-1}^{(k-1)} \lambda^{k+1-2r} \right)$$
$$= M_0^{(k)} \lambda^{k+1} + \sum_{j=1}^{r-1} \left(M_j^{(k)} - n_k^2 M_{j-1}^{(k-1)} \right) \lambda^{k+1-2j} - n_k^2 M_{r-1}^{(k-1)}.$$

We have $M_0^{(k)} = M_0^{(k+1)} = 1$, $-n_k^2 M_{j-1}^{(k-1)} = M_r^{(k+1)}$ and $M_j^{(k)} - n_k^2 M_{j-1}^{(k-1)} = M_j^{(k+1)}$.
So, we get

$$P_{k+1}(\lambda) = \sum_{j=0}^{r} M_j^{(k+1)} \lambda^{(k+1)-2j}.$$

Case 2: $k+1$ is odd. Let $k+1 = 2r+1$. Then, $k = 2r$ is even and $k-1 = 2r-1$ is odd.
By the inductive hypothesis:

$$P_k(\lambda) = \sum_{j=0}^{r} M_j^{(k)} \lambda^{k-2j},$$
$$P_{k-1}(\lambda) = -\lambda \left(\sum_{j=0}^{r-1} M_j^{(k-1)} \lambda^{k-2-2j} \right).$$

Using the recurrence relation:

$$P_{k+1}(\lambda) = -\lambda \left[\sum_{j=0}^{r} M_j^{(k)} \lambda^{k-2j} \right] - n_k^2 \left[-\lambda \sum_{j=0}^{r-1} M_j^{(k-1)} \lambda^{k-2-2j} \right]$$
$$= -\lambda \left[\sum_{j=0}^{r} M_j^{(k)} \lambda^{k-2j} - n_k^2 \sum_{j=0}^{r-1} M_j^{(k-1)} \lambda^{k-2-2j} \right].$$

Reindex the second sum with $l = j+1$:

$$P_{k+1}(\lambda) = -\lambda \left[\sum_{j=0}^{r} M_j^{(k)} \lambda^{k-2j} - \sum_{l=1}^{r} n_k^2 M_{l-1}^{(k-1)} \lambda^{k-2l} \right]$$
$$= -\lambda \left[M_0^{(k)} \lambda^k + \sum_{j=1}^{r} \left(M_j^{(k)} - n_k^2 M_{j-1}^{(k-1)} \right) \lambda^{k-2j} \right].$$

We have $M_0^{(k)} = M_0^{(k+1)} = 1$, and $M_j^{(k)} - n_k^2 M_{j-1}^{(k-1)} = M_j^{(k+1)}$. So, we get

$$P_{k+1}(\lambda) = -\lambda \left(\sum_{j=0}^{r} M_j^{(k+1)} \lambda^{k-2j} \right).$$

Hence, the proof is completed. ■

Theorem 3.5. *Let $G = P_{k+1}^{n_1,n_2,\cdots,n_k}$ be a non-uniform path semigraph with $N = \sum_{i=1}^{k} n_i + 1$ vertices and k edges with $n_i \geq 1$ for all $i = 1, \ldots, k$. The nullity of G, denoted $\eta(G)$, is given by*

$$\eta(G) = \begin{cases} \sum_{i=1}^{k}(n_i - 1), & \text{if } k \text{ is odd}, \\ 1 + \sum_{i=1}^{k}(n_i - 1), & \text{if } k \text{ is even}. \end{cases}$$

Proof. Let A_G be the adjacency matrix of the semigraph $G = P_{k+1}^{n_1,n_2,\cdots,n_k}$. Assume that $P(\lambda) = \det(A_G - \lambda I)$ is the characteristic polynomial of G. The number of pure middle vertices of G is $\sum_{i=1}^{k}(n_i - 1)$ in number. Delete these zero rows and corresponding columns in A_G we get the matrix

$$A_{sub} = \begin{pmatrix} 0 & n_1 & 0 & \cdots & 0 & 0 \\ n_1 & 0 & n_2 & \cdots & 0 & 0 \\ 0 & n_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n_k \\ 0 & 0 & 0 & \cdots & n_k & 0 \end{pmatrix}.$$

Thus, the total nullity of G is given by

$$\eta(G) = \eta(A_{sub}) + \sum_{i=1}^{k}(n_i - 1).$$

Claim:

$$\eta(A_{sub}) = \begin{cases} 0, & \text{if } k+1 \text{ is even}, \\ 1, & \text{if } k+1 \text{ is odd}. \end{cases}$$

Case 1: $k + 1$ is even.
From Theorem 3.4, we get the characteristic polynomial of A_{sub} as:

$$P_{k+1}(\lambda) = \sum_{j=0}^{r} M_j \lambda^{(k+1)-2j},$$

where $k + 1 = 2r$, the coefficients M_j for $P_{k+1}(\lambda)$ are defined by

$$M_j = \sum_{\substack{1 \leq i_1 < i_2 < \cdots < i_j \leq k \\ |i_p - i_q| \geq 2 \text{ for } p \neq q}} \left(\prod_{l=1}^{j} (-n_{i_l}^2) \right),$$

for $j = 1, 2, \cdots, r$ and $M_0 = 1$.
We are searching for a number of zero eigenvalues of this characteristic polynomial $P_{k+1}(\lambda)$ when $k + 1$ is even. Put $\lambda = 0$ in the characteristic polynomial, we get:

$$P_{k+1}(0) = M_k = (-n_1^2)(-n_2^2)(-n_3^2) \cdots (-n_k^2) \neq 0, \quad \text{since every } n_i > 0.$$

So, the nullity of A_{sub} is zero when $k+1$ is even.
Case 2: $k+1$ is odd.
From Theorem 3.4, we get the characteristic polynomial of A_{sub} as:

$$P_{k+1}(\lambda) = -\lambda \left(\sum_{j=0}^{r} M_j \lambda^{k-2j} \right),$$

where $k+1 = 2r+1$, the coefficients M_j for $P_{k+1}(\lambda)$ are defined by

$$M_j = \sum_{\substack{1\le i_1<i_2<\cdots<i_j\le k \\ |i_p-i_q|\ge 2 \text{ for } p\ne q}} \left(\prod_{l=1}^{j} (-n_{i_l}^2) \right),$$

for $j = 1, 2, \ldots, r$ and $M_0 = 1$. We are searching for a number of zero eigenvalues of the characteristic polynomial $P_{k+1}(\lambda)$ when $k+1$ is odd. Put $\lambda = 0$ in the characteristic polynomial, we get

$$P_{k+1}(0) = 0.$$

So, the nullity of A_{sub} is at least one. To find the exact nullity, take the derivative of $P_{k+1}(\lambda)$ with respect to λ. We get

$$P'_{k+1}(\lambda) = -\lambda \left(\sum_{j=0}^{r} M_j \lambda^{k-2j} \right)' - \left(\sum_{j=0}^{r} M_j \lambda^{k-2j} \right),$$

$$P'_{k+1}(0) = -M_r \neq 0, \quad \text{since } n_i > 0.$$

So, the nullity of A_{sub} is exactly one, when $k+1$ is odd. Hence, the claim is proven. Combining the above results we get

- If k is odd: $\eta(G) = 0 + \sum_{i=1}^{k}(n_i - 1) = \sum_{i=1}^{k}(n_i - 1)$.
- If k is even: $\eta(G) = 1 + \sum_{i=1}^{k}(n_i - 1)$.

This completes the proof. ■

Theorem 3.6. *Let $G = P_{k+1}^{n_1,n_2,\cdots,n_k}$ be a non-uniform path semigraph. Let A_{sub} be the $(k+1)\times(k+1)$ tridiagonal matrix*

$$A_{sub} = \begin{pmatrix} 0 & n_1 & 0 & \cdots & 0 & 0 \\ n_1 & 0 & n_2 & \cdots & 0 & 0 \\ 0 & n_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n_k \\ 0 & 0 & 0 & \cdots & n_k & 0 \end{pmatrix},$$

whose characteristic polynomial is $P_{k+1}(\lambda)$. The energy of the semigraph G is equal to the energy of the matrix A_{sub} and is given by

$$E(G) = \sum_{j=1}^{k+1} |\lambda_j|,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_{k+1}$ are the roots of the characteristic polynomial $P_{k+1}(\lambda)$.

Proof. The energy of a semigraph G is defined as $E(G) = \sum_{i=1}^{N} |\mu_i|$, where μ_i are the eigenvalues of the adjacency matrix A_G and $N = \sum_{i=1}^{k} n_i + 1$. The eigenvalues that are zero do not contribute anything to the sum of absolute values. Therefore, the energy of the semigraph G is determined entirely by the eigenvalues of the submatrix A_{sub}, obtained by deleting zero rows and corresponding columns of A_G. So

$$E(G) = E(A_{sub}) = \sum_{\lambda \in \mathrm{Spec}(A_{sub})} |\lambda|.$$

Let the eigenvalues of the $(k+1) \times (k+1)$ matrix A_{sub} be $\lambda_1, \lambda_2, \ldots, \lambda_{k+1}$. So, the energy of the semigraph is the sum of the absolute values of the roots of the polynomial $P_{k+1}(\lambda)$. Thus,

$$E(G) = \sum_{j=1}^{k+1} |\lambda_j|.$$

■

Corollary 3.7. *Let $G = P_{k+1}^{n_1, n_2, \cdots, n_k}$ be a non-uniform path semigraph. Let A_{sub} be the $(k+1) \times (k+1)$ tridiagonal matrix as stated in Theorem 3.6. Then, the energy of the semigraph G is equal to the energy of the matrix A_{sub} and is given by*

$$E(G) = 2 \sum_{j=1}^{\lfloor \frac{k+1}{2} \rfloor} |\lambda_j|,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_{k+1}$ are the roots of the characteristic polynomial $P_{k+1}(\lambda)$ in increasing order.

Proof. From Theorem 3.6, we have the energy of the semigraph G is equal to the energy of the matrix A_{sub}, and is given by

$$E(G) = \sum_{j=1}^{k+1} |\lambda_j|,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_{k+1}$ are the roots of the characteristic polynomial $P_{k+1}(\lambda)$. From Theorem 3.3, we get the eigenvalues of A_{sub} are symmetric about origin. To prove the result we split the order of matrix into two cases

- **Case** 1 : $k+1$ **is odd.** Since $k+1$ is odd, one eigenvalue of A_{sub} is always zero, which is $\lambda_{\frac{k}{2}+1}$. This zero eigenvalue does not contribute anything to the energy of the semigraph and the remaining eigenvalues are symmetric with respect to origin, we have $|\lambda_1| = \lambda_{\frac{k}{2}+2}, \cdots, |\lambda_{\frac{k}{2}}| = \lambda_{k+1}$. So, we get:

$$E(G) = 2 \sum_{j=1}^{\frac{k}{2}} |\lambda_j|.$$

- **Case** 2**:** $k+1$ **is even.** Since $k+1$ is even and all eigenvalues are symmetric about origin, we have $|\lambda_1| = \lambda_{\frac{k+1}{2}+1}, \cdots, |\lambda_{\frac{k+1}{2}}| = \lambda_{k+1}$. So, we get

$$E(G) = 2 \sum_{j=1}^{\frac{k+1}{2}} |\lambda_j|.$$

Combining these cases we get the required result. ■

3.2 Non-uniform cycle semigraphs, $C_k^{n_1,n_2,\cdots,n_k}$

Definition 3.8. (**Non-uniform cycle semigraphs**): Non-uniform cycle semigraph with $\sum_{i=1}^{k} n_i$ vertices, $v_{1,1}, v_{1,2}, \cdots, v_{1,n_1}, v_{2,1}, \cdots, v_{k,n_k}$ and k edges
$e_1 = (v_{1,1}, v_{1,2}, \cdots, v_{2,1}), e_2 = (v_{2,1}, v_{2,2}, \cdots, v_{3,1}) \cdots, e_k = (v_{k,1}, v_{k,2}, \cdots, v_{1,1})$, are denoted by $C_k^{n_1,n_2,\cdots,n_k}$, and is given in Figure 2.

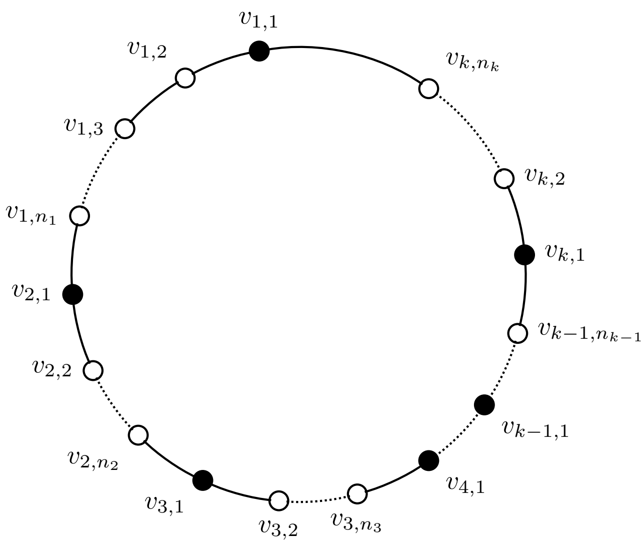

Figure 2: Non-uniform Cycle Semigraph, $C_k^{n_1,\ldots,n_k}$.

The adjacency matrix $A_{C_k^{n_1,n_2,\cdots,n_k}}$ of this semigraph is given by

$$
\begin{array}{c}
 \\ v_{11} \\ v_{12} \\ v_{13} \\ \vdots \\ v_{1n_1} \\ v_{21} \\ v_{22} \\ \vdots \\ v_{2n_k} \\ v_{31} \\ v_{32} \\ \vdots \\ v_{3n_k} \\ v_{41} \\ \vdots \\ v_{k1} \\ v_{k2} \\ v_{k3} \\ \vdots \\ v_{kn_k}
\end{array}
\begin{array}{c}
\begin{array}{ccccccccccccccccc}
v_{11} & v_{12} & \cdots & v_{1n_1} & v_{21} & v_{22} & \cdots & v_{2n_2} & v_{31} & \cdots & v_{3n_3} & \cdots & v_{k1} & v_{k2} & \cdots & v_{k(n_k-1)} & v_{kn_k}
\end{array} \\
\begin{pmatrix}
0 & 1 & \cdots & n_1-1 & n_1 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & n_k & n_k-1 & \cdots & 2 & 1 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
n_1 & n_1-1 & \cdots & 1 & 0 & 1 & \cdots & n_2-1 & n_2 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & n_2 & n_2-1 & \cdots & 1 & 0 & \cdots & n_3-1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0\cdots & & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
n_k & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & n_k-2 & n_k-1 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}
\end{array}.
$$

Theorem 3.9. *Let A be the $(k+1)\times(k+1)$ real symmetric matrix given by*

$$A = \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_k \\ v_{k+1} \end{array} \overset{\begin{array}{cccccc} v_1 & v_2 & v_3 & \cdots & v_k & v_{k+1} \end{array}}{\begin{pmatrix} 0 & n_1 & 0 & \cdots & 0 & n_{k+1} \\ n_1 & 0 & n_2 & \cdots & 0 & 0 \\ 0 & n_2 & 0 & \vdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n_k \\ n_{k+1} & 0 & 0 & \cdots & n_k & 0 \end{pmatrix}},$$

where all n_i's are non negative integers. Let $C_{k+1}(\lambda) = \det(A - \lambda I_{k+1})$ be the characteristic polynomial of A. Let P be the symmetric tridiagonal matrix obtained from A with $n_{k+1} = 0$ ie,

$$P = \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_k \\ v_{k+1} \end{array} \overset{\begin{array}{cccccc} v_1 & v_2 & v_3 & \cdots & v_k & v_{k+1} \end{array}}{\begin{pmatrix} 0 & n_1 & 0 & \cdots & 0 & 0 \\ n_1 & 0 & n_2 & \cdots & 0 & 0 \\ 0 & n_2 & 0 & \vdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n_k \\ 0 & 0 & 0 & \cdots & n_k & 0 \end{pmatrix}}.$$

Then the characteristic polynomial $C_{k+1}(\lambda)$, with $m = k+1$ is given by

$$C_m(\lambda) = P_m(\lambda) - n_m^2 P_{m-2}(\lambda) + 2(-1)^{m+1} \prod_{i=1}^{m} n_i,$$

where, $P_m(\lambda)$ is the $m \times m$ principal submatrix of P satisfies the recurrence relation $P_m(\lambda) = -\lambda P_{m-1}(\lambda) - n_{m-1}^2 P_{m-2}(\lambda)$ with initial conditions $P_0(\lambda) = 1$ and $P_1(\lambda) = -\lambda$.

Proof. Consider the characteristic polynomial $C_m(\lambda) = \det(A - \lambda I_m)$.

$$A - \lambda I_m = \begin{pmatrix} -\lambda & n_1 & 0 & \cdots & 0 & n_m \\ n_1 & -\lambda & n_2 & \cdots & 0 & 0 \\ 0 & n_2 & -\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda & n_{m-1} \\ n_m & 0 & 0 & \cdots & n_{m-1} & -\lambda \end{pmatrix}.$$

Find cofactor expansion along the last row of the matrix $A - \lambda I_m$.

$$\begin{aligned} C_m(\lambda) =& (-1)^{m+1} n_m \cdot \det(M_{m,1}) + (-1)^{m+(m-1)} n_{m-1} \cdot \det(M_{m,m-1}) \\ &+ (-1)^{m+m}(-\lambda) \cdot \det(M_{m,m}), \end{aligned}$$

where $M_{i,j}$ is the matrix obtained by deleting row i and column j. Consider the determinant of each minor.

First, consider the minor $M_{m,m}$. We get

$$M_{m,m} = \begin{pmatrix} -\lambda & n_1 & 0 & \cdots & 0 & 0 \\ n_1 & -\lambda & n_2 & \cdots & 0 & 0 \\ 0 & n_2 & -\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & n_{m-2} & -\lambda \end{pmatrix}_{(m-1)\times(m-1)},$$

and

$$\det(M_{m,m}) = P_{m-1}(\lambda).$$

Then, consider the minor $M_{m,1}$,

$$M_{m,1} = \begin{pmatrix} n_1 & 0 & 0 & \cdots & 0 & n_m \\ -\lambda & n_2 & 0 & \cdots & 0 & 0 \\ n_2 & -\lambda & n_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda & n_{m-1} \end{pmatrix}.$$

Expand its determinant along the last column.

$$\det(M_{m,1}) = (-1)^{1+(m-1)} n_m \det \begin{pmatrix} -\lambda & n_2 & 0 & \cdots & 0 & 0 \\ n_2 & -\lambda & n_3 & \cdots & 0 & 0 \\ 0 & n_3 & -\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda & n_{m-2} \\ 0 & 0 & 0 & \cdots & n_{m-2} & -\lambda \end{pmatrix}$$

$$+ (-1)^{(m-1)+(m-1)} n_{m-1} \det \begin{pmatrix} n_1 & 0 & 0 & \cdots & 0 & 0 \\ -\lambda & n_2 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda & n_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n_{m-3} & 0 \\ 0 & 0 & 0 & \cdots & -\lambda & n_{m-2} \end{pmatrix}.$$

We get:

$$\begin{aligned} \det(M_{m,1}) &= (-1)^m n_m P_{m-2}(\lambda) + (n_1 n_2 \cdots n_{m-2}) n_{m-1} \\ &= (-1)^m n_m P_{m-2}(\lambda) + \prod_{i=1}^{m-1} n_i. \end{aligned}$$

Consider the minor $M_{m,m-1}$.

$$M_{m,m-1} = \begin{pmatrix} -\lambda & n_1 & 0 & \cdots & 0 & n_m \\ n_1 & -\lambda & n_2 & \cdots & 0 & 0 \\ 0 & n_2 & -\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda & 0 \\ 0 & 0 & 0 & \cdots & n_{m-2} & n_{m-1} \end{pmatrix}_{(m-1)\times(m-1)}.$$

Expand along the last column we get:

$$\det(M_{m,m-1}) = (-1)^{1+(m-1)} n_m \det \begin{pmatrix} n_1 & -\lambda & n_2 & 0 & \cdots & 0 \\ 0 & n_2 & -\lambda & n_3 & \cdots & 0 \\ 0 & 0 & n_3 & -\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\lambda \\ 0 & 0 & 0 & 0 & \cdots & n_{m-2} \end{pmatrix}$$

$$+ (-1)^{(m-1)+(m-1)} n_{m-1} \det \begin{pmatrix} -\lambda & n_1 & 0 & \cdots & 0 & 0 \\ n_1 & -\lambda & n_2 & \cdots & 0 & 0 \\ 0 & n_2 & -\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda & n_{m-3} \\ 0 & 0 & 0 & \cdots & n_{m-3} & -\lambda \end{pmatrix}.$$

Finally, we obtain:

$$\det(M_{m,m-1}) = (-1)^m n_m (n_1 n_2 \cdots n_{m-2}) + n_{m-1} P_{m-2}(\lambda).$$

Substitute these determinants back into the expansion for $C_m(\lambda)$. We get

$$C_m(\lambda) = (-1)^{m+1} n_m \left[(-1)^m n_m P_{m-2}(\lambda) + \prod_{i=1}^{m-1} n_i \right] - n_{m-1} \left[(-1)^m n_m \prod_{i=1}^{m-2} n_i + n_{m-1} P_{m-2}(\lambda) \right] - \lambda P_{m-1}(\lambda).$$

Arrange the terms

$$\begin{aligned} C_m(\lambda) &= (-1)^{2m+1} n_m^2 P_{m-2}(\lambda) + (-1)^{m+1} n_m \prod_{i=1}^{m-1} n_i \\ &\quad - n_{m-1}(-1)^m n_m \prod_{i=1}^{m-2} n_i - n_{m-1}^2 P_{m-2}(\lambda) \\ &\quad - \lambda P_{m-1}(\lambda) \\ &= \underbrace{\left(-\lambda P_{m-1}(\lambda) - n_{m-1}^2 P_{m-2}(\lambda)\right)}_{\text{This is } P_m(\lambda) \text{ by the recurrence}} - n_m^2 P_{m-2}(\lambda) \\ &\quad + (-1)^{m+1} \prod_{i=1}^{m} n_i - (-1)^m \prod_{i=1}^{m} n_i \\ &= P_m(\lambda) - n_m^2 P_{m-2}(\lambda) + (-1)^{m+1} \prod_{i=1}^{m} n_i + (-1)^{m+1} \prod_{i=1}^{m} n_i \\ &= P_m(\lambda) - n_m^2 P_{m-2}(\lambda) + 2(-1)^{m+1} \prod_{i=1}^{m} n_i. \end{aligned}$$

This completes the proof. ■

Theorem 3.10. *Let A be the $m \times m$ real symmetric matrix defined by*

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_k \\ v_{k+1} \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & \cdots & v_k & v_{k+1} \end{array} \\ \begin{pmatrix} 0 & n_1 & 0 & \cdots & 0 & n_{k+1} \\ n_1 & 0 & n_2 & \cdots & 0 & 0 \\ 0 & n_2 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & n_{k-1} & 0 \\ 0 & 0 & 0 & n_{k-1} & 0 & n_k \\ n_{k+1} & 0 & 0 & \cdots & n_k & 0 \end{pmatrix} \end{array},$$

where $m = k + 1$. Then all eigenvalues of A are real. Furthermore, the spectrum is symmetric with respect to the origin if and only if the order of the matrix is even.

Proof. Since A is a real symmetric matrix, all its eigenvalues are real by the Spectral Theorem. To prove the spectral symmetry property, we will show that $C_m(\lambda)$ is an even function if and only if m is even. This implies that if λ_j is a root, then $C_m(-\lambda_j) = C_m(\lambda_j) = 0$, so $-\lambda_j$ is also a root.
From Theorem 3.9, we have

$$C_m(\lambda) = P_m(\lambda) - n_m^2 P_{m-2}(\lambda) + 2(-1)^{m+1} \prod_{i=1}^{m} n_i.$$

Also, from Theorem 3.3, we have

- $P_m(\lambda)$ is an even function if m is even $(P_m(-\lambda) = P_m(\lambda))$.
- $P_m(\lambda)$ is an odd function if m is odd $(P_m(-\lambda) = -P_m(\lambda))$.

We are going to divide m into two cases, even and odd.
Case 1: m is even.
Let $m = 2r$ for some integer $r \geq 1$. Then, m is even and $m-2$ is even. Therefore, $P_m(\lambda)$ and $P_{m-2}(\lambda)$ are both even functions.

$$\begin{aligned} C_m(-\lambda) &= P_m(-\lambda) - n_m^2 P_{m-2}(-\lambda) + 2(-1)^{m+1}\prod_{i=1}^{m} n_i \\ &= P_m(\lambda) - n_m^2 P_{m-2}(\lambda) + 2(-1)^{2r+1}\prod_{i=1}^{m} n_i \\ &= P_m(\lambda) - n_m^2 P_{m-2}(\lambda) - 2\prod_{i=1}^{m} n_i. \end{aligned}$$

when $m = 2r$, $C_m(\lambda)$ is

$$C_m(\lambda) = P_m(\lambda) - n_m^2 P_{m-2}(\lambda) + 2(-1)^{2r+1}\prod_{i=1}^{m} n_i = P_m(\lambda) - n_m^2 P_{m-2}(\lambda) - 2\prod_{i=1}^{m} n_i.$$

Since $C_m(-\lambda) = C_m(\lambda)$, the characteristic polynomial is an even function when m is even. Thus, the spectrum is symmetric with respect to the origin.
Case 2: m is odd.
Let $m = 2r+1$ for some integer $r \geq 1$. Then, m is odd and $m-2$ is odd. Therefore, $P_m(\lambda)$ and $P_{m-2}(\lambda)$ are both odd functions.

$$\begin{aligned} C_m(-\lambda) &= P_m(-\lambda) - n_m^2 P_{m-2}(-\lambda) + 2(-1)^{m+1}\prod_{i=1}^{m} n_i \\ &= (-P_m(\lambda)) - n_m^2(-P_{m-2}(\lambda)) + 2(-1)^{2r+2}\prod_{i=1}^{m} n_i \\ &= -P_m(\lambda) + n_m^2 P_{m-2}(\lambda) + 2\prod_{i=1}^{m} n_i \\ &= -\left(P_m(\lambda) - n_m^2 P_{m-2}(\lambda)\right) + 2\prod_{i=1}^{m} n_i. \end{aligned}$$

The original expression for $C_m(\lambda)$, when $m = 2r+1$, is

$$C_m(\lambda) = P_m(\lambda) - n_m^2 P_{m-2}(\lambda) + 2(-1)^{2r+2}\prod_{i=1}^{m} n_i = P_m(\lambda) - n_m^2 P_{m-2}(\lambda) + 2\prod_{i=1}^{m} n_i.$$

Since $C_m(-\lambda) \neq C_m(\lambda)$ and $C_m(-\lambda) \neq -C_m(\lambda)$ the characteristic polynomial is neither an even nor an odd function. Therefore, its roots are not symmetric with respect to origin. We conclude that the spectrum is symmetric if and only if m is even. ■

Theorem 3.11. *Let A be the $m \times m$ matrix defined by*

$$A = \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{m-1} \\ v_m \end{array} \overset{\begin{array}{cccccc} v_1 & v_2 & v_3 & \cdots & v_{m-1} & v_m \end{array}}{\begin{pmatrix} 0 & n_1 & 0 & \cdots & 0 & n_m \\ n_1 & 0 & n_2 & \cdots & 0 & 0 \\ 0 & n_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n_{m-1} \\ n_m & 0 & 0 & \cdots & n_{m-1} & 0 \end{pmatrix}},$$

The characteristic polynomial $C_m(\lambda) = \det(A - \lambda I)$ is defined as: ***If** $m = 2k$ **is even**, then $C_m(\lambda)$ is $C_m(\lambda) = \lambda^m + M_1\lambda^{m-2} + M_2\lambda^{m-4} + \cdots + M_k = \sum_{j=0}^{k} M_j\lambda^{m-2j}$.* ***If** $m = 2k+1$ **is odd**, then $C_m(\lambda)$ is $C_m(\lambda) = -\lambda\left(\lambda^{m-1} + M_1\lambda^{m-3} + \cdots + M_k\right) + 2\prod_{i=1}^{m} n_i = -\lambda\left(\sum_{j=0}^{k} M_j\lambda^{m-1-2j}\right) + 2\prod_{i=1}^{m} n_i$, where the coefficients M_j are defined by*

$$M_j = \sum_{\substack{1\le i_1<i_2<\cdots<i_j\le m \\ |i_r-i_s|\ge 2 \text{ for } r\neq s}} \left(\prod_{l=1}^{j}(-n_{i_l}^2)\right),$$

where $j = 1, 2, \cdots, m-1$ and $M_0 = 1$.

Proof. From Theorem 3.9, we have

$$C_m(\lambda) = P_m(\lambda) - n_m^2 P_{m-2}(\lambda) + 2(-1)^{m+1}\prod_{i=1}^{m} n_i.$$

Let $M_j^{(P,k)}$ be the coefficients of the polynomial $P_k(\lambda)$. We are going to divide m in to two cases, even and odd.

Case 1: m is even.

Let $m = 2k$. From the Theorem 3.2, we have

$$P_{2k}(\lambda) = \sum_{j=0}^{k} M_j^{(P,2k)}\lambda^{2k-2j},$$
$$P_{2k-2}(\lambda) = \sum_{j=0}^{k-1} M_j^{(P,2k-2)}\lambda^{2k-2-2j}.$$

Substituting these into the recurrence relation for $C_{2k}(\lambda)$ we get

$$\begin{aligned} C_{2k}(\lambda) &= \sum_{j=0}^{k} M_j^{(P,2k)}\lambda^{2k-2j} - n_{2k}^2\sum_{j=0}^{k-1} M_j^{(P,2k-2)}\lambda^{2k-2-2j} + 2(-1)^{2k+1}\prod_{i=1}^{2k} n_i \\ &= \sum_{j=0}^{k} M_j^{(P,2k)}\lambda^{2k-2j} - n_{2k}^2\sum_{j=0}^{k-1} M_j^{(P,2k-2)}\lambda^{2k-2(j+1)} - 2\prod_{i=1}^{2k} n_i. \end{aligned}$$

Reindex the second sum by setting $l = j+1$:

$$C_{2k}(\lambda) = \sum_{j=0}^{k} M_j^{(P,2k)}\lambda^{2k-2j} - n_{2k}^2\sum_{l=1}^{k} M_{l-1}^{(P,2k-2)}\lambda^{2k-2l} - 2\prod_{i=1}^{2k} n_i.$$

Arrange, it we get:

$$C_{2k}(\lambda) = M_0^{(P,2k)}\lambda^{2k} + \sum_{j=1}^{k}\left(M_j^{(P,2k)} - n_{2k}^2 M_{j-1}^{(P,2k-2)}\right)\lambda^{2k-2j} - 2\prod_{i=1}^{2k} n_i.$$

This is an even polynomial in λ with

$$M_j^{(C,2k)} = M_j^{(P,2k)} - n_{2k}^2 M_{j-1}^{(P,2k-2)} \text{ for } 1 \leq j \leq k-1,$$

and the constant term as:

$$M_k^{(C,2k)} = M_k^{(P,2k)} - n_{2k}^2 M_{k-1}^{(P,2k-2)} - 2\prod_{i=1}^{2k} n_i.$$

So, the result is true for even m.
Case 2: m is odd.
Let $m = 2k+1$. From the Theorem 3.2, we have:

$$P_{2k+1}(\lambda) = -\lambda \sum_{j=0}^{k} M_j^{(P,2k+1)}\lambda^{2k-2j},$$
$$P_{2k-1}(\lambda) = -\lambda \sum_{j=0}^{k-1} M_j^{(P,2k-1)}\lambda^{2k-2-2j}.$$

Substituting these into the recurrence for $C_{2k+1}(\lambda)$:

$$\begin{aligned}C_{2k+1}(\lambda) &= \left(-\lambda \sum_{j=0}^{k} M_j^{(P,2k+1)}\lambda^{2k-2j}\right) - n_{2k+1}^2\left(-\lambda \sum_{j=0}^{k-1} M_j^{(P,2k-1)}\lambda^{2k-2-2j}\right)\\ &+ 2(-1)^{2k+2}\prod_{i=1}^{2k+1} n_i\\ &= -\lambda\left[\sum_{j=0}^{k} M_j^{(P,2k+1)}\lambda^{2k-2j} - n_{2k+1}^2 \sum_{j=0}^{k-1} M_j^{(P,2k-1)}\lambda^{2k-2(j+1)}\right] + 2\prod_{i=1}^{2k+1} n_i.\end{aligned}$$

Reindex the second sum with $l = j+1$:

$$C_{2k+1}(\lambda) = -\lambda\left[\sum_{j=0}^{k} M_j^{(P,2k+1)}\lambda^{2k-2j} - n_{2k+1}^2 \sum_{l=1}^{k} M_{l-1}^{(P,2k-1)}\lambda^{2k-2l}\right] + 2\prod_{i=1}^{2k+1} n_i.$$

Combining the sums we get:

$$C_{2k+1}(\lambda) = -\lambda\left[M_0^{(P,2k+1)}\lambda^{2k} + \sum_{j=1}^{k}\left(M_j^{(P,2k+1)} - n_{2k+1}^2 M_{j-1}^{(P,2k-1)}\right)\lambda^{2k-2j}\right] + 2\prod_{i=1}^{2k+1} n_i.$$

We have $M_j^{(C,2k+1)} = \left(M_j^{(P,2k+1)} - n_{2k+1}^2 M_{j-1}^{(P,2k-1)}\right)$. This expression has the exact form required by the theorem. This completes the proof. ■

Theorem 3.12. *Let $G = C_k^{n_1,n_2,\cdots,n_k}$ be a non-uniform cycle semigraph with $N = \sum_{i=1}^{k} n_i$ vertices and k edges with $n_i \geq 1$ for all $i = 1, \ldots, k$. The nullity of G, denoted $\eta(G)$, is given by*

$$\eta(G) = \begin{cases} 2 + \sum_{i=1}^{k}(n_i - 1), & \text{if } k \equiv 0 \pmod 4 \text{ and} \\ & \left(\prod_{j=1}^{k/2-1} n_{2j-1}^2\right)(n_{k-1}^2 + n_k^2) = 2\prod_{i=1}^{k} n_i, \\ \sum_{i=1}^{k}(n_i - 1), & \text{otherwise.} \end{cases}$$

Proof. Let A_G be the adjacency matrix of the semigraph $G = C_k^{n_1,n_2,\cdots,n_k}$. $\sum_{i=1}^{k}(n_i - 1)$ zero rows are present in A_G. Deleting these zero rows and their corresponding columns yields the $k \times k$ submatrix A_{sub},

$$A_{sub} = \begin{array}{c} \\ v_{1,1} \\ v_{2,1} \\ v_{3,1} \\ \vdots \\ v_{k,1} \end{array} \begin{array}{c} \begin{array}{ccccc} v_{1,1} & v_{2,1} & v_{3,1} & \cdots & v_{k,1} \end{array} \\ \begin{pmatrix} 0 & n_1 & 0 & \cdots & n_k \\ n_1 & 0 & n_2 & \cdots & 0 \\ 0 & n_2 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & n_{k-1} \\ n_k & 0 & 0 & \cdots & 0 \end{pmatrix}_{k\times k} \end{array}.$$

So, the nullity of G is

$$\eta(G) = \eta(A_{sub}) + \sum_{i=1}^{k}(n_i - 1).$$

From Theorem 3.9, the characteristic polynomial of the $k \times k$ matrix A_{sub} is given by

$$C_k(\lambda) = P_k(\lambda) - n_k^2 P_{k-2}(\lambda) + 2(-1)^{k+1}\prod_{i=1}^{k} n_i.$$

where, $P_m(\lambda)$ satisfies the recurrence relation $P_m(\lambda) = -\lambda P_{m-1}(\lambda) - n_{m-1}^2 P_{m-2}(\lambda)$ with $P_0(\lambda) = 1$ and $P_1(\lambda) = -\lambda$. Now, we are going to evaluate $C_k(0)$. Divide k in to two cases even and odd.

Case 1: k is odd. Let $k = 2r + 1$. Then $k - 2 = 2r - 1$ is also odd. From Theorem 3.4, $P_k(0) = 0$ and $P_{k-2}(0) = 0$.

$$C_k(0) = 0 - n_k^2(0) + 2(-1)^{k+1}\prod_{i=1}^{k} n_i = 2(-1)^{2r+2}\prod_{i=1}^{k} n_i = 2\prod_{i=1}^{k} n_i.$$

Since $n_i \geq 1$ for all i, $C_k(0) \neq 0$. Therefore, the nullity of A_{sub} is 0.

Case 2: k is even. Let $k = 2r$. Then, $k - 2 = 2r - 2$ is also even. From Theorem 3.4, we have $P_{2r}(0) = (-1)^r \prod_{j=1}^{r} n_{2j-1}^2$ and $P_{2r-2}(0) = (-1)^{r-1}\prod_{j=1}^{r-1} n_{2j-1}^2$.

$$\begin{aligned} C_k(0) &= P_k(0) - n_k^2 P_{k-2}(0) + 2(-1)^{k+1}\prod_{i=1}^{k} n_i \\ &= (-1)^r \prod_{j=1}^{r} n_{2j-1}^2 - n_{2r}^2\left((-1)^{r-1}\prod_{j=1}^{r-1} n_{2j-1}^2\right) + 2(-1)^{2r+1}\prod_{i=1}^{2r} n_i \end{aligned}$$

$$= (-1)^r \left(\prod_{j=1}^{r} n_{2j-1}^2 + n_{2r}^2 \prod_{j=1}^{r-1} n_{2j-1}^2 \right) - 2 \prod_{i=1}^{2r} n_i$$
$$= (-1)^r \left(\prod_{j=1}^{r-1} n_{2j-1}^2 \right) (n_{2r-1}^2 + n_{2r}^2) - 2 \prod_{i=1}^{2r} n_i.$$

If r is odd, then $C_k(0) \neq 0$. Thus, $\eta(A_{sub}) = 0$ when r is odd, i.e., if $k \equiv 2 \mod 4$, then $\eta(A_{sub}) = 0$.
If r is even then $C_k(0)$ can be zero if $\left(\prod_{j=1}^{r-1} n_{2j-1}^2\right)(n_{2r-1}^2 + n_{2r}^2) = 2\prod_{i=1}^{2r} n_i$. i.e., if $k \equiv 0 \mod 4$ and $\left(\prod_{j=1}^{k/2-1} n_{2j-1}^2\right)(n_{k-1}^2 + n_k^2) = 2\prod_{i=1}^{k} n_i$ then $\eta(A_{sub}) \geq 1$. We need to find exact nullity of A_{sub}, so consider $C_k'(\lambda)$ when $\lambda = 0$. $C_k'(0) = 0$ because $C_k'(\lambda)$ is an odd function. So $\eta(A_{sub}) \geq 2$.
To show the nullity is exactly 2, we must show that $C_k''(0) \neq 0$.

$$C_k''(\lambda) = P_k''(\lambda) - n_k^2 P_{k-2}''(\lambda) \implies C_k''(0) = P_k''(0) - n_k^2 P_{k-2}''(0).$$

Since k and r are even either we get $P_k''(0) > 0$ or $P_k''(0) < 0$. If $P_k''(0) < 0$ then $P_{k-2}''(0) > 0$ and $P_k''(0) > 0$ then $P_{k-2}''(0) < 0$. So, $C_k''(0) \neq 0$ in both cases. Since $C_k''(0) \neq 0$, the multiplicity of the eigenvalue $\lambda = 0$ is exactly 2. Thus, $\eta(A_{\text{sub}}) = 2$.
Combining the cases we get

- If k is odd or $k \equiv 2 \pmod 4$, then $\eta(A_{sub}) = 0$.
- If $k \equiv 0 \pmod 4$ and $\left(\prod_{j=1}^{k/2-1} n_{2j-1}^2\right)(n_{k-1}^2 + n_k^2) = 2\prod_{i=1}^{k} n_i$ then we get $\eta(A_{sub}) = 2$.

■

Theorem 3.13. *Let $G = C_k^{n_1,n_2,\cdots,n_k}$ be a non-uniform cycle semigraph. Let A_{sub} be the $k \times k$ matrix defined by*

$$A_{sub} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_k \\ v_{k+1} \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & \cdots & v_k & v_{k+1} \end{array} \\ \begin{pmatrix} 0 & n_1 & 0 & \cdots & 0 & n_k \\ n_1 & 0 & n_2 & \cdots & 0 & 0 \\ 0 & n_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n_{k-1} \\ n_k & 0 & 0 & \cdots & n_{k-1} & 0 \end{pmatrix} \end{array}.$$

Then, the energy of the semigraph G is equal to the energy of the matrix A_{sub}.

$$E(G) = E(A_{sub}) = \sum_{j=1}^{k} |\lambda_j|,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the eigenvalues of A_{sub}, which are the roots of the characteristic polynomial $C_k(\lambda)$.

Proof. The energy of a semigraph G with N vertices is defined as $E(G) = \sum_{i=1}^{N} |\mu_i|$, where μ_i are the eigenvalues of the adjacency matrix A_G and $N = \sum_{i=1}^{k} n_i$. The pure middle vertices contribute $\sum_{i=1}^{k}(n_i - 1)$ zero eigenvalues to the spectrum of A_G. Since zero eigenvalues do not contribute anything to the sum of absolute values, the energy of G is determined entirely by the non-zero eigenvalues, which are the non-zero eigenvalues of A_{sub}. So,

$$E(G) = \sum_{\mu \in \mathrm{Spec}(A_G)} |\mu| = \sum_{\lambda \in \mathrm{Spec}(A_{sub})} |\lambda| = E(A_{sub}).$$

If $\lambda_1, \ldots, \lambda_k$ are the eigenvalues of A_{sub} then the energy is

$$E(G) = \sum_{j=1}^{k} |\lambda_j|.$$

■

Corollary 3.14. *Let $G = C_k^{n_1,n_2,\cdots,n_k}$ be a non-uniform cycle semigraph. Let A_{sub} be the $k \times k$ matrix with k even, defined in Theorem 3.13. Then, the energy of the semigraph G is equal to the energy of the matrix A_{sub}.*

$$E(G) = E(A_{sub}) = 2\sum_{j=1}^{\frac{k}{2}} |\lambda_j|,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_{\frac{k}{2}}, \cdots, \lambda_k$ are the eigenvalues of A_{sub} in the increasing order, which are the roots of the characteristic polynomial $C_k(\lambda)$.

Proof. From Theorem 3.13, the energy of the semigraph G is equal to the energy of the matrix A_{sub}.

$$E(G) = E(A_{sub}) = \sum_{j=1}^{k} |\lambda_j|,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the eigenvalues of A_{sub}, which are the roots of the characteristic polynomial $C_k(\lambda)$. Since the eigenvalues of matrix A_{sub} is symmetric with respect to origin from Theorem 3.10 we have if λ is an eigenvalue of A_{sub} then $-\lambda$ is also an eigenvalue. So, we get

$$E(G) = E(A_{sub}) = 2\sum_{j=1}^{\frac{k}{2}} |\lambda_j|.$$

■

4 Conclusion

In this paper, we have conducted a detailed spectral analysis of non-uniform Path and Cycle Semigraphs. We successfully derived formulas for their nullity and energy. During that process we used the core matrix to derive the results concisely. Our work demonstrates that the spectral properties of usual graphs can be extended to these more generalized graph structures called semigraphs.

The key findings for the Non-Uniform Path Semigraph, $P_{k+1}^{n_1,\ldots,n_k}$, revealed that its spectrum is always symmetric about the origin. This fundamental property led to a concise formula for its nullity. For the Non-Uniform Cycle Semigraph, $C_k^{n_1,\ldots,n_k}$, the analysis proved to be more complex. We found that its spectrum is symmetric only when the number of segments is even. The nullity is typically equal to the number of pure middle vertices, $\sum(n_i - 1)$. However, a particularly interesting phenomenon occurs when the number of segments is a multiple of four; in this case, the nullity increases by two, provided a specific multiplicative condition on $\{n_i\}$ is satisfied.

This research opens several avenues for future investigation. One natural direction is to extend this analysis to other families of semigraphs. Another promising area is the study of other spectral invariants, like the Laplacian spectrum, Laplacian energy etc. Exploring applications of these non-uniform models in physics, chemistry, or computer science could reveal new connections between their spectral properties and real-world phenomena.

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References

[1] I. Gutman, *The energy of a graph*, *Ber. Math.- Statist. Sekt. Forschungsz. Graz* **103** (1978) 1–22.

[2] I. Gutman and B. Furtula, Survey of graph energies, *Math. Interdisc. Res.* **2** (2017) 85–129, https://doi.org/10.22052/MIR.2017.81507.1057.

[3] X. Li, Y. Shi and I. Gutman, *Graph Energy*, Springer, New York, 2012.

[4] D. Cvetković, P. Rowlinson and S. Simić, *An Introduction to the Theory of Graph Spectra*, Cambridge University Press, 2010.

[5] E. Sampathkumar, C. M. Deshpande, B. Y. Bam, L. Pushpalatha and V. Swaminathan, *Semigraphs and Their Applications*, India: Academy of Discrete Mathematics and Applications India, 2019.

[6] A. K. Nandi, I. Gutman and S. K. Nath, On color matrix and energy of semigraphs, *Iranian J. Math. Chem.* **13** (2022) 291-299, https://doi.org/10.22052/IJMC.2022.246186.1616.

[7] G. H. Shirdel, A. Mortezaee and L. Alaameri, Randić matrix and Randić energy of uniform Hypergraphs, *Math. Interdisc. Res.* **9** (2024) 269–288, https://doi.org/10.22052/MIR.2024.254116.1456.

[8] C. Krattenthaler, Advanced determinant calculus: a complement, *Linear Algebra Appl.* **411** (2005) 68–166.

[9] I. Gutman and B. Zhou, Laplacian energy of a graph, *Linear Algebra Appl.* **414** (2006) 29–37.

[10] C. M. Deshpande and Y. S. Gaidhani, About adjacency matrix of semigraphs, *Int. J. Appl. Phys. Math.* **2** (2012) 250–252, https://doi.org/10.7763/IJAPM.2012.V2.103.

[11] S. Arumugam, K. Arathi Bhat, I. Gutman, M. P. Karantha and R. Poojary, *Nullity of Graphs-A Survey and Some New Results*, Applied Linear Algebra, Probability and Statistics. Indian Statistical Institute Series. Springer, Singapore, 2023, https://doi.org/10.1007/978-981-99-2310-6_8.

[12] Y. S. Gaidhani, C. M. Deshpande and S. Pirzada, Energy of a semigraph, *AKCE Int. J. Graphs Comb.* **16** (2019) 41–49, https://doi.org/10.1016/j.akcej.2018.06.006.

[13] R. B. Bapat, *Graphs and Matrices*, Springer, London, 2014, https://doi.org/10.1007/978-1-4471-6569-9.

[14] N. D. Cahill, J. R. DErrico and J. P. Spence, Complex factorization of the Fibonacci and Lucas numbers, *Fibonacci Quart.* **41** (2003) 13–19.