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Research Paper

On Closed Derivation Formulas over the M-polynomial for the Elliptic-Type Indices of a Graph

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Abstract

A polynomial function that provides information about the molecular structure of a graph is known as the M-polynomial of a graph. This polynomial helps us to understand the characteristics of chemical compounds and their relationships. Very recently, in 2024, the elliptic Sombor (ESO), reduced elliptic (RE) and modified reduced elliptic (${}^{m}RE$) indices of a graph were proposed and their values were calculated for some standard graphs, jagged-rectangle benzenoid systems and polycyclic aromatic hydrocarbons. In this work, we establish closed derivation formulas for the above-mentioned elliptic-type indices of a graph based on its M-polynomial. Moreover, we enumerate the elliptic-type indices of the above family of chemical graphs.

1 Introduction

Chemical graph theory is a branch of mathematical chemistry that applies graph theory concepts to investigate the properties, behaviors, and structures of chemical compounds. Topological indices are numerical values obtained using mathematical invariants from the graphical illustration of the molecular structure. They have an essential importance in quantitative structure-property relationships ($QSPR$) and quantitative structure-activity relationships ($QSAR$) analysis [1]. Let G be a simple and connected graph having $V(G)$ and $E(G)$ as the set of vertices and the set of edges of G, respectively. The *degree* of a vertex $v \in V(G)$ is represented by $d(v)$, which counts the first neighbors of the vertex v [1].

In 2024, Gutman et al. [2] introduced the elliptic Sombor index of a graph G, which is defined as:

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$$ESO(G) = \sum_{uv \in E(G)} (d(u) + d(v))\sqrt{d^2(u) + d^2(v)}.$$

Motivated by the above index, the reduced elliptic (RE) index and the modified reduced elliptic (^mRE) index of a graph G are proposed by V.R. Kulli [3] in 2024 and are defined as

$$RE(G) = \sum_{uv \in E(G)} \{d(u) - 1 + d(v) - 1\}\sqrt{(d(u) - 1)^2 + (d(v) - 1)^2},$$

and

$$^mRE(G) = \sum_{uv \in E(G)} \frac{1}{\{d(u) - 1 + d(v) - 1\}\sqrt{(d(u) - 1)^2 + (d(v) - 1)^2}}.$$

Topological indices are often computed independently using the corresponding mathematical formulas. The polynomial approach has drawn interest in estimating various classes of topological indices instead of computing each separately [4–8]. One can employ the single algebraic equation of the Hosoya polynomial [7] to compute the distance-based topological indices (namely Wiener index, hyper-Wiener index, Tratch-Stankevitch-Zefirov, index and Harary index) of a graph. The NM-polynomial [9] is used to compute the neighbourhood degree sum-based topological indices of a graph. To find the degree-based topological indices, the idea of M-polynomial is proposed in [6] by Deutsch and Klavžar in 2015.

Definition 1.1 ([6]). The M-polynomial of a graph G is defined as

$$M(G; x, y) = \sum_{\delta \le i \le j \le \Delta} m_{ij} x^i y^j,$$

where $\delta = \min\{d(v) : v \in V(G)\}$, $\Delta = \max\{d(v) : v \in V(G)\}$ and m_{ij} is the number of edges $uv \in E(G)$ such that $d(u) = i, d(v) = j$, where $i, j \ge 1$.

A degree-based topological index for a simple connected graph G, as specified in [10] is denoted as $I(G)$ and stated as

$$I(G) = \sum_{uv \in E(G)} f(d(u), d(v)), \tag{1}$$

where $f(d(u), d(v))$ is a function of $d(u)$ and $d(v)$, which related to the corresponding topological indices. It is possible to rewrite the Equation (1) by counting the edges which has same end degrees of vertices as

$$I(G) = \sum_{\delta \le i \le j \le \Delta} m_{ij} f(i, j). \tag{2}$$

There are some well-known operators mentioned in [6, 7, 10–24], which we will use to formulate the M-polynomial of elliptic-type indices. They are

$D_x(f(x,y)) = x\frac{\partial(f(x,y))}{\partial x}$, $\quad D_y(f(x,y)) = y\frac{\partial(f(x,y))}{\partial y}$,

$D_x^{1/2}(f(x,y)) = \sqrt{x\frac{\partial(f(x,y))}{\partial x}} \cdot \sqrt{f(x,y)}$, $\quad D_y^{1/2}(f(x,y)) = \sqrt{y\frac{\partial(f(x,y))}{\partial y}} \cdot \sqrt{f(x,y)}$,

$S_x(f(x,y)) = \int_0^x \frac{f(t,y)}{t} dt$, $\quad S_x^{1/2} = \sqrt{\int_0^x \frac{f(t,y)}{t} dt} \cdot \sqrt{f(x,y)}$,

$P_x(f(x^s, y^t)) = f(x^{s^2}, y^t)$, $\quad P_y(f(x^s, y^t)) = f(x^s, y^{t^2})$ where $s, t \in \mathbb{N} \cup \{0\}$,

$Q_{x(k)}(f(x,y)) = x^k(f(x,y))$, $\quad Q_{y(k)}(f(x,y)) = y^k(f(x,y))$ where $k \in \mathbb{Z} \setminus \{0\}$,

$J(f(x,y)) = f(x,x)$.

2 Methodology

We provide and demonstrate closed derivation formulas in Section 3, associated with the aforementioned elliptic-type indices (elliptic Sombor (ESO), reduced elliptic (RE) and modified reduced elliptic (^mRE) indices), which can be applied over the M-polynomial of a graph to compute the indices. In Section 4, we compute the values of the above elliptic-type indices of some well-known graphs (such as a complete graph, complete bipartite graph, star graph, r-regular graph, cycle graph, and path graph) by using the M-polynomial-based derivation formulas. Furthermore, we list the M-polynomial formulas for the family of benzenoid systems $B_{m,n}$ and polycyclic aromatic compounds PAH_n and then employ our proposed closed derivation formulas to determine the elliptic-type indices of our interest. Furthermore, we use MATLAB R2019a software to depict the surface representation of the elliptic-type indices of $B_{m,n}$ and PAH_n for different parameters associated with them ($B_{m,n}$ and PAH_n). Section 5 concludes our work.

3 Closed derivation formulas over the M-polynomial for the Elliptic-type indices

In this section, we present the closed derivation formulas for the elliptic Sombor, reduced elliptic, and modified reduced elliptic indices. In order to do so, we construct three new operators that are required to establish the closed derivation formulas shown below.

$$\overline{P}_x(f(x^{s^2}, y^t)) = f(x^s, y^t),$$

$$\overline{P}_y(f(x^s, y^{t^2})) = f(x^s, y^t),$$

$$\overline{J}(f(x,x)) = f(x,y), \text{where } s,t \in \mathbb{N} \cup \{0\}.$$

Theorem 3.1. *Let $G = (V(G), E(G))$ be a graph and its elliptic Sombor index is*

$$ESO(G) = \sum_{uv \in E(G)} f(d(u), d(v)), \text{ where } f(x,y) = (x+y)\sqrt{x^2+y^2},$$

then

$$ESO(G) = D_x^{1/2} J P_y P_x (D_x + D_y)(M(G; x, y))|_{x=1},$$

where $M(G(x,y))$ is the M-polynomial of G.

Proof. Let $M(G; x, y)$ be the M-polynomial of G, then

$$\begin{aligned}
&D_x^{1/2} J P_y P_x (D_x + D_y)(M(G; x, y)) \\
&= D_x^{1/2} J P_y P_x (D_x + D_y)\Big\{\sum_{\delta \le i \le j \le \Delta} m_{ij} x^i y^j\Big\} \\
&= D_x^{1/2} J P_y P_x D_x \Big\{\sum_{\delta \le i \le j \le \Delta} m_{ij} x^i y^j\Big\} + D_x^{1/2} J P_y P_x D_y \Big\{\sum_{\delta \le i \le j \le \Delta} m_{ij} x^i y^j\Big\} \\
&= \sum_{\delta \le i \le j \le \Delta} D_x^{1/2} J P_y P_x D_x \{m_{ij} x^i y^j\} + \sum_{\delta \le i \le j \le \Delta} D_x^{1/2} J P_y P_x Dy)\{m_{ij} x^i y^j\} \\
&= \sum_{\delta \le i \le j \le \Delta} D_x^{1/2} J P_y P_x \{i m_{ij} x^i y^j\} + \sum_{\delta \le i \le j \le \Delta} D_x^{1/2} J P_y P_x \{j m_{ij} x^i y^j\}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\delta \le i \le j \le \Delta} D_x^{1/2} J\{i m_{ij} x^{i^2} y^{j^2}\} + \sum_{\delta \le i \le j \le \Delta} D_x^{1/2} J\{j m_{ij} x^{i^2} y^{j^2}\} \\
&= \sum_{\delta \le i \le j \le \Delta} D_x^{1/2} \{i m_{ij} x^{i^2+j^2}\} + \sum_{\delta \le i \le j \le \Delta} D_x^{1/2} \{j m_{ij} x^{i^2+j^2}\} \\
&= \sum_{\delta \le i \le j \le \Delta} i\sqrt{i^2+j^2}\; m_{ij} x^{i^2+j^2} + \sum_{\delta \le i \le j \le \Delta} j\sqrt{i^2+j^2}\; m_{ij} x^{i^2+j^2} \\
&= \sum_{\delta \le i \le j \le \Delta} (i+j)\sqrt{i^2+j^2}\; m_{ij} x^{i^2+j^2}.
\end{aligned}
$$

$$
\begin{aligned}
\therefore\; D_x^{1/2} J P_y P_x (D_x + D_y)(M(G;x,y))|_{x=1} &= \sum_{\delta \le i \le j \le \Delta} (i+j)\sqrt{i^2+j^2}\; m_{ij} \\
&= \sum_{\delta \le i \le j \le \Delta} m_{ij} \cdot f(i,j). \qquad (3)
\end{aligned}
$$

From the Equations (1) and (2), we get

$$ESO(G) = \sum_{uv \in E(G)} f(d(u), d(v)) = \sum_{\delta \le i \le j \le \Delta} m_{ij} \cdot f(i,j). \qquad (4)$$

Therefore, Equations (3) and (4) give the required result. ■

Theorem 3.2. *Let $G = (V(G), E(G))$ be a graph and its reduced elliptic index is*

$$RE(G) = \sum_{uv \in E(G)} f(d(u), d(v)), \text{ where } f(x,y) = (x-1+y-1)\sqrt{(x-1)^2 + (y-1)^2},$$

then

$$RE(G) = D_x^{1/2} J P_y P_x (D_x + D_y) Q_{y(-1)} Q_{x(-1)} (M(G;x,y))|_{x=1},$$

where $M(G(x,y))$ is the M-polynomial of G.

Proof. Let $M(G;x,y)$ be the M-polynomial of G, then

$$
\begin{aligned}
&D_x^{1/2} J P_y P_x (D_x + D_y) Q_{y(-1)} Q_{x(-1)} (M(G;x,y)) \\
&= D_x^{1/2} J P_y P_x (D_x + D_y) Q_{y(-1)} Q_{x(-1)} \left\{ \sum_{\delta \le i \le j \le \Delta} m_{ij} x^i y^j \right\} \\
&= \sum_{\delta \le i \le j \le \Delta} D_x^{1/2} J P_y P_x (D_x + D_y) Q_{y(-1)} Q_{x(-1)} \{m_{ij} x^i y^j\} \\
&= \sum_{\delta \le i \le j \le \Delta} D_x^{1/2} J P_y P_x (D_x + D_y) \{m_{ij} x^{i-1} y^{j-1}\} \\
&= \sum_{\delta \le i \le j \le \Delta} D_x^{1/2} J P_y P_x D_x \{m_{ij} x^{i-1} y^{j-1}\} + \sum_{\delta \le i \le j \le \Delta} D_x^{1/2} J P_y P_x D_y \{m_{ij} x^{i-1} y^{j-1}\} \\
&= \sum_{\delta \le i \le j \le \Delta} D_x^{1/2} J P_y P_x \{(i-1) m_{ij} x^{i-1} y^{j-1}\} + \sum_{\delta \le i \le j \le \Delta} D_x^{1/2} J P_y P_x \{(j-1) m_{ij} x^{i-1} y^{j-1}\} \\
&= \sum_{\delta \le i \le j \le \Delta} D_x^{1/2} J \{(i-1) m_{ij} x^{(i-1)^2} y^{(j-1)^2}\} + \sum_{\delta \le i \le j \le \Delta} D_x^{1/2} J \{(j-1) m_{ij} x^{(i-1)^2} y^{(j-1)^2}\}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\delta\le i\le j\le\Delta} D_x^{1/2}\{(i-1)m_{ij}x^{(i-1)^2+(j-1)^2}\} + \sum_{\delta\le i\le j\le\Delta} D_x^{1/2}\{(j-1)m_{ij}x^{(i-1)^2+(j-1)^2}\}\\
&= \sum_{\delta\le i\le j\le\Delta} (i-1)\sqrt{(i-1)^2+(j-1)^2}\ m_{ij}x^{(i-1)^2+(j-1)^2}\\
&\quad + \sum_{\delta\le i\le j\le\Delta} (j-1)\sqrt{(i-1)^2+(j-1)^2}\ m_{ij}x^{(i-1)^2+(j-1)^2}\\
&= \sum_{\delta\le i\le j\le\Delta} (i-1+j-1)\sqrt{(i-1)^2+(j-1)^2}\ m_{ij}x^{(i-1)^2+(j-1)^2}.
\end{aligned}
$$

$$
\begin{aligned}
&\therefore\ D_x^{1/2}JP_yP_x(D_x+D_y)Q_{y(-1)}Q_{x(-1)}(M(G;x,y))|_{x=1}\\
&= \sum_{\delta\le i\le j\le\Delta} (i-1+j-1)\sqrt{(i-1)^2+(j-1)^2}\ m_{ij} = \sum_{\delta\le i\le j\le\Delta} m_{ij}\cdot f(i,j).
\end{aligned}
\tag{5}
$$

From the Equations (1) and (2), we get

$$
RE(G) = \sum_{uv\in E(G)} f(d(u),d(v)) = \sum_{\delta\le i\le j\le\Delta} m_{ij}\cdot f(i,j). \tag{6}
$$

Therefore, Equations (5) and (6) give the required result. ■

Theorem 3.3. *Let $G=(V(G),E(G))$ be a graph and its modified reduced elliptic index is*

$$
{}^{m}RE(G) = \sum_{uv\in E(G)} f(d(u),d(v)),\ \text{where } f(x,y) = \frac{1}{(x-1+y-1)\sqrt{(x-1)^2+(y-1)^2}},
$$

then

$$
{}^{m}RE(G) = S_xJ\overline{P}_y\overline{P}_x\overline{J}S_x^{1/2}JP_yP_xQ_{y(-1)}Q_{x(-1)}(M(G;x,y))|_{x=1},
$$

where $M(G(x,y))$ is the M-polynomial of G.

Proof. Let $M(G;x,y)$ be the M-polynomial of G, then

$$
\begin{aligned}
&S_xJ\overline{P}_y\overline{P}_x\overline{J}S_x^{1/2}JP_yP_xQ_{y(-1)}Q_{x(-1)}(M(G;x,y))\\
&= S_xJ\overline{P}_y\overline{P}_x\overline{J}S_x^{1/2}JP_yP_xQ_{y(-1)}Q_{x(-1)}\left\{\sum_{\delta\le i\le j\le\Delta} m_{ij}x^iy^j\right\}\\
&= \sum_{\delta\le i\le j\le\Delta} S_xJ\overline{P}_y\overline{P}_x\overline{J}S_x^{1/2}JP_yP_xQ_{y(-1)}Q_{x(-1)}\{m_{ij}x^iy^j\}\\
&= \sum_{\delta\le i\le j\le\Delta} S_xJ\overline{P}_y\overline{P}_x\overline{J}S_x^{1/2}JP_yP_x\{m_{ij}x^{i-1}y^{j-1}\}\\
&= \sum_{\delta\le i\le j\le\Delta} S_xJ\overline{P}_y\overline{P}_x\overline{J}S_x^{1/2}J\{m_{ij}x^{(i-1)^2}y^{(j-1)^2}\}\\
&= \sum_{\delta\le i\le j\le\Delta} S_xJ\overline{P}_y\overline{P}_x\overline{J}S_x^{1/2}\{m_{ij}x^{\{(i-1)^2+(j-1)^2\}}\}\\
&= \sum_{\delta\le i\le j\le\Delta} S_xJ\overline{P}_y\overline{P}_x\overline{J}\left\{\frac{1}{\sqrt{(i-1)^2+(j-1)^2}}\ m_{ij}x^{\{(i-1)^2+(j-1)^2\}}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\delta \le i \le j \le \Delta} S_x J \overline{P}_y \overline{P}_x \left\{ \frac{1}{\sqrt{(i-1)^2 + (j-1)^2}} \, m_{ij} x^{(i-1)^2} y^{(j-1)^2} \right\} \\
&= \sum_{\delta \le i \le j \le \Delta} S_x J \left\{ \frac{1}{\sqrt{(i-1)^2 + (j-1)^2}} \, m_{ij} x^{i-1} y^{j-1} \right\} \\
&= \sum_{\delta \le i \le j \le \Delta} S_x \left\{ \frac{1}{\sqrt{(i-1)^2 + (j-1)^2}} \, m_{ij} x^{(i-1+j-1)} \right\} \\
&= \sum_{\delta \le i \le j \le \Delta} \frac{1}{(i-1+j-1)\sqrt{(i-1)^2 + (j-1)^2}} \, m_{ij} x^{(i-1+j-1)},
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ & S_x J \overline{P}_y \overline{P}_x \overline{J} S_x^{1/2} J P_y P_x Q_{y(-1)} Q_{x(-1)} (M(G; x, y))|_{x=1} \\
&= \sum_{\delta \le i \le j \le \Delta} \frac{1}{(i-1+j-1)\sqrt{(i-1)^2 + (j-1)^2}} \, m_{ij} = \sum_{\delta \le i \le j \le \Delta} m_{ij} \cdot f(i,j). \qquad (7)
\end{aligned}
$$

From the Equations (1) and (2), we get

$$
{}^{m}RE(G) = \sum_{uv \in E(G)} f(d(u), d(v)) = \sum_{\delta \le i \le j \le \Delta} m_{ij} \cdot f(i,j). \qquad (8)
$$

Therefore, Equations (7) and (8) give the required result. ■

4 Elliptic-type indices for some well-known graphs, $B_{m,n}$ and PAH_n

This section is divided into three subsections where we calculate the elliptic indices for some standard families of graphs, chemical families of $B_{m,n}$ and chemical families of PAH_n using our closed-form derivation formulas for the indices over their respective M-polynomials. Also, we show some of the numerical calculations and graphical representations of the elliptic-type indices of $B_{m,n}$ and PAH_n for different type parameters associated with them, which are acquired through MATLAB R2019a software.

4.1 Elliptic-type indices for some well-known graphs

Let us now find the values of the ESO, RE, and ${}^{m}RE$ indices using their derivation formulas of some well-known graphs, such as a complete graph, complete bipartite graph, star graph, r-regular graph, cycle graph, and path graph.

Theorem 4.1 ([13])**.** *Let G be a complete bipartite graph $K_{m,n}$ having $m+n$ vertices $1 \le m \le n$ and $n \ge 2$. Then the M-polynomial of the graph G is $M(G; x, y) = mnx^m y^n$.*

Theorem 4.2. *If G be a complete bipartite graph $K_{m,n}$ with $1 \le m \le n$ and $n \ge 2$, then*

$$
\begin{aligned}
&(i)\ ESO(G) = mn(m+n)\sqrt{m^2+n^2}, \\
&(ii)\ RE(G) = mn(m-1+n-1)\sqrt{(m-1)^2+(n-1)^2}, \\
&(iii)\ {}^{m}RE(G) = \frac{mn}{(m-1+n-1)\sqrt{(m-1)^2+(n-1)^2}}.
\end{aligned}
$$

Proof. The M-polynomial of the complete bipartite graph $K_{m,n}$ is $M(G;x,y) = mnx^m x^n$ as given in Theorem 4.1. Then

(i) **For elliptic Sombor index of $G = K_{m,n}$**

$$\begin{aligned}
ESO(G) &= D_x^{1/2} J P_y P_x (D_x + D_y)(M(G;x,y))|_{x=1} \\
&= D_x^{1/2} J P_y P_x (D_x + D_y)(mnx^m y^n)|_{x=1} \\
&= D_x^{1/2} J P_y P_x D_x (mnx^m y^n)|_{x=1} + D_x^{1/2} J P_y P_x D_y (mnx^m y^n)|_{x=1} \\
&= m^2 n \cdot D_x^{1/2} J P_y P_x \{x^m y^n\}|_{x=1} + mn^2 \cdot D_x^{1/2} J P_y P_x \{x^m y^n\}|_{x=1} \\
&= m^2 n \cdot D_x^{1/2} J \{x^{m^2} y^{n^2}\}|_{x=1} + mn^2 \cdot D_x^{1/2} J \{x^{m^2} y^{n^2}\}|_{x=1} \\
&= m^2 n \cdot D_x^{1/2} \{x^{m^2+n^2}\}|_{x=1} + mn^2 \cdot D_x^{1/2} \{x^{m^2+n^2}\}|_{x=1} \\
&= m^2 n \sqrt{m^2+n^2} \cdot \{x^{m^2+n^2}\}|_{x=1} + mn^2 \sqrt{m^2+n^2} \cdot \{x^{m^2+n^2}\}|_{x=1} \\
&= m^2 n \sqrt{m^2+n^2} + mn^2 \sqrt{m^2+n^2} \\
&= mn(m+n)\sqrt{m^2+n^2}.
\end{aligned}$$

(ii) **For reduced elliptic index of $G = K_{m,n}$**

$$\begin{aligned}
&RE(G) \\
&= D_x^{1/2} J P_y P_x (D_x + D_y) Q_{y(-1)} Q_{x(-1)} (M(G;x,y))|_{x=1} \\
&= D_x^{1/2} J P_y P_x (D_x + D_y) Q_{y(-1)} Q_{x(-1)} (mnx^m y^n)|_{x=1} \\
&= D_x^{1/2} J P_y P_x (D_x + D_y)(mnx^{m-1} y^{n-1})|_{x=1} \\
&= mn \cdot D_x^{1/2} J P_y P_x D_x (x^{m-1} y^{n-1})|_{x=1} + mn \cdot D_x^{1/2} J P_y P_x D_y (x^{m-1} y^{n-1})|_{x=1} \\
&= mn(m-1) \cdot D_x^{1/2} J P_y P_x (x^{m-1} y^{n-1})|_{x=1} + mn(n-1) \cdot D_x^{1/2} J P_y P_x (x^{m-1} y^{n-1})|_{x=1} \\
&= mn(m-1) \cdot D_x^{1/2} J (x^{(m-1)^2} y^{(n-1)^2})|_{x=1} + mn(n-1) \cdot D_x^{1/2} J (x^{(m-1)^2} y^{(n-1)^2})|_{x=1} \\
&= mn(m-1) \cdot D_x^{1/2} (x^{(m-1)^2+(n-1)^2})|_{x=1} + mn(n-1) \cdot D_x^{1/2} (x^{(m-1)^2+(n-1)^2})|_{x=1} \\
&= mn(m-1)\sqrt{(m-1)^2+(n-1)^2} \cdot (x^{(m-1)^2+(n-1)^2})|_{x=1} \\
&\quad + mn(n-1)\sqrt{(m-1)^2+(n-1)^2} \cdot (x^{(m-1)^2+(n-1)^2})|_{x=1} \\
&= mn(m-1)\sqrt{(m-1)^2+(n-1)^2} + mn(n-1)\sqrt{(m-1)^2+(n-1)^2} \\
&= mn(m-1+n-1)\sqrt{(m-1)^2+(n-1)^2}.
\end{aligned}$$

(iii) **For modified reduced elliptic index of $G = K_{m,n}$**

$$\begin{aligned}
{}^m RE(G) &= S_x J \overline{P}_y \overline{P}_x \overline{J} S_x^{1/2} J P_y P_x Q_{y(-1)} Q_{x(-1)} (M(G;x,y))|_{x=1} \\
&= S_x J \overline{P}_y \overline{P}_x \overline{J} S_x^{1/2} J P_y P_x Q_{y(-1)} Q_{x(-1)} (mnx^m y^n)|_{x=1} \\
&= mn \cdot S_x J \overline{P}_y \overline{P}_x \overline{J} S_x^{1/2} J P_y P_x (x^{m-1} y^{n-1})|_{x=1} \\
&= mn \cdot S_x J \overline{P}_y \overline{P}_x \overline{J} S_x^{1/2} J (x^{(m-1)^2} y^{(n-1)^2})|_{x=1} \\
&= mn \cdot S_x J \overline{P}_y \overline{P}_x \overline{J} S_x^{1/2} (x^{(m-1)^2+(n-1)^2})|_{x=1} \\
&= \frac{mn}{\sqrt{(m-1)^2+(n-1)^2}} \cdot S_x J \overline{P}_y \overline{P}_x \overline{J} (x^{(m-1)^2+(n-1)^2})|_{x=1} \\
&= \frac{mn}{\sqrt{(m-1)^2+(n-1)^2}} \cdot S_x J \overline{P}_y \overline{P}_x (x^{(m-1)^2} y^{(n-1)^2})|_{x=1}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{mn}{\sqrt{(m-1)^2+(n-1)^2}} \cdot S_x J(x^{m-1}y^{n-1})|_{x=1} \\
&= \frac{mn}{\sqrt{(m-1)^2+(n-1)^2}} \cdot S_x (x^{(m-1+n-1)})|_{x=1} \\
&= \frac{mn}{(m-1+n-1)\sqrt{(m-1)^2+(n-1)^2}} \cdot (x^{(m-1+n-1)})|_{x=1} \\
&= \frac{mn}{(m-1+n-1)\sqrt{(m-1)^2+(n-1)^2}}.
\end{aligned}
$$

∎

Corollary 4.3 can be derived from Theorem 4.2 by putting $m = n = r$. Similarly, Corollary 4.4 can be achieved by setting $m = 1$ and $n = r - 1$ in Theorem 4.2.

Corollary 4.3. *If G be a complete bipartite graph $K_{r,r}$ with $r \geq 2$, then*

$$
\begin{aligned}
&(i)\ ESO(G) = 2\sqrt{2}r^4, \\
&(ii)\ RE(G) = 2\sqrt{2}r^2(r-1)^2, \\
&(iii)\ {}^{m}RE(G) = \frac{r^2}{2\sqrt{2}(r-1)^2}.
\end{aligned}
$$

Corollary 4.4. *If G be a star graph $K_{1,r-1}$ with $r \geq 2$, then*

$$
\begin{aligned}
&(i)\ ESO(G) = r(r-1)\sqrt{r^2-2r+2}, \\
&(ii)\ RE(G) = (r-1)(r-2)^2, \\
&(iii)\ {}^{m}RE(G) = \frac{(r-1)}{(r-2)^2}.
\end{aligned}
$$

Theorem 4.5 ([12])**.** *Let G be a r-regular graph having n vertices and $r \geq 2$. Then, the M-polynomial of the r-regular graph is given by $M(G;x,y) = \frac{nr}{2}x^r y^r$.*

Theorem 4.6. *If G be a r-regular graph having n vertices and $r \geq 2$, then*

$$
\begin{aligned}
&(i)\ ESO(G) = nr^3\sqrt{2}, \\
&(ii)\ RE(G) = nr\sqrt{2}(r-1)^2, \\
&(iii)\ {}^{m}RE(G) = \frac{nr}{4(r-1)\sqrt{r-1}}.
\end{aligned}
$$

Proof. The M-polynomial of the r-regular graph is $M(G;x,y) = \frac{nr}{2}x^r y^r$ as given in Theorem 4.5. Then
(i) **For elliptic Sombor index of $G = r - regular\ graph$**

$$
\begin{aligned}
ESO(G) &= D_x^{1/2} J P_y P_x (D_x + D_y)(M(G;x,y))|_{x=1} \\
&= D_x^{1/2} J P_y P_x (D_x + D_y)\left(\frac{nr}{2}x^r y^r\right)|_{x=1} \\
&= D_x^{1/2} J P_y P_x D_x\left(\frac{nr}{2}x^r y^r\right)|_{x=1} + D_x^{1/2} J P_y P_x D_y\left(\frac{nr}{2}x^r y^r\right)|_{x=1} \\
&= \left(\frac{nr^2}{2}\right) \cdot D_x^{1/2} J P_y P_x (x^r y^r)|_{x=1} + \left(\frac{nr^2}{2}\right) \cdot D_x^{1/2} J P_y P_x (x^r y^r)|_{x=1}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{nr^2}{2}\right) \cdot D_x^{1/2} J(x^{r^2} y^{r^2})|_{x=1} + \left(\frac{nr^2}{2}\right) \cdot D_x^{1/2} J(x^{r^2} y^{r^2})|_{x=1} \\
&= \left(\frac{nr^2}{2}\right) \cdot D_x^{1/2} (x^{2r^2})|_{x=1} + \left(\frac{nr^2}{2}\right) \cdot D_x^{1/2} (x^{2r^2})|_{x=1} \\
&= \left(\frac{nr^2}{2}\right) \sqrt{2r^2} \cdot (x^{2r^2})|_{x=1} + \left(\frac{nr^2}{2}\right) \sqrt{2r^2} \cdot (x^{2r^2})|_{x=1} \\
&= \left(\frac{nr^2}{2}\right) \sqrt{2r^2} + \left(\frac{nr^2}{2}\right) \sqrt{2r^2} \\
&= nr^3 \sqrt{2}.
\end{aligned}
$$

(ii) **For reduced elliptic index of $G = r - regular\ graph$**

$$
\begin{aligned}
&RE(G) \\
&= D_x^{1/2} J P_y P_x (D_x + D_y) Q_{y(-1)} Q_{x(-1)} (M(G; x, y))|_{x=1} \\
&= D_x^{1/2} J P_y P_x (D_x + D_y) Q_{y(-1)} Q_{x(-1)} \left(\frac{nr}{2} x^r y^r\right)|_{x=1} \\
&= D_x^{1/2} J P_y P_x (D_x + D_y) \left(\frac{nr}{2} x^{r-1} y^{r-1}\right)|_{x=1} \\
&= \left(\frac{nr}{2}\right) D_x^{1/2} J P_y P_x D_x (x^{r-1} y^{r-1})|_{x=1} + \left(\frac{nr}{2}\right) D_x^{1/2} J P_y P_x D_y (x^{r-1} y^{r-1})|_{x=1} \\
&= \left(\frac{nr(r-1)}{2}\right) D_x^{1/2} J P_y P_x (x^{r-1} y^{r-1})|_{x=1} + \left(\frac{nr(r-1)}{2}\right) D_x^{1/2} J P_y P_x (x^{r-1} y^{r-1})|_{x=1} \\
&= \left(\frac{nr(r-1)}{2}\right) \cdot D_x^{1/2} J (x^{(r-1)^2} y^{(r-1)^2})|_{x=1} \\
&\quad + \left(\frac{nr(r-1)}{2}\right) \cdot D_x^{1/2} J (x^{(r-1)^2} y^{(r-1)^2})|_{x=1} \\
&= \left(\frac{nr(r-1)}{2}\right) \cdot D_x^{1/2} (x^{2(r-1)^2})|_{x=1} + \left(\frac{nr(r-1)}{2}\right) \cdot D_x^{1/2} (x^{2(r-1)^2})|_{x=1} \\
&= \left(\frac{nr(r-1)}{2}\right) \sqrt{2(r-1)^2} \cdot (x^{2(r-1)^2})|_{x=1} \\
&\quad + \left(\frac{nr(r-1)}{2}\right) \sqrt{2(r-1)^2} \cdot (x^{2(r-1)^2})|_{x=1} \\
&= \left(\frac{nr(r-1)}{2}\right) \sqrt{2(r-1)^2} + \left(\frac{nr(r-1)}{2}\right) \sqrt{2(r-1)^2} \\
&= nr\sqrt{2}(r-1)^2.
\end{aligned}
$$

(iii) **For modified reduced elliptic index of $G = r - regular\ graph$**

$$
\begin{aligned}
{}^m RE(G) &= S_x J \overline{P}_y \overline{P}_x \overline{J} S_x^{1/2} J P_y P_x Q_{y(-1)} Q_{x(-1)} (M(G; x, y))|_{x=1} \\
&= S_x J \overline{P}_y \overline{P}_x \overline{J} S_x^{1/2} J P_y P_x Q_{y(-1)} Q_{x(-1)} \left(\frac{nr}{2} x^r y^r\right)|_{x=1} \\
&= \left(\frac{nr}{2}\right) \cdot S_x J \overline{P}_y \overline{P}_x \overline{J} S_x^{1/2} J P_y P_x (x^{r-1} y^{r-1})|_{x=1} \\
&= \left(\frac{nr}{2}\right) \cdot S_x J \overline{P}_y \overline{P}_x \overline{J} S_x^{1/2} J (x^{(r-1)^2} y^{(r-1)^2})|_{x=1}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{nr}{2}\right) \cdot S_x J \overline{P}_y \overline{P}_x \overline{J} S_x^{1/2} (x^{2(r-1)^2})|_{x=1} \\
&= \left(\frac{nr}{2\sqrt{2(r-1)^2}}\right) \cdot S_x J \overline{P}_y \overline{P}_x \overline{J} (x^{2(r-1)^2})|_{x=1} \\
&= \left(\frac{nr}{2\sqrt{2(r-1)^2}}\right) \cdot S_x J \overline{P}_y \overline{P}_x (x^{(r-1)^2} y^{(r-1)^2})|_{x=1} \\
&= \left(\frac{nr}{2\sqrt{2(r-1)^2}}\right) \cdot S_x J (x^{r-1} y^{r-1})|_{x=1} \\
&= \left(\frac{nr}{2\sqrt{2(r-1)^2}}\right) \cdot S_x (x^{2(r-1)})|_{x=1} \\
&= \left(\frac{nr}{4(r-1)\sqrt{r-1}}\right) \cdot (x^{2(r-1)})|_{x=1} \\
&= \frac{nr}{4(r-1)\sqrt{r-1}}.
\end{aligned}
$$

■

The proof of the following Corollaries 4.8 and 4.10 are quite evident and can be solved similarly to Theorems 4.2 and 4.6. They are left as an exercise for the readers.

Theorem 4.7 ([13]). *Let G be a cycle graph C_n with $n(\geq 3)$ vertices, then the M-polynomial of G is $M(G; x, y) = nx^2y^2$.*

Corollary 4.8. *If G be a cycle graph C_n with $n(\geq 3)$ vertices, then*

$$
\begin{aligned}
&(i)\ ESO(G) = 8\sqrt{2}n, \\
&(ii)\ RE(G) = 2\sqrt{2}n, \\
&(iii)\ {}^{m}RE(G) = \frac{n}{2}.
\end{aligned}
$$

Theorem 4.9 ([13]). *Consider a complete graph K_n, a $(n-1)$-regular graph with $n(\geq 3)$ vertices. Then the M-polynomial of the graph G is $M(G; x, y) = \frac{n(n-1)}{2} x^{n-1} y^{n-1}$.*

Corollary 4.10. *If G be a complete graph K_n with $n(\geq 3)$ vertices, then*

$$
\begin{aligned}
&(i)\ ESO(G) = n(n-1)^3\sqrt{2}, \\
&(ii)\ RE(G) = \sqrt{2}n(n-1)(n-2)^2, \\
&(iii)\ {}^{m}RE(G) = \frac{n(n-1)}{4(n-2)\sqrt{n-2}}.
\end{aligned}
$$

Theorem 4.11 ([13]). *Let G be a path graph P_n with $n(\geq 3)$ vertices, then the M-polynomial of G is $M(G; x, y) = 2x^1y^2 + (n-3)x^2y^2$.*

Theorem 4.12. *If G be a path graph P_n with $n(\geq 3)$ vertices, then*

$$
\begin{aligned}
&(i)\ ESO(G) = 6\sqrt{5} + 8\sqrt{2}(n-3), \\
&(ii)\ RE(G) = 2 + 2\sqrt{2}(n-3), \\
&(iii)\ {}^{m}RE(G) = 2 + \frac{(n-3)}{\sqrt{2}}.
\end{aligned}
$$

Proof. The M-polynomial of the path graph is $M(G;x,y) = 2x^1y^2 + (n-3)x^2y^2$ as given in Theorem 4.11. Then

(i) **For elliptic Sombor index of $G = P_n$**

$$\begin{aligned}
&ESO(G)\\
&= D_x^{1/2}JP_yP_x(D_x+D_y)(M(G;x,y))|_{x=1}\\
&= D_x^{1/2}JP_yP_x(D_x+D_y)\Big(2x^1y^2+(n-3)x^2y^2\Big)|_{x=1}\\
&= D_x^{1/2}JP_yP_xD_x\Big(2x^1y^2+(n-3)x^2y^2\Big)|_{x=1}\\
&\quad + D_x^{1/2}JP_yP_xD_y\Big(2x^1y^2+(n-3)x^2y^2\Big)|_{x=1}\\
&= D_x^{1/2}JP_yP_x\Big(2x^1y^2+2(n-3)x^2y^2\Big)|_{x=1} + D_x^{1/2}JP_yP_x\Big(4x^1y^2+2(n-3)x^2y^2\Big)|_{x=1}\\
&= D_x^{1/2}J\Big(2x^1y^4+2(n-3)x^4y^4\Big)|_{x=1} + D_x^{1/2}J\Big(4x^1y^4+2(n-3)x^4y^4\Big)|_{x=1}\\
&= D_x^{1/2}\Big(2x^5+2(n-3)x^8\Big)|_{x=1} + D_x^{1/2}\Big(4x^5+2(n-3)x^8\Big)|_{x=1}\\
&= \Big(2\sqrt{5}x^5+2\sqrt{8}(n-3)x^8\Big)|_{x=1} + \Big(4\sqrt{5}x^5+2\sqrt{8}(n-3)x^8\Big)|_{x=1}\\
&= 2\sqrt{5}+2\sqrt{8}(n-3)+4\sqrt{5}+2\sqrt{8}(n-3)\\
&= 6\sqrt{5}+8\sqrt{2}(n-3).
\end{aligned}$$

(ii) **For reduced elliptic index of $G = P_n$**

$$\begin{aligned}
&RE(G)\\
&= D_x^{1/2}JP_yP_x(D_x+D_y)Q_{y(-1)}Q_{x(-1)}(M(G;x,y))|_{x=1}\\
&= D_x^{1/2}JP_yP_x(D_x+D_y)Q_{y(-1)}Q_{x(-1)}\Big(2x^1y^2+(n-3)x^2y^2\Big)|_{x=1}\\
&= D_x^{1/2}JP_yP_x(D_x+D_y)\Big(2y^1+(n-3)x^1y^1\Big)|_{x=1}\\
&= D_x^{1/2}JP_yP_xD_x\Big(2y^1+(n-3)x^1y^1\Big)|_{x=1} + D_x^{1/2}JP_yP_xD_y\Big(2y^1+(n-3)x^1y^1\Big)|_{x=1}\\
&= D_x^{1/2}JP_yP_x\Big((n-3)x^1y^1\Big)|_{x=1} + D_x^{1/2}JP_yP_x\Big(2y^1+(n-3)x^1y^1\Big)|_{x=1}\\
&= D_x^{1/2}J\Big((n-3)x^1y^1\Big)|_{x=1} + D_x^{1/2}J\Big(2y^1+(n-3)x^1y^1\Big)|_{x=1}\\
&= D_x^{1/2}\Big((n-3)x^2\Big)|_{x=1} + D_x^{1/2}\Big(2x^1+(n-3)x^2\Big)|_{x=1}\\
&= \Big(\sqrt{2}(n-3)x^2\Big)|_{x=1} + \Big(2x+\sqrt{2}(n-3)x^2\Big)|_{x=1}\\
&= \sqrt{2}(n-3)+2+\sqrt{2}(n-3)\\
&= 2+2\sqrt{2}(n-3).
\end{aligned}$$

(iii) **For modified reduced elliptic index of $G = P_n$**

$$\begin{aligned}
{}^{m}RE(G) &= S_xJ\overline{P}_y\overline{P}_x\overline{J}S_x^{1/2}JP_yP_xQ_{y(-1)}Q_{x(-1)}(M(G;x,y))|_{x=1}\\
&= S_xJ\overline{P}_y\overline{P}_x\overline{J}S_x^{1/2}JP_yP_xQ_{y(-1)}Q_{x(-1)}\Big(2x^1y^2+(n-3)x^2y^2\Big)|_{x=1}\\
&= S_xJ\overline{P}_y\overline{P}_x\overline{J}S_x^{1/2}JP_yP_x\Big(2y^1+(n-3)x^1y^1\Big)|_{x=1}
\end{aligned}$$

$$= S_x J \overline{P}_y \overline{P}_x \overline{J} S_x^{1/2} J\Big(2y^1 + (n-3)x^1y^1\Big)|_{x=1}$$
$$= S_x J \overline{P}_y \overline{P}_x \overline{J} S_x^{1/2}\Big(2x^1 + (n-3)x^2\Big)|_{x=1}$$
$$= S_x J \overline{P}_y \overline{P}_x \overline{J}\Big(2x^1 + \sqrt{2}(n-3)x^2\Big)|_{x=1}$$
$$= S_x J \overline{P}_y \overline{P}_x\Big(2y^1 + \sqrt{2}(n-3)x^1y^1\Big)|_{x=1}$$
$$= S_x J\Big(2y^1 + \sqrt{2}(n-3)x^1y^1\Big)|_{x=1}$$
$$= S_x\Big(2x^1 + \sqrt{2}(n-3)x^2\Big)|_{x=1}$$
$$= \Big(2x^1 + \frac{\sqrt{2}}{2}(n-3)x^2\Big)|_{x=1}$$
$$= 2 + \frac{(n-3)}{\sqrt{2}}.$$

■

4.2 Elliptic-type indices for jagged-rectangle benzenoid system $B_{m,n}$

This subsection calculates the elliptic Sombor, reduced elliptic, and modified reduced elliptic indices for the jagged-rectangle benzenoid system $B_{m,n}$. Note that, $B_{m,n}$ contains $4mn+4m+2n-2$ and $6mn+5m+n-4$ number of vertices and edges, respectively, as shown in Figure 1, while the degree of each vertex is either 2 or 3.

Theorem 4.13 ([25])**.** *Let G be the family of jagged-rectangle benzenoid system $B_{m,n}$ with $m \in \mathbb{N} \setminus \{1\}$ and $n \in \mathbb{N}$. Then,*

$$M(G;x,y) = (2n+4)x^2y^2 + (4m+4n-4)x^2y^3 + (6mn+m-5n-4)x^3y^3.$$

Theorem 4.14. *Let G be the jagged-rectangle benzenoid system $B_{m,n}$ with $m \in \mathbb{N} \setminus \{1\}$ and $n \in \mathbb{N}$. Then the elliptic topological indices are given by*

$$(i)\ ESO(G) = 36\sqrt{18}mn + (6\sqrt{18} + 20\sqrt{13})m + (8\sqrt{8} + 20\sqrt{13} - 30\sqrt{18})n$$
$$+ 16\sqrt{8} - 20\sqrt{13} - 24\sqrt{18},$$
$$(ii)\ RE(G) = 24\sqrt{8}mn + (12\sqrt{5} + 4\sqrt{8})m + (4\sqrt{2} + 12\sqrt{5} - 20\sqrt{8})n + 8\sqrt{2}$$
$$- 12\sqrt{5} - 16\sqrt{8},$$
$$(iii)\ {}^{m}RE(G) = \frac{3}{4\sqrt{2}}mn + \Big(\frac{4}{3\sqrt{5}} + \frac{1}{8\sqrt{2}}\Big)m + \Big(\frac{1}{\sqrt{2}} + \frac{4}{3\sqrt{5}} - \frac{5}{8\sqrt{2}}\Big)n + \sqrt{2}$$
$$- \frac{4}{3\sqrt{5}} - \frac{1}{2\sqrt{2}}.$$

Proof. The M-polynomial of $B_{m,n}$ is given by

$$M(G;x,y) = (2n+4)x^2y^2 + (4m+4n-4)x^2y^3 + (6mn+m-5n-4)x^3y^3.$$

Then

(i) $ESO(G)$

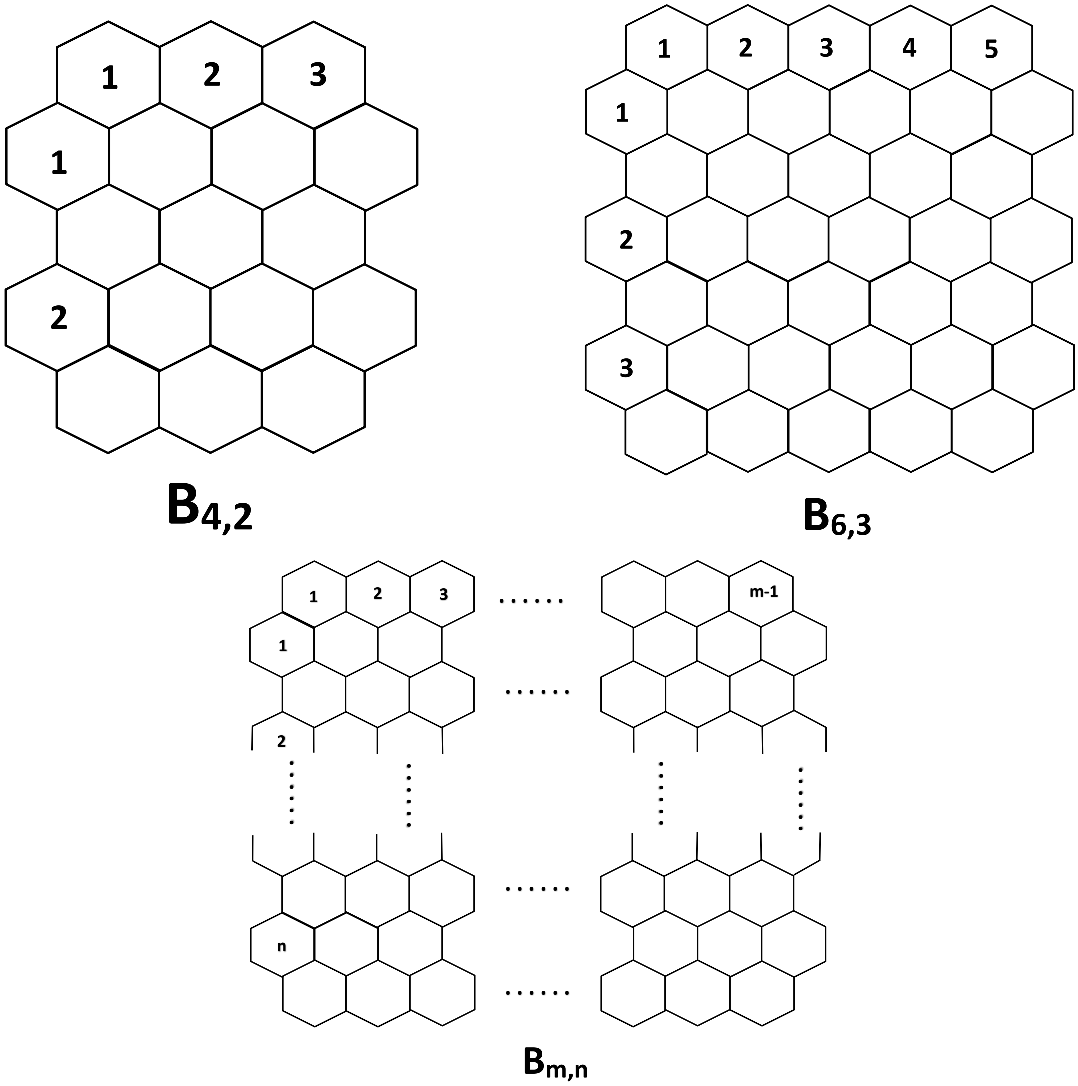


Figure 1: Chemical structure of the benzenoid system $B_{m,n}$.

$$
\begin{aligned}
&= D_x^{1/2} J P_y P_x (D_x + D_y)(M(G;x,y))|_{x=1}\\
&= D_x^{1/2} J P_y P_x (D_x + D_y)\Big((2n+4)x^2y^2 + (4m+4n-4)x^2y^3\\
&\quad + (6mn+m-5n-4)x^3y^3\Big)|_{x=1}\\
&= D_x^{1/2} J P_y P_x D_x\Big((2n+4)x^2y^2 + (4m+4n-4)x^2y^3 + (6mn+m-5n\\
&\quad -4)x^3y^3\Big)|_{x=1} + D_x^{1/2} J P_y P_x D_y\Big((2n+4)x^2y^2 + (4m+4n-4)x^2y^3 + (6mn+m\\
&\quad -5n-4)x^3y^3\Big)|_{x=1}
\end{aligned}
$$

$$
\begin{aligned}
&= D_x^{1/2} J P_y P_x \Big(2(2n+4)x^2y^2 + 2(4m+4n-4)x^2y^3 + 3(6mn+m-5n \\
&\quad -4)x^3y^3 \Big)|_{x=1} + D_x^{1/2} J P_y P_x \Big(2(2n+4)x^2y^2 + 3(4m+4n-4)x^2y^3 + 3(6mn+m \\
&\quad -5n-4)x^3y^3 \Big)|_{x=1} \\
&= D_x^{1/2} J \Big(2(2n+4)x^4y^4 + 2(4m+4n-4)x^4y^9 + 3(6mn+m-5n-4)x^9y^9 \Big)|_{x=1} \\
&\quad + D_x^{1/2} J \Big(2(2n+4)x^4y^4 + 3(4m+4n-4)x^4y^9 + 3(6mn+m-5n-4)x^9y^9 \Big)|_{x=1} \\
&= D_x^{1/2} \Big(2(2n+4)x^8 + 2(4m+4n-4)x^{13} + 3(6mn+m-5n-4)x^{18} \Big)|_{x=1} \\
&\quad + D_x^{1/2} \Big(2(2n+4)x^8 + 3(4m+4n-4)x^{13} + 3(6mn+m-5n-4)x^{18} \Big)|_{x=1} \\
&= \Big(2\sqrt{8}(2n+4)x^8 + 2\sqrt{13}(4m+4n-4)x^{13} + 3\sqrt{18}(6mn+m-5n-4)x^{18} \Big)|_{x=1} \\
&\quad + \Big(2\sqrt{8}(2n+4)x^8 + 3\sqrt{13}(4m+4n-4)x^{13} + 3\sqrt{18}(6mn+m-5n-4)x^{18} \Big)|_{x=1} \\
&= \Big(2\sqrt{8}(2n+4) + 2\sqrt{13}(4m+4n-4) + 3\sqrt{18}(6mn+m-5n-4) \Big) \\
&\quad + \Big(2\sqrt{8}(2n+4) + 3\sqrt{13}(4m+4n-4) + 3\sqrt{18}(6mn+m-5n-4) \Big) \\
&= 36\sqrt{18}mn + (6\sqrt{18} + 20\sqrt{13})m + (8\sqrt{8} + 20\sqrt{13} - 30\sqrt{18})n + 16\sqrt{8} \\
&\quad - 20\sqrt{13} - 24\sqrt{18}.
\end{aligned}
$$

(ii) $RE(G)$

$$
\begin{aligned}
&= D_x^{1/2} J P_y P_x (D_x + D_y) Q_{y(-1)} Q_{x(-1)} (M(G;x,y))|_{x=1} \\
&= D_x^{1/2} J P_y P_x (D_x + D_y) Q_{y(-1)} Q_{x(-1)} \Big((2n+4)x^2y^2 + (4m+4n-4)x^2y^3 + (6mn \\
&\quad + m - 5n - 4)x^3y^3 \Big)|_{x=1} \\
&= D_x^{1/2} J P_y P_x (D_x + D_y) \Big((2n+4)x^1y^1 + (4m+4n-4)x^1y^2 + (6mn+m-5n \\
&\quad -4)x^2y^2 \Big)|_{x=1} \\
&= D_x^{1/2} J P_y P_x D_x \Big((2n+4)x^1y^1 + (4m+4n-4)x^1y^2 + (6mn+m-5n \\
&\quad -4)x^2y^2 \Big)|_{x=1} + D_x^{1/2} J P_y P_x D_y \Big((2n+4)x^1y^1 + (4m+4n-4)x^1y^2 + (6mn+m \\
&\quad -5n-4)x^2y^2 \Big)|_{x=1} \\
&= D_x^{1/2} J P_y P_x \Big((2n+4)x^1y^1 + (4m+4n-4)x^1y^2 + 2(6mn+m-5n-4)x^2y^2 \Big)|_{x=1} \\
&\quad + D_x^{1/2} J P_y P_x \Big((2n+4)x^1y^1 + 2(4m+4n-4)x^1y^2 + 2(6mn+m-5n \\
&\quad -4)x^2y^2 \Big)|_{x=1} \\
&= D_x^{1/2} J \Big((2n+4)x^1y^1 + (4m+4n-4)x^1y^4 + 2(6mn+m-5n-4)x^4y^4 \Big)|_{x=1} \\
&\quad + D_x^{1/2} J \Big((2n+4)x^1y^1 + 2(4m+4n-4)x^1y^4 + 2(6mn+m-5n-4)x^4y^4 \Big)|_{x=1}
\end{aligned}
$$

$$
\begin{aligned}
&= D_x^{1/2}\Big((2n+4)x^2+(4m+4n-4)x^5+2(6mn+m-5n-4)x^8\Big)|_{x=1}\\
&\quad + D_x^{1/2}\Big((2n+4)x^2+2(4m+4n-4)x^5+2(6mn+m-5n-4)x^8\Big)|_{x=1}\\
&= \Big(\sqrt{2}(2n+4)x^2+\sqrt{5}(4m+4n-4)x^5+2\sqrt{8}(6mn+m-5n-4)x^8\Big)|_{x=1}\\
&\quad + \Big(\sqrt{2}(2n+4)x^2+2\sqrt{5}(4m+4n-4)x^5+2\sqrt{8}(6mn+m-5n-4)x^8\Big)|_{x=1}\\
&= \Big(\sqrt{2}(2n+4)+\sqrt{5}(4m+4n-4)+2\sqrt{8}(6mn+m-5n-4)\Big)\\
&\quad + \Big(\sqrt{2}(2n+4)+2\sqrt{5}(4m+4n-4)+2\sqrt{8}(6mn+m-5n-4)\Big)\\
&= 24\sqrt{8}mn+(12\sqrt{5}+4\sqrt{8})m+(4\sqrt{2}+12\sqrt{5}-20\sqrt{8})n+8\sqrt{2}-12\sqrt{5}\\
&\quad -16\sqrt{8}.
\end{aligned}
$$

(iii) ${}^{m}RE(G)$

$$
\begin{aligned}
&= S_xJ\overline{P}_y\overline{P}_x\overline{J}S_x^{1/2}JP_yP_xQ_{y(-1)}Q_{x(-1)}(M(G;x,y))|_{x=1}\\
&= S_xJ\overline{P}_y\overline{P}_x\overline{J}S_x^{1/2}JP_yP_xQ_{y(-1)}Q_{x(-1)}\Big((2n+4)x^2y^2+(4m+4n-4)x^2y^3+(6mn+m\\
&\quad -5n-4)x^3y^3\Big)|_{x=1}\\
&= S_xJ\overline{P}_y\overline{P}_x\overline{J}S_x^{1/2}JP_yP_x\Big((2n+4)x^1y^1+(4m+4n-4)x^1y^2+(6mn+m-5n\\
&\quad -4)x^2y^2\Big)|_{x=1}\\
&= S_xJ\overline{P}_y\overline{P}_x\overline{J}S_x^{1/2}J\Big((2n+4)x^1y^1+(4m+4n-4)x^1y^4+(6mn+m-5n\\
&\quad -4)x^4y^4\Big)|_{x=1}\\
&= S_xJ\overline{P}_y\overline{P}_x\overline{J}S_x^{1/2}\Big((2n+4)x^2+(4m+4n-4)x^5+(6mn+m-5n-4)x^8\Big)|_{x=1}\\
&= S_xJ\overline{P}_y\overline{P}_x\overline{J}\Big(\Big(\frac{2n+4}{\sqrt{2}}\Big)(2n+4)x^2+\Big(\frac{4m+4n-4}{\sqrt{5}}\Big)x^5\\
&\quad +\Big(\frac{6mn+m-5n-4}{\sqrt{8}}\Big)x^8\Big)|_{x=1}\\
&= S_xJ\overline{P}_y\overline{P}_x\Big(\Big(\frac{2n+4}{\sqrt{2}}\Big)x^1y^1+\Big(\frac{4m+4n-4}{\sqrt{5}}\Big)x^1y^4+\Big(\frac{6mn+m-5n-4}{\sqrt{8}}\Big)x^4y^4\Big)|_{x=1}\\
&= S_xJ\Big(\Big(\frac{2n+4}{\sqrt{2}}\Big)x^1y^1+\Big(\frac{4m+4n-4}{\sqrt{5}}\Big)x^1y^2+\Big(\frac{6mn+m-5n-4}{\sqrt{8}}\Big)x^2y^2\Big)|_{x=1}\\
&= S_x\Big(\Big(\frac{2n+4}{\sqrt{2}}\Big)x^2+\Big(\frac{4m+4n-4}{\sqrt{5}}\Big)x^3+\Big(\frac{6mn+m-5n-4}{\sqrt{8}}\Big)x^4\Big)|_{x=1}\\
&= \Big(\Big(\frac{2n+4}{2\sqrt{2}}\Big)x^2+\Big(\frac{4m+4n-4}{3\sqrt{5}}\Big)x^3+\Big(\frac{6mn+m-5n-4}{4\sqrt{8}}\Big)x^4\Big)|_{x=1}\\
&= \frac{3}{4\sqrt{2}}mn+\Big(\frac{4}{3\sqrt{5}}+\frac{1}{8\sqrt{2}}\Big)m+\Big(\frac{1}{\sqrt{2}}+\frac{4}{3\sqrt{5}}-\frac{5}{8\sqrt{2}}\Big)n+\sqrt{2}-\frac{4}{3\sqrt{5}}-\frac{1}{2\sqrt{2}}.
\end{aligned}
$$

■

In Table 1, we list all the numerical values of $ESO(G)$, $RE(G)$ and ${}^{m}RE(G)$ indices of $G=B_{m,n}$ where $m=n$. Similarly, Tables 2 and 3 contains the values of $ESO(G)$, $RE(G)$ and ${}^{m}RE(G)$

indices of $G = B_{m,n}$, where $2 \leq m \leq 10$, $n = 10$ and $m = 10$, $2 \leq n \leq 10$, respectively. Moreover, we provide the graphical interpretations of the numerical computation of $ESO(G)$, $RE(G)$ and ${}^{m}RE(G)$ indices of $G = B_{m,n}$ for different values of m and n using MATLAB R2019a software. See in Figure 2.

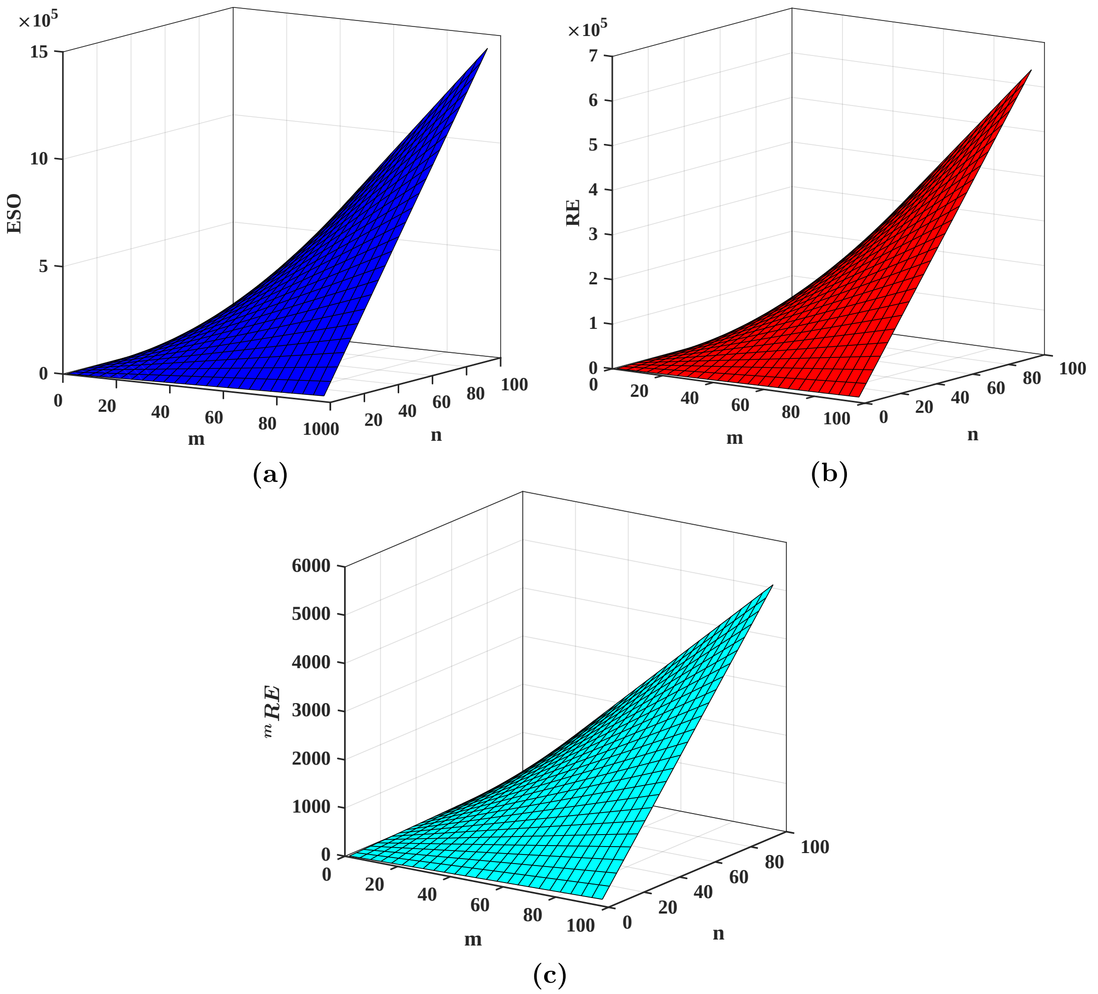


Figure 2: Graphical representation of (a) ESO(G), (b) RE(G), (c) ${}^{m}RE(G)$ indices of $G = B_{m,n}$ with $1 \leq m \leq 100$ and $1 \leq n \leq 100$.

Table 1: Numerical calculation of $ESO(G)$, $RE(G)$ and ${}^{m}RE(G)$ indices of $G = B_{m,n}$ where $m = n$.

[m,n]	[2,2]	[3,3]	[4,4]	[5,5]	[6,6]	[7,7]	[8,8]	[9,9]	[10,10]
$ESO(G)$	612.3129	1441.0143	2575.1858	4014.8275	5759.9393	7810.5212	10166.5733	12828.0955	15795.0878
$RE(G)$	238.8904	592.3693	1081.6127	1706.6206	2467.3930	3363.9299	4396.2313	5564.2973	6868.1277
${}^{m}RE(G)$	3.2928	7.4906	12.7490	19.0681	26.4479	34.8883	44.3893	54.9511	66.5735

Table 2: Numerical calculation of $ESO(G)$, $RE(G)$ and ${}^{m}RE(G)$ indices of $G = B_{m,n}$ where $2 \leq m \leq 10$ and $n = 10$.

[m,n]	[2,10]	[3,10]	[4,10]	[5,10]	[6,10]	[7,10]	[8,10]	[9,10]	[10,10]
$ESO(G)$	3836.6089	5331.4187	6826.2286	8321.0385	9815.8483	11310.6582	12805.4681	14300.2780	15795.0878
$RE(G)$	1630.1786	2284.9222	2939.6659	3594.4095	4249.1531	4903.8968	5558.6404	6213.3840	6868.1277
${}^{m}RE(G)$	17.2555	23.4202	29.5850	35.7497	41.9145	48.0792	54.2440	60.4087	66.5735

Table 3: Numerical calculation of $ESO(G)$, $RE(G)$ and ${}^{m}RE(G)$ indices of $G = B_{m,n}$ where $m = 10$ and $2 \leq n \leq 10$.

[m,n]	[10,2]	[10,3]	[10,4]	[10,5]	[10,6]	[10,7]	[10,8]	[10,9]	[10,10]
$ESO(G)$	2795.7477	4420.6652	6045.5827	7670.5002	9295.4177	10920.3353	12545.2528	14170.1703	15795.0878
$RE(G)$	1132.3754	1849.3444	2566.3135	3283.2825	4000.2515	4717.2206	5434.1896	6151.1586	6868.1277
${}^{m}RE(G)$	18.6699	24.6577	30.6456	36.6336	42.6216	48.6096	54.5975	60.5855	66.5735

4.3 Elliptic-type indices for polycyclic aromatic hydrocarbons PAH_n

This subsection calculates the elliptic Sombor, reduced elliptic, and modified reduced elliptic indices for the polycyclic aromatic hydrocarbons PAH_n. The molecular structure of the polycyclic aromatic hydrocarbons PAH_n contains $6n^2 + 6n$ and $9n^2 + 3n$ number of vertices and edges, respectively, as shown in Figure 3.

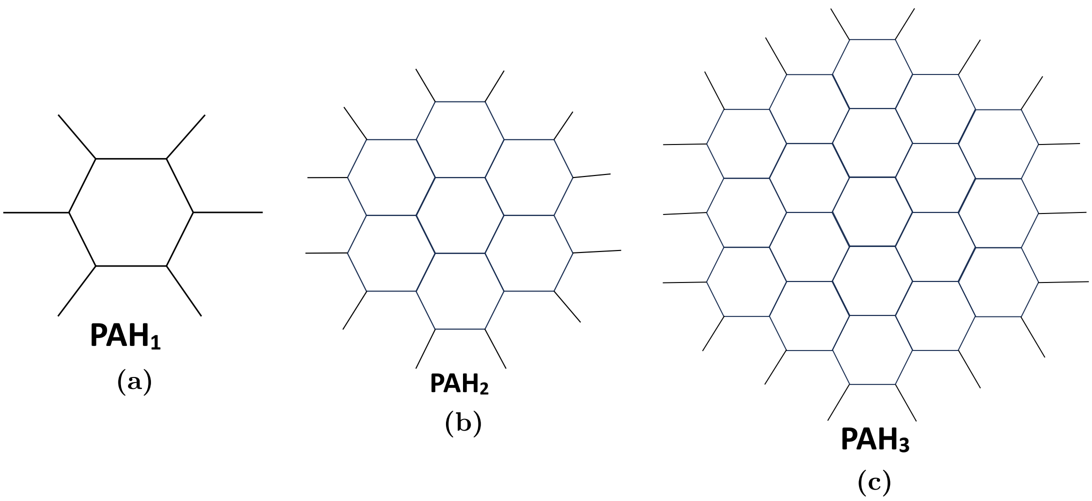


Figure 3: Molecular structure of the polycyclic aromatic hydrocarbons PAH_n for $n = 1, 2, 3$.

Theorem 4.15. *Consider G be the family of polycyclic aromatic hydrocarbons PAH_n. Then,*

$$M(G; x, y) = 6nx^1y^3 + (9n^2 - 3n)x^3y^3.$$

Proof. The graph G contains two types of edges in terms of the degree of end vertices as follows:

$$E_{1,3} = \{uv \in E(G) : d(u) = 1, d(v) = 3\}, \qquad |E_{1,3}| = 6n,$$

$$E_{3,3} = \{uv \in E(G) : d(u) = d(v) = 3\}, \qquad |E_{3,3}| = 9n^2 - 3n.$$

Then, by applying the Definition 1.1, we get the M-polynomial of polycyclic aromatic hydrocarbons PAH_n. ■

Theorem 4.16. *Let G be the family of polycyclic aromatic hydrocarbons PAH_n. Then, the elliptic topological indices are given by*

$$(i)\ ESO(G) = 54\sqrt{18}n^2 + (24\sqrt{10} - 18\sqrt{18})n,$$
$$(ii)\ RE(G) = 72\sqrt{2}n^2 + (1 - \sqrt{2})24n,$$
$$(iii)\ {}^{m}RE(G) = \Big(\frac{9}{8\sqrt{2}}\Big)n^2 + \Big(\frac{3}{2} - \frac{3}{8\sqrt{2}}\Big)n.$$

Proof. The M-polynomial of PAH_n is given by

$$M(G; x, y) = 6nx^1y^3 + (9n^2 - 3n)x^3y^3.$$

Then

$$
\begin{aligned}
&(i)\ ESO(G)\\
&= D_x^{1/2}JP_yP_x(D_x + D_y)(M(G; x, y))|_{x=1}\\
&= D_x^{1/2}JP_yP_x(D_x + D_y)\Big(6nx^1y^3 + (9n^2 - 3n)x^3y^3\Big)|_{x=1}\\
&= D_x^{1/2}JP_yP_xD_x\Big(6nx^1y^3 + (9n^2 - 3n)x^3y^3\Big)|_{x=1} + D_x^{1/2}JP_yP_xD_y\Big(6nx^1y^3 + (9n^2\\
&\quad - 3n)x^3y^3\Big)|_{x=1}\\
&= D_x^{1/2}JP_yP_x\Big(6nx^1y^3 + 3(9n^2 - 3n)x^3y^3\Big)|_{x=1} + D_x^{1/2}JP_yP_x\Big(18nx^1y^3 + 3(9n^2\\
&\quad - 3n)x^3y^3\Big)|_{x=1}\\
&= D_x^{1/2}J\Big(6nx^1y^9 + 3(9n^2 - 3n)x^9y^9\Big)|_{x=1} + D_x^{1/2}J\Big(18nx^1y^9 + 3(9n^2 - 3n)x^9y^9\Big)|_{x=1}\\
&= D_x^{1/2}\Big(6nx^{10} + 3(9n^2 - 3n)x^{18}\Big)|_{x=1} + D_x^{1/2}\Big(18nx^{10} + 3(9n^2 - 3n)x^{18}\Big)|_{x=1}\\
&= \Big(6\sqrt{10}nx^{10} + 3\sqrt{18}(9n^2 - 3n)x^{18}\Big)|_{x=1} + \Big(18\sqrt{10}nx^{10} + 3\sqrt{18}(9n^2\\
&\quad - 3n)x^{18}\Big)|_{x=1}\\
&= \Big(6\sqrt{10}n + 3\sqrt{18}(9n^2 - 3n)\Big) + \Big(18\sqrt{10}n + 3\sqrt{18}(9n^2 - 3n)\Big)\\
&= 54\sqrt{18}n^2 + (24\sqrt{10} - 18\sqrt{18})n.
\end{aligned}
$$

$$
\begin{aligned}
&(ii)\ RE(G)\\
&= D_x^{1/2}JP_yP_x(D_x + D_y)Q_{y(-1)}Q_{x(-1)}(M(G; x, y))|_{x=1}\\
&= D_x^{1/2}JP_yP_x(D_x + D_y)Q_{y(-1)}Q_{x(-1)}\Big(6nx^1y^3 + (9n^2 - 3n)x^3y^3\Big)|_{x=1}\\
&= D_x^{1/2}JP_yP_x(D_x + D_y)\Big(6ny^2 + (9n^2 - 3n)x^2y^2\Big)|_{x=1}\\
&= D_x^{1/2}JP_yP_xD_x\Big(6ny^2 + (9n^2 - 3n)x^2y^2\Big)|_{x=1} + D_x^{1/2}JP_yP_xD_y\Big(6ny^2 + (9n^2
\end{aligned}
$$

$$
\begin{aligned}
&- 3n)x^2y^2\Big)|_{x=1}\\
&= D_x^{1/2}JP_yP_x\Big(2(9n^2-3n)x^2y^2\Big)|_{x=1} + D_x^{1/2}JP_yP_x\Big(12ny^2+2(9n^2-3n)x^2y^2\Big)|_{x=1}\\
&= D_x^{1/2}J\Big(2(9n^2-3n)x^4y^4\Big)|_{x=1} + D_x^{1/2}J\Big(12ny^4+2(9n^2-3n)x^4y^4\Big)|_{x=1}\\
&= D_x^{1/2}\Big(2(9n^2-3n)x^8\Big)|_{x=1} + D_x^{1/2}\Big(12nx^4+2(9n^2-3n)x^8\Big)|_{x=1}\\
&= \Big(2\sqrt{8}(9n^2-3n)x^8\Big)|_{x=1} + \Big(24nx^4+2\sqrt{8}(9n^2-3n)x^8\Big)|_{x=1}\\
&= 2\sqrt{8}(9n^2-3n)+24n+2\sqrt{8}(9n^2-3n)\\
&= 72\sqrt{2}n^2+(1-\sqrt{2})24n.
\end{aligned}
$$

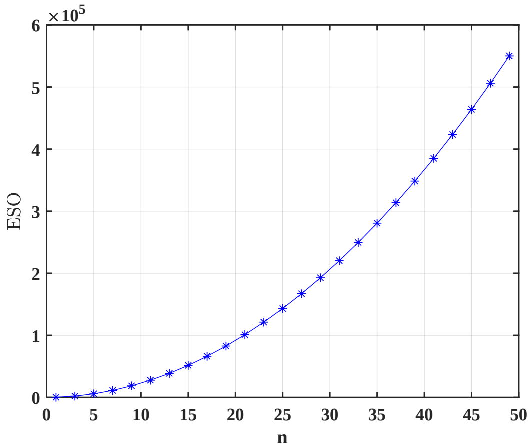
$$
\begin{aligned}
(iii)\ {}^{m}RE(G) &= S_xJ\overline{P}_y\overline{P}_x\overline{J}S_x^{1/2}JP_yP_xQ_{y(-1)}Q_{x(-1)}(M(G;x,y))|_{x=1}\\
&= S_xJ\overline{P}_y\overline{P}_x\overline{J}S_x^{1/2}JP_yP_xQ_{y(-1)}Q_{x(-1)}\Big(6nx^1y^3+(9n^2-3n)x^3y^3\Big)|_{x=1}\\
&= S_xJ\overline{P}_y\overline{P}_x\overline{J}S_x^{1/2}JP_yP_x\Big(6ny^2+(9n^2-3n)x^2y^2\Big)|_{x=1}\\
&= S_xJ\overline{P}_y\overline{P}_x\overline{J}S_x^{1/2}J\Big(6ny^4+(9n^2-3n)x^4y^4\Big)|_{x=1}\\
&= S_xJ\overline{P}_y\overline{P}_x\overline{J}S_x^{1/2}\Big(6nx^4+(9n^2-3n)x^8\Big)|_{x=1}\\
&= S_xJ\overline{P}_y\overline{P}_x\overline{J}\Big(3nx^4+\Big(\frac{9n^2-3n}{\sqrt{8}}\Big)x^8\Big)|_{x=1}\\
&= S_xJ\overline{P}_y\overline{P}_x\Big(3ny^4+\Big(\frac{9n^2-3n}{\sqrt{8}}\Big)x^4y^4\Big)|_{x=1}\\
&= S_xJ\Big(3ny^2+\Big(\frac{9n^2-3n}{\sqrt{8}}\Big)x^2y^2\Big)|_{x=1}\\
&= S_x\Big(3nx^2+\Big(\frac{9n^2-3n}{\sqrt{8}}\Big)x^4\Big)|_{x=1}\\
&= \Big(\frac{3}{2}nx^2+\Big(\frac{9n^2-3n}{4\sqrt{8}}\Big)x^4\Big)|_{x=1}\\
&= \frac{3}{2}n+\Big(\frac{9n^2-3n}{4\sqrt{8}}\Big) = \Big(\frac{9}{8\sqrt{2}}\Big)n^2+\Big(\frac{3}{2}-\frac{3}{8\sqrt{2}}\Big)n.
\end{aligned}
$$

■

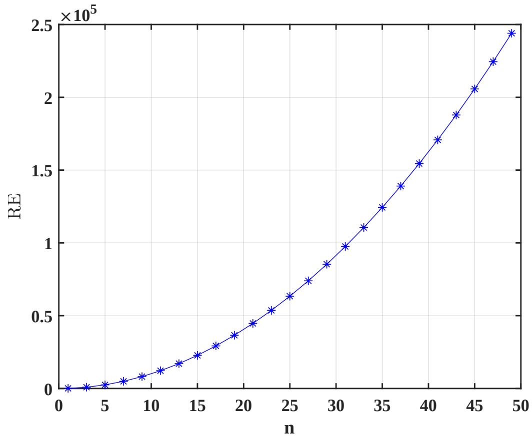
In Table 4, we lists all the numerical values of $ESO(G)$, $RE(G)$ and ${}^{m}RE(G)$ indices of $G = PAH_n$ where $n = 10$. Additionally, we provide the graphical interpretations of the numerical computation of $ESO(G)$, $RE(G)$ and ${}^{m}RE(G)$ indices of $G = PAH_n$ in Figure 4 using MATLAB R2019a software.

5 Concluding remarks

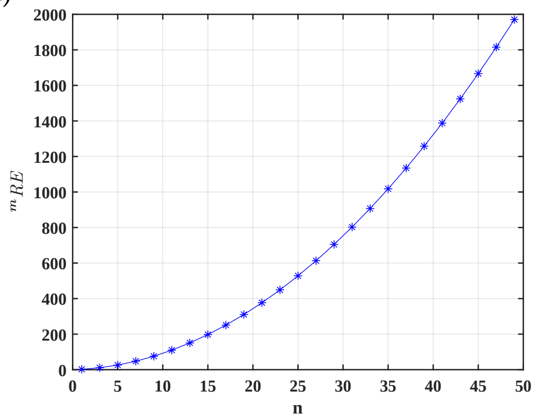
We suggested the closed derivation formulae in this work to calculate the recently announced elliptic Sombor index, reduced elliptic, and modified reduced elliptic indices of a graph, which are written in terms of a graph's M-polynomial. Additionally, we used our closed derivation formulas to evaluate these elliptic-type indices for some standard families of graphs, $B_{m,n}$ and



(a)



(b)



(c)

Figure 4: Graphical representation of (a) ESO(G), (b) RE(G), (c) ${}^{m}RE(G)$ indices of $G = PAH_n$ with $1 \leq n \leq 50$.

Table 4: Numerical computation of $ESO(G)$, $RE(G)$ and ${}^{m}RE(G)$ indices of $G = PAH_n$ where $n = 10$.

n	1	2	3	4	5	6	7	8	9	10
$ESO(G)$	228.6297	915.4647	2060.5048	3663.7501	5725.2006	8244.8563	11222.7172	14658.7833	18553.0546	22905.5310
$RE(G)$	91.8823	387.4113	886.5870	1589.4095	2495.8788	3605.9948	4919.7576	6437.1671	8158.2234	10082.9264
${}^{m}RE(G)$	2.0303	5.6517	10.8640	17.6673	26.0616	36.0468	47.6231	60.7904	75.5486	91.8979

PAH_n, by using their respective M-polynomials. Also, we have depicted the graphical representation of ESO, RE and ${}^{m}RE$ indices of $B_{m,n}$ and PAH_n within the range of $1 \leq m, n \leq 100$ and $1 \leq n \leq 50$, respectively. We have discovered that our proposed closed derivation formulas for the M-polynomial approach yield identical numerical outcomes for the RE and ${}^{m}RE$ indices of $B_{m,n}$ and PAH_n as those originally calculated by V.R. Kulli in [3]. Furthermore, it can be observed that the M-polynomial technique based on the derivation formulas is simpler, quicker, and more compact for estimating the elliptic indices.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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