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Research Paper

Emergence of Periodic Wave Solutions in a Diffusive Four-Component Brusselator Model

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Abstract

In the kinetic theory of chemical processes, cyclic reactions play a significant role. This study addresses a four-component non-linear Brusselator reaction-diffusion model for cyclic reactions to investigate the existence of periodic wave solutions to the model. The local behavior of the non-diffusive model solutions is investigated first. Then, the existence and uniqueness of the solution in the diffusive model are discussed. Furthermore, the emergence of periodic wave solutions in the diffusive model is also analyzed analytically. To validate our theoretical investigation, we also perform some numerical simulations in one and two space dimensions of the diffusive model. To demonstrate the originality of this study, we conclude by summarizing our findings and drawing comparisons with previous research.

1 Introduction

A theoretical structure for chemical processes called the Brusselator model can exhibit several dynamical states based on the parameters governing the diffusion coefficients and reaction rates. A reaction-diffusion (R-D) system can emerge from a homogeneous state and exhibit periodic patterns in either a stable or dynamic environment [1]. Numerous dynamical models associated with wave phenomena have been studied in recent years [2–10]. Particularly within the landscape of chemical R-D patterns, there is a rich tapestry of mathematical intrigue and real-world relevance. Analysis of the PWS was presented for chemical reactions [11–13], ecological [14–16], physical [17–19], and biological systems [20, 21]. The creation of patterns in a

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number of attributes was examined using the Brunsselator model in [22–25]. The PWS stability analysis for the diffusive 2-component Brusselator model was examined in [26, 27]. In a recent study, Nazimuddin et al. [28, 29] analyzed a diffusive 2-component Brusselator system to show the instability of PWS. Also, a Brusselator model related to three components is investigated in [30] to reveal the instability of PWS. A novel non-diffusive four-component Brusselator model was initially suggested in [31], where the authors demonstrated that the non-diffusive model experienced a closed trajectory. No further investigations using that model have been published since then.

In this paper, we investigate this novel four-component Brusselator model incorporating diffusion terms. We investigate the existence of PWS in the diffusive four-component chemical Brusselator model. The PWS of the R-D model is strongly associated with the formation of periodic patterns in the two-space dimension. The non-diffusive system is analyzed first by ignoring diffusion terms. The persistence of a stable limit cycle is then proved analytically and numerically utilizing the diffusive four-component Brusselator model. A comparison table has also been presented to show the originality of this research work.

Section 2 of this paper describes a diffusive four-component Brusselator model. The investigation of the local behavior of solutions in the non-diffusive system is presented in Section 3. In Section 4, the persistence and uniqueness of solutions in the diffusive model are illustrated. In Section 5, the limit cycle behavior in the diffusive model is discussed. Section 6 presents the numerical simulation result of the diffusive model. We discuss the originality of the ongoing research in Section 7. Our investigation concludes with the new discoveries presented in Section 8.

2 Diffusive four-component Brusselator model

Dissipative structures in chemical systems were the subject of numerous investigations in the 1960s. Four intermediate reaction processes explored by Prigogine and Lefever [32] can reflect the mechanism of the cyclic reaction, as well as the widely recognized Brusselator model:

$$\begin{cases} P \to X, \\ Q + X \to Y + R, \\ 2X + Y \to 3X, \\ X \to S. \end{cases}$$

The concentrations of the inputs, P, Q; outputs, R, S; and two intermediate reactants, X, Y, are involved in the chemical processes mentioned above. Following is the dimensionless version of the two-component diffusive Brusselator system.

$$\begin{cases} \frac{\partial \alpha}{\partial t} = d_1 \nabla^2 \alpha + a_1 - (b_1 + 1)\alpha + \alpha^2 \beta, \\ \frac{\partial \beta}{\partial t} = d_2 \nabla^2 \beta + b\alpha - \alpha^2 \beta. \end{cases} \tag{1}$$

For the two-component diffusive Brusselator model (1), the instability of PWS was investigated in [28, 29].

The following is the possible representation of the three-component diffusive Brusselator model.

$$\begin{cases} \frac{\partial \alpha}{\partial t} = d_1 \nabla^2 \alpha + a_1 - (b_1 + \gamma)\alpha + \alpha^2 \beta, \\ \frac{\partial \beta}{\partial t} = d_2 \nabla^2 \beta + \alpha\gamma - \alpha^2 \beta, \\ \frac{\partial \gamma}{\partial t} = d_3 \nabla^2 \gamma - \alpha\gamma + c_1. \end{cases} \tag{2}$$

In [30], the stability analysis of PWS in the three-component diffusive Brusselator model (2) was investigated. The diffusive dimensionless form of the four-component Brusselator model may be obtained by adding diffusion terms to the non-diffusive four-component Brusselator system [31].

$$\begin{cases} \frac{\partial\alpha}{\partial t} = d_1\nabla^2\alpha + a_1\delta - (b_1+\gamma)\alpha + \alpha^2\beta, \\ \frac{\partial\beta}{\partial t} = d_2\nabla^2\beta + \alpha\gamma - \alpha^2\beta, \\ \frac{\partial\gamma}{\partial t} = d_3\nabla^2\gamma - \alpha\gamma + c_1, \\ \frac{\partial\delta}{\partial t} = d_4\nabla^2\delta - a_1\delta + \alpha\gamma, \end{cases} \tag{3}$$

where the input concentrations are denoted by α, β, γ, and δ. The rate constants are represented by the kinetic parameters a_1, b_1, and c_1.

3 Local dynamical behavior in the non-diffusive model

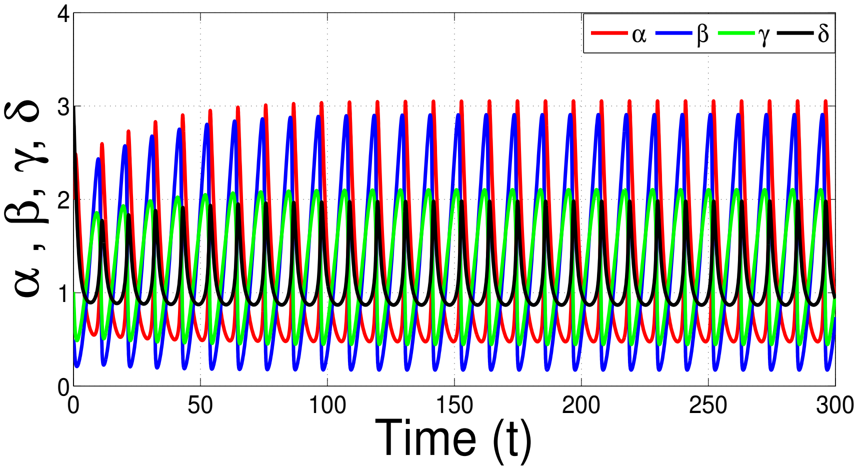
The following collection of ordinary differential equations (ODEs) generated after eliminating diffusion elements from the diffusive model (3):

$$\begin{cases} \frac{d\alpha}{dt} = a_1\delta - (b_1+\gamma)\alpha + \alpha^2\beta, \\ \frac{d\beta}{dt} = \alpha\gamma - \alpha^2\beta, \\ \frac{d\gamma}{dt} = -\alpha\gamma + c_1, \\ \frac{d\delta}{dt} = -a_1\delta + \alpha\gamma. \end{cases} \tag{4}$$

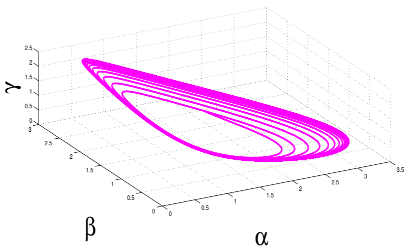
For system (4), $(\alpha,\beta,\gamma,\delta) = (\frac{c_1}{b_1}, \frac{b_1^2}{c_1}, b_1, \frac{c_1}{a_1})$ is the equilibrium point. The stable limit cycle solution $(\alpha,\beta,\gamma,\delta)$ of Equation (4) suggests a periodic solution. When $a_1 = 1, b_1 = 1$, and a certain range of values of c_1 [31], the system (4) displays a periodic solution. Our updated version of Equation (4) is as follows:

$$\begin{cases} \frac{d\alpha}{dt} = \delta - (1+\gamma)\alpha + \alpha^2\beta, \\ \frac{d\beta}{dt} = \alpha\gamma - \alpha^2\beta, \\ \frac{d\gamma}{dt} = -\alpha\gamma + c_1, \\ \frac{d\delta}{dt} = -\delta + \alpha\gamma. \end{cases} \tag{5}$$

The system (5) has a single equilibrium point, $(c_1, \frac{1}{c_1}, 1, c_1)$. The stability matrix of the system (5) has the characteristic equation $K^4 + K^3(1+c_1+c_1^2) + (-1+2c_1^2+c_1^3)K^2 + (-c_1+2c_1^3)K + c_1^3 = 0$. For the system (5), we detect a hopf bifurcation point for $c_1 = 1.17308$ (approximately). For $c_1 > 1.17308$, the equilibrium point $(c_1, \frac{1}{c_1}, 1, c_1)$ of the system (5) is stable and for $c_1 < 1.17308$, the equilibrium point $(c_1, \frac{1}{c_1}, 1, c_1)$ of the system (5) is unstable. Thus, we obtain a closed trajectory of the non-diffusive four-component Brusselator model (5).
Figure 1 displays an unstable solution profile for the system (5), where c_1 serves as a free parameter. It can be seen in Figure 1, we take $c_1 = 1.12$ and get an unstable solution profile. As a result, we find a closed trajectory of (5). An open trajectory of (5) is therefore obtained by taking $c_1 = 1.2$, which gives us a stable fixed point and is shown in Figure 2. Additionally, we note that periodic solutions with low amplitude are produced when the value of c_1 is close to 1.17308, while periodic solutions with large amplitude are produced when the value of c_1 is decreased from this value.



(a)



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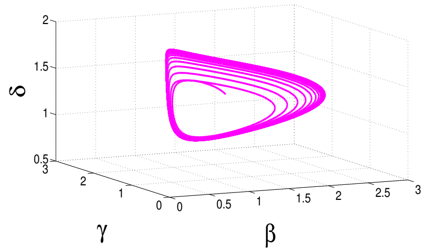

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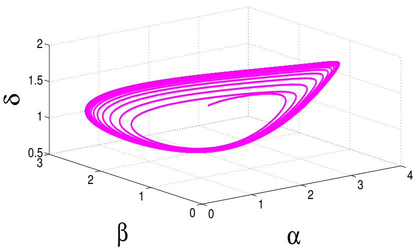

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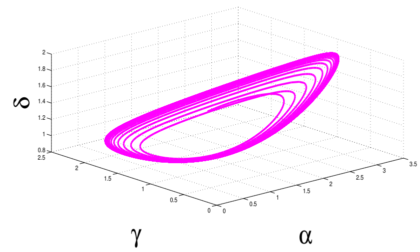

(e)

Figure 1: Solution behavior of the system (5), where $a_1 = 1$, $b_1 = 1$, and $c_1 = 1.12$, (a) time series solution; (b) possible phase portraits.

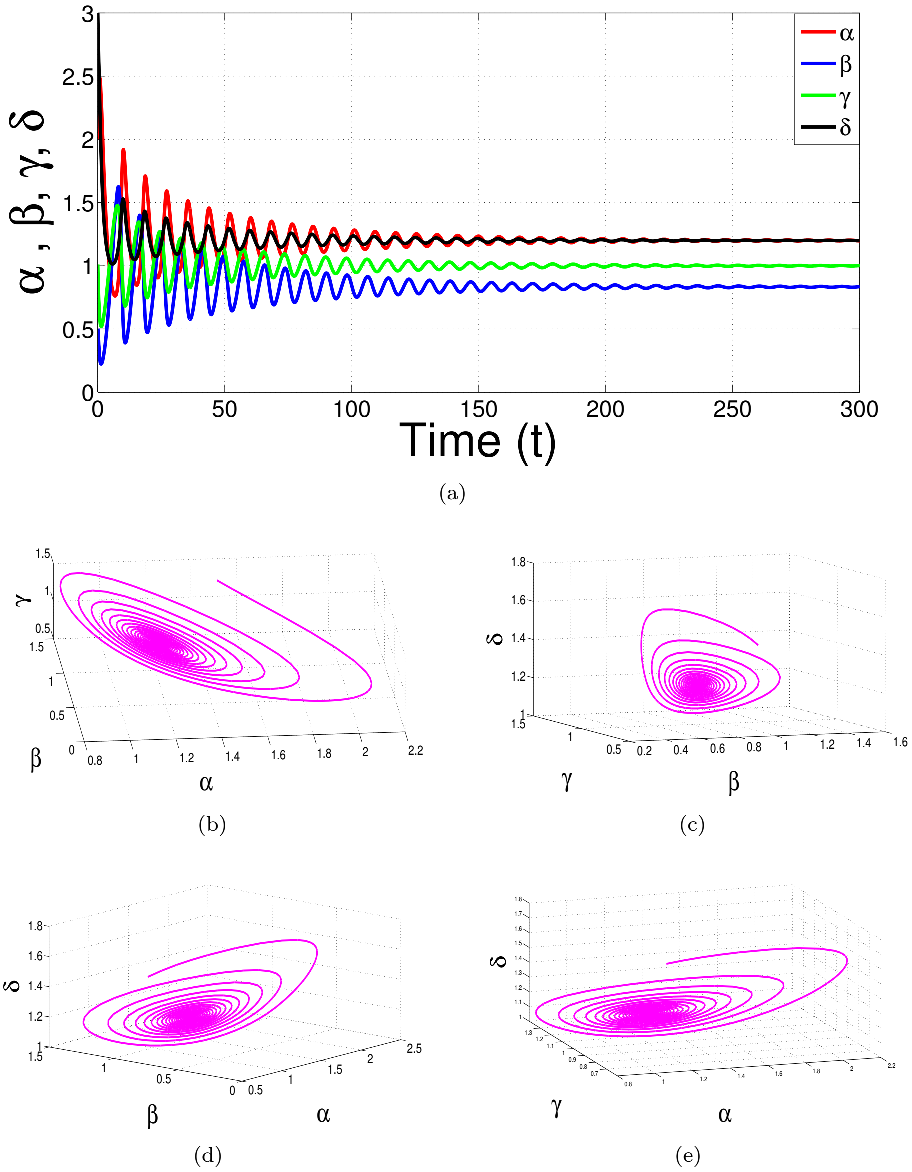

Figure 2: Solution behavior of the system (5), where $a_1 = 1$, $b_1 = 1$, and $c_1 = 1.2$, (a) time series solution; (b) possible phase portraits.

4 Emergence and uniqueness of solutions in the diffusive model

Let us consider the diffusive four-component Brussselator model stated in (3). Firstly, we introduce the Riemann-Liouville integral operator in Equation (3), we get
$\alpha(x,t) - \alpha(x,0) = I_t^1(d_1\,\alpha_{xx} + a_1\delta - (b_1+\gamma)\alpha + \alpha^2\beta)$

$$= \int_0^t (d_1\,\alpha_{xx} + a_1\delta(x,t) - (b_1+\gamma(x,t))\alpha(x,t) + \alpha^2(x,t)\beta(x,t))ds, \tag{6}$$

$$\beta(x,t) - \beta(x,0) = \int_0^t (d_2\,\beta_{xx} + \alpha(x,t)\gamma(x,t) - \alpha^2(x,t)\beta(x,t))ds, \tag{7}$$

$$\gamma(x,t) - \gamma(x,0) = \int_0^t (d_3\,\gamma_{xx} - \alpha(x,t)\gamma(x,t) + c_1)ds, \tag{8}$$

and

$$\delta(x,t) - \delta(x,0) = \int_0^t (d_4\,\delta_{xx} - a_1\delta(x,t) + \alpha(x,t)\gamma(x,t))ds. \tag{9}$$

Let,

$$L_1(t,\alpha(x,t)) = d_1\alpha_{xx} + a_1\delta - (b_1+\gamma)\alpha + \alpha^2\beta, \tag{10}$$

$$L_2(t,\beta(x,t)) = d_2\beta_{xx} + \alpha\gamma - \alpha^2\beta, \tag{11}$$

$$L_3(t,\gamma(x,t)) = d_3\gamma_{xx} - \alpha\gamma + c_1, \tag{12}$$

and

$$L_4(t,\delta(x,t)) = d_4\delta_{xx} - a_1\delta + \alpha\gamma. \tag{13}$$

Now, if $\alpha(x,t)$, $\alpha_1(x,t)$, $\beta(x,t)$, $\beta_1(x,t)$, $\gamma(x,t), \gamma_1(x,t)$, $\delta(x,t)$, and $\delta_1(x,t)$ are continuous functions and are in $\omega^2\left((0,1)\times(0,1)\right)$, then there exists $t_1>0$, $t_2>0$, $t_3>0$, and $t_4>0$ such that

$$\begin{cases} \| \alpha_{xx} - (\alpha_1)_{xx} \| \le t_1 \| \alpha - (\alpha_1) \|, \\ \| \beta_{xx} - (\beta_1)_{xx} \| \le t_2 \| \beta - (\beta_1) \|, \\ \| \gamma_{xx} - (\gamma_1)_{xx} \| \le t_3 \| \gamma - (\gamma_1) \|, \\ \| \delta_{xx} - (\delta_1)_{xx} \| \le t_4 \| \delta - (\delta_1) \| . \end{cases} \tag{14}$$

Now, we will prove that $L_1(t,\alpha(x,t))$, $L_2(t,\beta(x,t))$, $L_3(t,\gamma(x,t))$, and $L_4(t,\delta(x,t))$ accomplish the Lipschitz condition. We have

$$\begin{aligned} \| L_1(t,\alpha) - L_1(t,\alpha_1) \| &= \| d_1\alpha_{xx} + a_1\delta - (b_1+\gamma)\alpha + \alpha^2\beta - d_1(\alpha_1)_{xx} - a_1\delta \\ &\quad + (b_1+\gamma)\alpha_1 - \alpha_1^2\beta \| \\ &\le | d_1 | t_1 \| \alpha - (\alpha_1) \| + (| b_1 | + | \gamma |) \| \alpha - (\alpha_1) \| + | \beta | \| \alpha^2 - \alpha_1^2 \| \end{aligned}$$

$$\leq (k_1 t_1 + (k_2 + k_3) + k_4(m_1 + m_2)) \parallel \alpha - (\alpha_1) \parallel .$$

Here, $\mid d_1 \mid \leq k_1$, $\mid b_1 \mid \leq k_2$, $\mid \gamma \mid \leq k_3$, $\mid \beta \mid \leq k_4$, $\mid \alpha \mid \leq m_1$, and $\mid \alpha_1 \mid \leq m_2$.

Considering, $N_1 = k_1 t_1 + (k_2 + k_3) + k_4(m_1 + m_2)$, we get

$$\parallel L_1(t, \alpha) - L_1(t, \alpha_1) \parallel \leq N_1 \parallel \alpha - \alpha_1 \parallel . \tag{15}$$

Similarly, for the functions $\beta(x, t), \gamma(x, t)$, and $\delta(x, t)$, we can write

$$\begin{cases} \parallel L_2(t, \beta) - L_2(t, \beta_1) \parallel \leq N_2 \parallel \beta - \beta_1 \parallel, \\ \parallel L_3(t, \gamma) - L_3(t, \gamma_1) \parallel \leq N_3 \parallel \gamma - \gamma_1 \parallel, \\ \parallel L_4(t, \delta) - L_4(t, \delta_1) \parallel \leq N_4 \parallel \delta - \delta_1 \parallel, \end{cases} \tag{16}$$

where $N_2 = l_1 t_2 + m_1^2$, $N_3 = l_2 t_3 + m_1$, and $N_4 = l_3 t_4 + m_3$.
Therefore, the Lipschitz condition is accomplished by the kernels $L_1(t, \alpha)$, $L_2(t, \beta)$, $L_3(t, \gamma)$, and $L_4(t, \delta)$.

5 Emergence of limit cycle solution in the diffusive model

This section analytically examines the presence of a stable closed trajectory in the diffusive model (3). One possible matrix form for the equation (3) is as follows:

$$S_t = M S_{xx} + N(S),$$

where

$$S = \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix}, \; M = \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}, \text{ and } \quad N(S) = \begin{bmatrix} a_1\delta - (b_1 + \gamma)\alpha + \alpha^2\beta \\ \alpha\gamma - \alpha^2\beta \\ -\alpha\gamma + c_1 \\ -a_1\delta + \alpha\gamma \end{bmatrix}.$$

Consider that (3) has a spatially homogeneous positive equilibrium point denoted by $S^* = (\alpha^*, \ \beta^*, \ \gamma^*, \ \delta^*)$, which is unique and is obtained by setting $N(S) = 0$. Let us now investigate the emergence of Hopf instability of the spatially constant steady-state S^* under the spatially homogeneous perturbation $S = S^* + \hat{S}e^{\lambda t + ipx}$, where λ act as the growth factor and p act as the wave number. The linearized version's characteristic matrix at the equilibrium point S^* is as follows:

$$J^* = \begin{bmatrix} 0 & c_1^2/b_1^2 & -c_1/b_1 & a_1 \\ -b_1 & -c_1^2/b_1^2 & c_1/b_1 & 0 \\ -b_1 & 0 & -c_1/b_1 & 0 \\ b_1 & 0 & c_1/b_1 & -a_1 \end{bmatrix}.$$

The characteristic polynomial can be found as follows:

$$\begin{vmatrix} \lambda - p^2 d_1 & c_1^2/b_1^2 & c_1/b_1 & -a_1 \\ b_1 & c_1^2/b_1^2 + \lambda - p^2 d_2 & -c_1/b_1 & 0 \\ b_1 & 0 & c_1/b_1 + \lambda - p^2 d_3 & 0 \\ -b_1 & 0 & -c_1/b_1 & a_1 + \lambda - p^2 d_4 \end{vmatrix}.$$

The corresponding characteristic equation can be expressed in the following explicit form.

$$S(\lambda, c_1) = \lambda^4 + \sigma_3\lambda^3 + \sigma_2\lambda^2 + \sigma_1\lambda + \sigma_0 = 0, \tag{17}$$

where
$\sigma_3 = \mu_3 - p^2\kappa_3,$
$\sigma_2 = \mu_2 - p^2\kappa_2 + p^4\psi_2,$
$\sigma_1 = \mu_1 - p^2\kappa_1 + p^4\psi_1 + p^6\xi_1,$
$\sigma_0 = \mu_0 + p^2\kappa_0 + p^4\psi_0 - p^6\xi_0 + p^8\tau_0,$
$\mu_3 = a_1 + c_1/b_1 + c_1^2/b_1^2,$
$\mu_2 = (-a_1b_1^4 + a_1b_1^2c_1 - b_1^3c_1 + a_1b_1c_1^2 + b_1^2c_1^2 + c_1^3)/b_1^3,$
$\mu_1 = (-a_1b_1^3c_1 + a_1c_1^3 + b_1c_1^3)/b_1^3,$
$\mu_0 = ac_1^3/b_1^2,$
$\kappa_3 = d_1 + d_2 + d_3 + d_4,$
$\kappa_2 = ((a_1b_1^2 + c_1(b_1 + c_1))d_1 + b_1(a_1b_1 + c_1)d_2 + a_1b_1^2d_3 + c_1^2d_3 + b_1c_1d_4 + c_1^2d_4)/b_1^2,$
$\psi_2 = d_3d_4 + d_2d_3 + d_2d_4 + d_1d_2 + d_1d_3 + d_1d_4,$
$\kappa_1 = (c_1(c_1^2 + a_1b_1(b_1 + c_1))d_1 - b_1^2(a_1b_1^2 - a_1c_1 + b_1c_1)d_2 - a_1b_1^4d_3 + a_1b_1c_1^2d_3 + b_1^2c_1^2d_3 - b_1^3c_1d_4 +$
$b^2c_1^2d_4 + c_1^3d_4)/b_1^3,$
$\psi_1 = (c_1^2d_3d_4 + b_1d_2(a_1b_1d_3 + c_1d_4) + d_1(b_1(a_1b_1 + c_1)d_2 + (a_1b^2 + c^2)d_3 + c_1(b_1 + c_1)d_4))/b^2,$
$\xi_1 = -d_2d_3d_4 - d_1(d_3d_4 + d_2(d_3 + d_4)),$
$\kappa_0 = -((a_1c_1^3d_1)/b_1^3) + a_1c_1d_2 - (c_1^3d_4)/b_1^2,$
$\psi_0 = (c_1d_1(a_1b_1^2d_2 + c_1(a_1b_1d_3 + c_1d_4)) - b_1^2(-c_1^2d_3d_4 + b_1d_2(a_1b_1d_3 + c_1d_4)))/b_1^3,$
$\xi_0 = -d_1(c_1^2d_3d_4 + b_1d_2(a_1b_1d_3 + c_1d_4))/b_1^2,$
$\tau_0 = d_1d_2d_3d_4.$

Now, the Routh-Hurwitz criteria can be used to determine the necessary and sufficient requirements for local stability of the system (3), which takes the following form:

$$\Re(\lambda) < 0 \iff \begin{cases} \sigma_0 > 0, \\ \sigma_1 > 0, \\ \sigma_3 > 0, \\ \sigma_1\sigma_2\sigma_3 > \sigma_1^2 + \sigma_0\sigma_3^2, \end{cases} \qquad \forall p. \tag{18}$$

When homogeneous perturbations ($p = 0$) occur, conditions (18) lead to

$$\Re(\lambda) < 0 \iff$$

$$\begin{cases} \frac{a_1c_1^3}{b_1^2} > 0, \\ \frac{a_1b_1^3c_1 - a_1c_1^3 - b_1c_1^3}{b_1^3} < 0, \\ a_1 + \frac{c_1}{b_1} + \frac{c_1^2}{b_1^2} > 0, \\ \frac{(a_1b_1^3c_1 - (a_1+b_1)c_1^3)^2 + a_1c_1^3(a_1b_1^2 + c_1(b_1+c_1))^2 - (-a_1b_1^4 + (a_1-b_1)b_1^2c + b_1(a_1+b_1)c^2 + c^3)(-a_1b_1^3c + (a_1+b_1)c^3)(a_1 + \frac{c(b_1+c)}{b_1^2})}{b_1^6} < 0. \end{cases} \tag{19}$$

It is widely accepted that the process for the emergence of a spatially-uniform time-periodic solution is Hopf instability and a formula for the Hopf bifurcation points locus can be expressed by setting $\Re(\lambda) = 0$:

$$\Gamma_8\, c_1^8 + \Gamma_7\, c_1^7 + \Gamma_6\, c_1^6 + \Gamma_5\, c_1^5 + \Gamma_4\, c_1^4 + \Gamma_3\, c_1^3 + \Gamma_2\, c_1^2 + \Gamma_1\, c_1 = 0,$$

where
$\Gamma_1 = a_1^3 b_1$,
$\Gamma_2 = a_1^2 - \frac{a_1^3}{b_1}$,
$\Gamma_3 = -\frac{a_1(a_1+3a_1^2-b_1+a_1b_1)}{b_1^2}$,
$\Gamma_4 = \frac{a_1^2(a_1-4b_1)}{b_1^4}$,
$\Gamma_5 = \frac{(1+a_1)(a_1^2-2a_1b_1-b_1^2)}{b_1^5}$,
$\Gamma_6 = \frac{2a_1^2-2a_1b_1-b_1^2}{b_1^6}$,
$\Gamma_7 = \frac{a_1+a_1^2+b_1+a_1b_1+b_1^2}{b_1^7}$,
$\Gamma_8 = \frac{a_1+b_1}{b_1^8}$.

Theorem 5.1. *If $\psi \neq 0$, $a_1 > 0$, $b_1 > 0$, and $c_1 > 0$, then the roots of (17) must satisfy the following condition:*

$$a_1 b_1 < c_1^2(a_1 + b_1).$$

Proof. Let the zeros of Equation (17) be of the form $\lambda = i\psi \quad (\psi \neq 0)$. So, we get from (17)

$$S(i\psi, c_1) = (i\psi)^4 + \sigma_3\,(i\psi)^3 + \sigma_2\,(i\psi)^2 + \sigma_1\, i\psi + \sigma_0 = 0.$$

When the real and imaginary components are divided from the above equation, we get

$$\psi^4 - \sigma_2\,\psi^2 + \sigma_0 = 0, \tag{20}$$

and

$$\sigma_3\, k^3 + \sigma_1\, k = 0. \tag{21}$$

Now, from Equation (21), we get

$$\psi^2 = -\left(\frac{a_1 b_1^3 c_1 - a_1 c_1^3 - b_1 c_1^3}{a_1 b_1^3 + b_1^2 c_1 + b_1 c_1^2}\right) > 0.$$

Therefore, the roots of (17) must satisfy the inequality $a_1 b_1 < c_1^2(a_1 + b_1)$ for $a_1 > 0$, $b_1 > 0$, and $c_1 > 0$. ■

Remark 1. (1) Since we consider the parameter c_1 as a bifurcation parameter, the system (3) has a limit cycle that passes through the critical value $c_1 = c_1^*$, as per the Hopf bifurcation theorem [33].
(2) For any wave speed value, there exists a constant $\rho_0 > 0$. Now, for every $0 < \rho < \rho_0$, the Equation (3) has a set of periodic solutions $l_n(\rho)$ with period $T_n = T_n(\rho)$, for each parameter value $c_1 = c_1^*$.

6 Numerical investigation of the diffusive model

In order to verify these theoretical estimations, we apply some numerical techniques for analyzing the diffusive four-component Brusselator model covered in this section. An implicit technique is used for the one-dimensional analysis and the alternating direction implicit (ADI) approach [34] is used for the two-dimensional analysis.

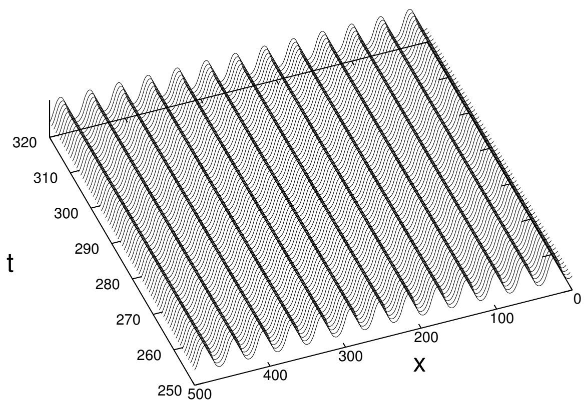

Figure 3: Stable periodic wave solution, α in (3).

6.1 One-space dimensional analysis

In this part, we run direct numerical simulations on (3) in one dimension in order to confirm our analytical result. We utilize the following periodic boundary conditions to generate an implicit scheme:

$$\begin{cases} \frac{\partial\alpha}{\partial t} = d_1\nabla^2\alpha + a_1\delta - (b_1+\gamma)\alpha + \alpha^2\beta, \quad t>0, \quad x\in\Lambda, \\ \frac{\partial\beta}{\partial t} = d_2\nabla^2\beta + \alpha\gamma - \alpha^2\beta, \quad t>0, \quad x\in\Lambda, \\ \frac{\partial\gamma}{\partial t} = d_3\nabla^2\gamma - \alpha\gamma + c_1, \quad t>0, \quad x\in\Lambda, \\ \frac{\partial\delta}{\partial t} = d_4\nabla^2\delta - a_1\delta + \alpha\gamma, \quad t>0, \quad x\in\Lambda. \end{cases}$$

We consider a space domain $[0,\Phi_x]$, defined by $\Phi_x = n\times r$, where the period and number of periodic pulses are denoted by r and n, respectively. In our simulation, we used 2500×2500 grid elements, space step $dx = 0.2$ and time step $dt = 0.01$. The parameter values of (3) are considered as $d_1 = 10, d_2 = 100, d_3 = 10, d_4 = 10, a_1 = 1, b_1 = 1$ and $c_1 = 1.12$.

We select $\Phi_x = 500$ as the system size with twelve periodic pulses, where we choose $p_1 = 41.67$ as the spatial period. The numerical simulation of (3) is performed for the time interval $250 < t < 320$. As seen in Figure 3, the outcome is a stable PWS for (3). As a consequence, the analytical outcome and the numerical outcome in the one-dimension result of the model (3) exhibit good agreement.

6.2 Two-space dimensional analysis

In this part, we run direct numerical simulation on (3) in two dimensions in order to confirm the analytical result. We consider the following in the square domain $\Gamma = (0,150)\times(0,150)$:

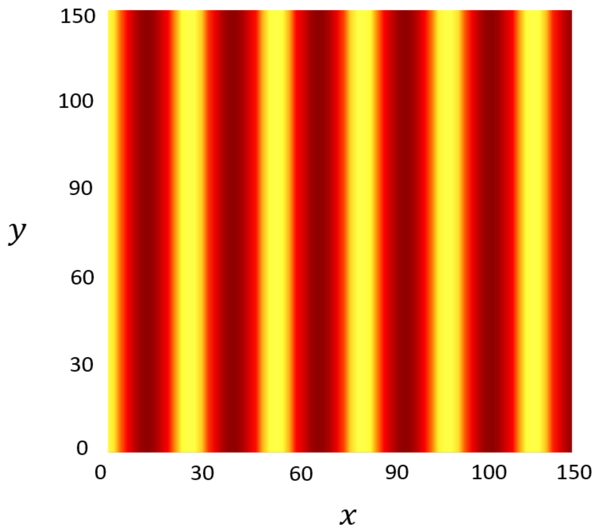

Figure 4: A stripe pattern formation of (3).

$$\begin{cases} \frac{\partial\alpha}{\partial t} = d_1\nabla^2\alpha + a_1\delta - (b_1+\gamma)\alpha + \alpha^2\beta, \quad t>0, \quad (x,y)\in\Gamma, \\ \frac{\partial\beta}{\partial t} = d_2\nabla^2\beta + \alpha\gamma - \alpha^2\beta, \quad t>0, \quad (x,y)\in\Gamma, \\ \frac{\partial\gamma}{\partial t} = d_3\nabla^2\gamma - \alpha\gamma + c_1, \quad t>0, \quad (x,y)\in\Gamma, \\ \frac{\partial\delta}{\partial t} = d_4\nabla^2\delta - a_1\delta + \alpha\gamma, \quad t>0, \quad (x,y)\in\Gamma. \end{cases}$$

We use Neumann boundary conditions to create an IBVP for the model (3). Furthermore, PBC might be applied here as well, as we have demonstrated the development of PWS in the 1D situation. We consider the parameter values of (3) as $d_1 = 10, d_2 = 100, d_3 = 10, d_4 = 10, a_1 = 1, b_1 = 1$, and $c_1 = 1.12$. We take into account the grid elements 1500×1500, the time interval $dt = 0.01$, and the step size $dx = dy = 0.1$. In this simulation, we obtain a stable stripe pattern for $c_1 = 1.12$ presented in as Figure 4.

7 Originality in the current investigation

This section discusses the novelty of this research work. In [31], a novel non-diffusive four-component Brusselator model was first proposed. The authors showed that the non-diffusive model followed a closed trajectory. Since then, no more studies employing that model have been released. Hence, we study this new four-component Brusselator model incorporating diffusion terms, in this research. We prove that the solutions in the diffusive model exist and are unique. The presence of PWS in the diffusive model is then demonstrated analytically. We also conduct numerical simulations of the diffusive model in one and two space dimensions to verify our theoretical analysis. We provide an overview of our results by comparing it with the earlier studies that are shown in Table 1.

Table 1: Comparison table.

Literature	Model	Methodology	Obtained Solutions
Akhmedov et al. [31]	Non-diffusive four-component Brusselator model.	Discrete-Numerical tracking method.	Existence of a closed trajectory.
Current study	Diffusive four-component Brusselator model.	(i) Theoretical investigation, (ii) Direct PDE simulation in 1-D space, (iii) Direct PDE simulation in 2-D space.	(i) Existence and Uniqueness of Solution, (ii) Existence of PWS.

8 Conclusion

In this paper, we studied the four-component Brusselator model incorporating diffusion terms. Understanding the kinetic properties of chemical reactions requires an understanding of the local behavior of four-component Brusselator model solutions. So, we examined the local behavior of the model's solutions first while ignoring the diffusion terms from the complete system. We then demonstrated the uniqueness and existence of solutions in the diffusive model. In the diffusive four-component Brusselator model, the presence of periodic wave solutions is illustrated analytically. We also carried out several numerical simulations in one and two spatial dimensions of the diffusive model to verify our theoretical analysis. In the future, the four-component Brusselator model may be investigated for the stability of PWS rigorously in both 1D and 2D cases, or the model may be expanded to include the additional input concentration in order to better understand the behavior of the system.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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