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Research Paper

Explicit Chebyshev Collocation Method for Multi-Order Fractional Nonlinear Boundary Value Problems in Mathematical Chemistry

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Abstract

This paper presents a numerical method for solving a class of nonlinear multi-order fractional differential equations using the first-kind Chebyshev polynomials. The proposed approach is based on a collocation framework that incorporates operational matrices of derivatives specifically tailored to the spectral properties of the Chebyshev polynomials on the interval $[0, 1]$. Two cases of interest are considered: the classical case with $\nu = 2$ and $\lambda = 1$, and the fractional-order case with $1 < \nu \leq 2$ and $0 < \lambda \leq 1$. To ensure high accuracy, an appropriate set of the shifted Chebyshev basis functions that satisfy the boundary conditions is utilized. The Caputo definition of fractional derivatives is adopted to handle the fractional operators. The resulting nonlinear algebraic system is solved efficiently using Newton's method. Numerical experiments confirm the proposed method's efficiency, stability, and accuracy in comparison with existing techniques.

1 Introduction

Fractional-order differential equations (FDEs) [1] have emerged as powerful tools for modeling complex systems with memory and hereditary characteristics, commonly observed in viscoelastic materials, anomalous diffusion, and chemical kinetics. These models are particularly useful when involving multi-order derivatives, capturing intricate temporal and spatial dynamics. Applications often arise in chemical reactors, thermal explosions, and porous media, extending classical models such as those in [2] to incorporate fractional-order dynamics [3].

Various numerical and analytical methods have been developed to address nonlinear fractional boundary value problems (BVPs). These include fixed-point techniques [4], Chebyshev

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finite difference methods [5], and homotopy-based approaches [6]. Spectral methods based on operational matrices and orthogonal polynomial bases, such as Jacobi, Laguerre, and Chebyshev, have gained wide attention due to their ability to achieve high accuracy within the collocation framework [7–12].

The mathematical foundations of fractional calculus are well established in references like [13], and recent work has extended their application to chemical and biological models, including fractional Michaelis–Menten kinetics [14], CO_2 absorption [15], and vibrational dynamics in NMR systems [16]. In optimization and variational settings, Caputo-type derivatives have been successfully applied [17, 18]. Orthogonal polynomial approximations using Jacobi and Chebyshev polynomials have been employed in a range of scientific and engineering applications [19–22].

Spectral methods [23–27] are particularly favored for their exponential convergence when applied to smooth problems. These approaches rely on global orthogonal basis functions, especially Chebyshev polynomials, to convert differential operators into algebraic systems that are highly efficient to solve [28, 29]. Consequently, they are well-suited for high-precision problems.

Recent developments have significantly expanded the use of spectral techniques in fractional problems. For example, modified shifted Chebyshev–Galerkin methods have been used for even-order PDEs [30]. Other studies extended the method to fractional models in fluid mechanics, biomechanics, and structural analysis using the Chebyshev collocation approach [31–33]. The Chebyshev-based tau method has been applied to Bagley–Torvik equations [34], while Petrov–Galerkin schemes have been adapted to singular-kernel problems [35]. Further advancements include third-kind Chebyshev methods for hyperbolic equations [36] and fractional diffusion models via Lucas polynomials [37]. Additionally, fixed-point approaches using G-metrics have been successfully integrated with spectral methods for fractal and chaotic models [38].

Emerging approaches using Bernoulli and Hahn polynomials [39, 40], along with improvements to nonlinear fractional models in mathematical chemistry [41–45], illustrate the versatility and effectiveness of polynomial-based techniques.

In this study, we introduce a novel Chebyshev spectral collocation method for solving nonlinear multi-order fractional BVPs, focusing on models from mathematical chemistry. The method employs newly constructed operational matrices for both integer and Caputo derivatives and utilizes the shifted Chebyshev polynomials to enhance both accuracy and computational efficiency.

Key contributions of this work:

- Accurate numerical solutions are achieved with a minimal number of basis functions.
- The method efficiently handles both linear and nonlinear systems.
- New operational matrices for multi-order Caputo derivatives are derived and embedded into the spectral collocation formulation.

Paper organization: Section 2 introduces properties of the first-kind Chebyshev polynomials. Section 3 outlines the development of the proposed numerical scheme. Section 4 provides an estimate of the truncation error. Section 5 provides numerical examples. Final remarks are discussed in Section 6.

2 Key properties of shifted Chebyshev polynomials of the first kind

The family of shifted Chebyshev polynomials of the first kind, denoted by $T^*_\ell(x)$, can be generated recursively using the following three-term relation:

$$T^*_{\ell+1}(x) = 2(2x-1)\,T^*_\ell(x) - T^*_{\ell-1}(x), \tag{1}$$

with the initial values $T^*_0(x) = 1$ and $T^*_1(x) = 2x - 1$.

These polynomials satisfy an orthogonality condition over the interval $[0, 1]$ with respect to the weight function $\hat{w}(x) = \frac{1}{\sqrt{x(1-x)}}$, as shown below [11, 12]:

$$\int_0^1 T^*_\ell(x)\,T^*_n(x)\,\hat{w}(x)\,dx = h_\ell\,\delta_{\ell,n}, \tag{2}$$

where

$$h_\ell = \begin{cases} \pi, & \text{if } \ell = 0, \\ \frac{\pi}{2}, & \text{if } \ell > 0, \end{cases} \tag{3}$$

and $\delta_{\ell,n}$ is the Kronecker delta function:

$$\delta_{\ell,n} = \begin{cases} 1, & \text{if } \ell = n, \\ 0, & \text{if } \ell \neq n. \end{cases} \tag{4}$$

Moreover, the shifted Chebyshev polynomials admit a power series representation [11, 12]:

$$T^*_\ell(x) = \sum_{k=0}^{\ell} A_{k,\ell}\,x^k, \quad \ell > 0, \tag{5}$$

where the coefficients $A_{k,\ell}$ are given by

$$A_{k,\ell} = \frac{\ell\,(-1)^{\ell-k}\,2^{2k}\,(\ell+k-1)!}{(\ell-k)!\,(2k)!}. \tag{6}$$

An inversion formula expresses powers of x in terms of Chebyshev polynomials:

$$x^\ell = 2^{1-2\ell}\,(2\ell)! \sum_{p=0}^{\ell} \frac{\epsilon_p}{(\ell-p)!(\ell+p)!}\,T^*_p(x), \quad \ell \geq 0, \tag{7}$$

where ϵ_p is defined as

$$\epsilon_p = \begin{cases} \frac{1}{2}, & \text{if } p = 0, \\ 1, & \text{if } p > 0. \end{cases} \tag{8}$$

Remark 1. The inversion identity in (7) can also be written in the following compact form:

$$x^r = \frac{4^{-r}(2r)!}{(r!)^2} + \sum_{p=1}^{r} B_{p,r}\,T^*_p(x), \quad r \geq 0, \tag{9}$$

where the coefficients $B_{p,r}$ are given by

$$B_{p,r} = \frac{2^{1-2r}\,(2r)!}{(r-p)!(r+p)!}. \tag{10}$$

Corollary 2.1. *([10]). For any positive integer q, the qth-order derivative of $T_\ell^*(x)$ can be expressed as a linear combination of the polynomials themselves:*

$$D^q\, T_\ell^*(x) = \sum_{\substack{p=0 \\ \ell+p+q\ even}}^{\ell-q} \varsigma_{\ell,p,q}\, T_p^*(x), \tag{11}$$

where the coefficients $\varsigma_{\ell,p,q}$ are defined by

$$\varsigma_{\ell,p,q} = \frac{\ell\, 2^{2q}\, \epsilon_p\, (q)_{\frac{1}{2}(\ell-p-q)}}{\left(\frac{1}{2}(\ell-p-q)\right)!\, \left(\frac{1}{2}(\ell+p+q)\right)_{1-q}},$$

and ϵ_p is given in Equation (8).

3 Collocation algorithm for the nonlinear multi-order fractional version

We consider the nonlinear multi-order fractional version

$$D^\nu\, u(x) - \alpha D^\lambda\, u(x) + \alpha\, \eta\, \big(\chi - u(x)\big)\, e^{u(x)} = 0, \qquad 0 < x < 1, \tag{12}$$

subject to

$$u'(0) = \alpha\, u(0), \quad u'(1) = 0, \tag{13}$$

where $\nu \in (1,2]$, $\lambda \in (0,1]$. Also, η stands for the Damkohler number, γ stands for adiabatic temperature rise, and α stands for the Peclet number.

3.1 Choice of the basis functions

Assuming the following basis functions

$$\rho_i(x) = T_{i+2}^*(x) - 2(i+2)^2 \left(\frac{(-1)^i + 1}{\alpha} + x\right) - (-1)^i.$$

Utilizing the derivative formula for Chebyshev polynomials given in Equation (11), we derive the following two important results.

Theorem 3.1. *The following formula holds*

$$D\, \rho_i(x) = \sum_{p=0}^{i+1} \mathcal{Z}_{i,p}\, T_p^*(x), \tag{14}$$

where

$$\mathcal{Z}_{i,p} = (2\,i+4) \begin{cases} -(1+i+\eta_i), & \text{if } p = 0, \\ 2(1-\eta_{i+p}), & \text{otherwise}, \end{cases} \tag{15}$$

and

$$\eta_i = \begin{cases} 1, & \text{if } i \text{ is even}, \\ 0, & \text{otherwise}. \end{cases} \tag{16}$$

Proof. By setting $q = 1, \ell = i+2$ in Equation (11) and rearranging the terms on the right-hand side, we obtain the desired result. ■

Theorem 3.2. *The following formula holds*

$$D^2\,\rho_i(x) = \sum_{p=0}^{i} \theta_{p,i}\, T_p^*(x), \tag{17}$$

where

$$\theta_{p,i} = (4\,i+8)\cdot\left((i+2)^2 - p^2\right)\,\epsilon_p\,\eta_{i+p}, \tag{18}$$

and

$$\epsilon_i = \begin{cases} \frac{1}{2}, & \text{if } i = 0,\\ 1, & \text{otherwise.}\end{cases} \tag{19}$$

Proof. By setting $q = 2, \ell = i+2$ in Equation (11) and rearranging the terms on the right-hand side, we obtain the desired result. ■

Remark 2. Let us consider the vector $\boldsymbol{\rho}(x)$ defined as

$$\boldsymbol{\rho}(x) = [\rho_0(x), \rho_1(x), \ldots, \rho_M(x)]^T, \tag{20}$$

then the matrix form of Theorems 3.1 and 3.2 can be written as

$$D\,\boldsymbol{\rho}(x) = \mathcal{H}\,\boldsymbol{\rho}(x), \tag{21}$$

$$D^2\,\boldsymbol{\rho}(x) = \mathcal{F}\,\boldsymbol{\rho}(x), \tag{22}$$

where $\mathcal{H} = (\mathcal{Z}_{i,p})$ and $\mathcal{F} = (\theta_{p,i})$ are operational matrices of derivatives of order $(M+1)^2$.

3.2 Derivation of the collocation technique for $\nu = 2$ and $\lambda = 1$

To handle Equation (12), we approximate $u(x)$ by $\boldsymbol{\rho}(x)$ as

$$u_M(x) = \sum_{i=0}^{M} c_i\,\rho_i(x) = \boldsymbol{C}^T\,\boldsymbol{\rho}(x), \tag{23}$$

where $\boldsymbol{C}^T = [c_0, c_1, c_2, \ldots, c_N]$ and $\boldsymbol{\rho}(x)$ is given in (20). Based on Remark 2 and (23), the residual of Equation (12) after putting $\nu = 2$ and $\lambda = 1$ can be written as

$$\begin{aligned} R(x) &= D^2\,u_M - \alpha D\,u_M + \alpha\,\eta\,(\chi - u_M)\,e^{u_M}\\ &= \boldsymbol{C}^T\,\mathcal{F}\,\boldsymbol{\rho}(x) - \alpha\,\boldsymbol{C}^T\,\mathcal{H}\,\boldsymbol{\rho}(x) + \alpha\,\eta\,(\chi - \boldsymbol{C}^T\,\boldsymbol{\rho}(x))\,e^{\boldsymbol{C}^T\,\boldsymbol{\rho}(x)}. \end{aligned} \tag{24}$$

Also the boundary conditions (13) yields,

$$\boldsymbol{C}^T\,D\,\boldsymbol{\rho}(0) = \alpha\,\boldsymbol{C}^T\,\boldsymbol{\rho}(0), \quad \boldsymbol{C}^T\,D\,\boldsymbol{\rho}(1) = 0, \tag{25}$$

To get approximate the solution $u_M(x)$, The residual of Equation (24) is enforced at the first $M-2$ roots of the polynomial $\rho_{M+2}(x)$ as follows

$$R(x_i) = 0, \quad i = 1, ..., M-2. \tag{26}$$

Combined the $M-2$ system of equations resulting from the last equation with the two boundary conditions given in (25), yields a system of $M+1$ nonlinear algebraic equations.

Remark 3. The resulting system is then solved using Newton's iterative scheme, allowing for the computation of the approximate solution $u_M(x)$ as expressed in (23).

3.3 Derivation of the collocation technique for $1 < \nu < 2$ and $0 < \lambda < 1$

This section presents a numerical procedure tailored to solve the nonlinear multi-order fractional problem in the case where the fractional orders satisfy $1 < \nu < 2$ and $0 < \lambda < 1$. To facilitate the formulation, we first summarize essential concepts from fractional calculus.

Definition 3.3 ([13])**.** For a sufficiently smooth function $h(\tau)$, the Caputo fractional derivative of order ζ is defined by

$$D^{\zeta} h(\tau) = \frac{1}{\Gamma(p-\zeta)} \int_0^{\tau} (\tau-\xi)^{p-\zeta-1} h^{(p)}(\xi)\, d\xi, \quad \zeta > 0, \quad \tau > 0, \tag{27}$$

where $p \in \mathbf{N}$ is the smallest integer such that $p-1 < \zeta \leq p$.

The Caputo operator D^{ζ} satisfies the following properties for $p-1 < \zeta \leq p$, $p \in \mathbf{N}$:

$$D^{\zeta} b = 0, \quad \text{for any constant } b, \tag{28}$$

$$D^{\zeta} \chi^{\kappa} = \begin{cases} 0, & \text{if } \kappa \in \mathbf{N}_0 \text{ and } \kappa < \lceil \zeta \rceil, \\ \frac{\kappa!}{\Gamma(\kappa-\zeta+1)} \chi^{\kappa-\zeta}, & \text{if } \kappa \in \mathbf{N}_0 \text{ and } \kappa \geq \lceil \zeta \rceil, \end{cases} \tag{29}$$

where $\mathbf{N} = \{1, 2, 3, \ldots\}$, $\mathbf{N}_0 = \{0\} \cup \mathbf{N}$, and $\lceil \zeta \rceil$ denotes the ceiling function.

Theorem 3.4. *The following formula holds for all $\lambda \in (0,1)$:*

$$\begin{aligned} D^{\lambda} \rho_j(x) =& x^{-\lambda} \left(\sum_{k=1}^{j+2} \sum_{p=k}^{j+2} \frac{p! A_{p,j+2}\, B_{k,p}}{(p-\lambda)!}\, T_k^*(x) + \frac{(-1)^j \Gamma(1-\lambda)}{\Gamma(-j-\lambda-1)\Gamma(j-\lambda+3)} \right. \\ & \left. + \frac{(\lambda-1)(-1)^j - 2(j+2)^2\, x}{\Gamma(2-\lambda)} \right), \end{aligned} \tag{30}$$

where $A_{i,j}$ and $B_{i,j}$ are defined respectively in (6) *and* (10)*.*

Proof. Based on relation (5), one gets

$$\rho_j(x) = \sum_{k=0}^{j+2} A_{k,j}\, x^k - 2(j+2)^2 \left(\frac{(-1)^j + 1}{\alpha} + x \right) - (-1)^j. \tag{31}$$

Now, the application of Caputo fractional derivative (27), we get

$$D^{\lambda} \rho_j(x) = x^{-\lambda} \left(\sum_{k=1}^{j+2} \frac{A_{k,j}\, k!}{\Gamma(k-\lambda+1)}\, x^k - \frac{2\,(j+2)^2\, x}{(1-\lambda)!} \right). \tag{32}$$

The previous equation can be rewritten after using (9) as

$$D^{\lambda} \rho_j(x) = x^{-\lambda} \left(\sum_{k=1}^{j+2} \sum_{p=1}^{k} \frac{k! A_{k,j+2}\, B_{p,k}}{(k-\lambda)!}\, T_p^*(x) + \sum_{k=1}^{j+2} \frac{k!\, 4^{-k}\, (2k)!\, A_{k,j+2}}{(k!)^2 (k-\lambda)!} - \frac{2\,(j+2)^2\, x}{(1-\lambda)!} \right). \tag{33}$$

The previous equation can be rewritten after rearranging the terms as

$$D^{\lambda} \rho_j(x) = x^{-\lambda} \left(\sum_{k=1}^{j+2} \sum_{p=k}^{j+2} \frac{p! A_{p,j+2} B_{k,p}}{(p-\lambda)!}\, T_k^*(x) + \sum_{k=1}^{j+2} \frac{k!\, 4^{-k} (2k)!\, A_{k,j+2}}{(k!)^2 (k-\lambda)!} - \frac{2\,(j+2)^2\, x}{(1-\lambda)!} \right). \tag{34}$$

Now, $\sum_{k=1}^{j+2} \frac{k!\,4^{-k}(2k)!\,A_{k,j+2}}{(k!)^2(k-\lambda)!}$ can be summed and simplified as

$$\sum_{k=1}^{j+2} \frac{k!\,4^{-k}(2k)!\,A_{k,j+2}}{(k!)^2(k-\lambda)!} = \frac{(-1)^j \left(\frac{\Gamma(1-\lambda)^2}{\Gamma(-j-\lambda-1)\Gamma(j-\lambda+3)} - 1\right)}{\Gamma(1-\lambda)}. \tag{35}$$

Therefore, we get the following relation after inserting Equation (35) into Equation (34)

$$\begin{aligned} D^{\lambda}\,\rho_j(x) =& x^{-\lambda}\left(\sum_{k=1}^{j+2}\sum_{p=k}^{j+2} \frac{p! A_{p,j+2}\,B_{k,p}}{(p-\lambda)!}\,T_k^*(x) + \frac{(-1)^j\Gamma(1-\lambda)}{\Gamma(-j-\lambda-1)\Gamma(j-\lambda+3)}\right.\\ &\left.+\frac{(\lambda-1)(-1)^j - 2(j+2)^2\,x}{\Gamma(2-\lambda)}\right), \end{aligned} \tag{36}$$

this completes the proof of this theorem. ■

Theorem 3.5. *The following formula holds for $\nu \in (1, 2)$:*

$$\begin{aligned} D^{\nu}\,\rho_j(x) =& x^{-\nu}\left(\sum_{k=1}^{j+2}\sum_{p=k}^{j+2} \frac{p! A_{p,j+2}\,B_{k,p}}{(p-\nu)!} T_k^*(x)\right.\\ &\left.+(-1)^j\left(\frac{\Gamma(1-\nu)}{\Gamma(-j-\nu-1)\Gamma(j-\nu+3)} + \frac{\nu+2(j+2)^2x-1}{\Gamma(2-\nu)}\right)\right). \end{aligned} \tag{37}$$

Proof. Using the same steps as in Theorem 3.4, the proof of this theorem is readily obtained. ■

Based on Theorems 3.4 and 3.5 and (23), the residual of Equation (12) when $1 < \nu < 2$ and $0 < \lambda < 1$ can be written as

$$\begin{aligned} R(x) =& D^{\nu}\,u_M - \alpha D^{\lambda}\,u_M + \alpha\,\eta\,(\chi - u_M)\,e^{u_M}\\ &= x^{-\nu}\sum_{j=0}^{M} c_j\left(\sum_{k=1}^{j+2}\sum_{p=k}^{j+2} \frac{p! A_{p,j+2}\,B_{k,p}}{(p-\nu)!} T_k^*(x)\right.\\ &\left.+(-1)^j\left(\frac{\Gamma(1-\nu)}{\Gamma(-j-\nu-1)\Gamma(j-\nu+3)} + \frac{\nu+2(j+2)^2x-1}{\Gamma(2-\nu)}\right)\right)\\ &-\alpha\,x^{-\lambda}\sum_{j=0}^{M} c_j\left(\sum_{k=1}^{j+2}\sum_{p=k}^{j+2} \frac{p! A_{p,j+2}\,B_{k,p}}{(p-\lambda)!}\,T_k^*(x) + \frac{(-1)^j\Gamma(1-\lambda)}{\Gamma(-j-\lambda-1)\Gamma(j-\lambda+3)}\right.\\ &\left.+\frac{(\lambda-1)(-1)^j - 2(j+2)^2\,x}{\Gamma(2-\lambda)}\right) + \alpha\,\eta\left(\chi - \sum_{j=0}^{M} c_j\,\rho_j(x)\right) e^{\sum_{j=0}^{M} c_j\,\rho_j(x)}. \end{aligned} \tag{38}$$

We may now enforce Equation (38) at $(M-2)$ points with conditions Equation (25) by applying the collocation method, which results in $(M+1)$ nonlinear equations that can be solved iteratively using Newton's method.

4 Error analysis

Theorem 4.1. *Suppose that* $\frac{d^i u_M(x)}{dx^i} \in \mathcal{C}([0,1])$ *for* $i = 0, 1, 2, \ldots, M+2$*, where* $u_M(x)$ *denotes the approximate solution. Define*

$$\varrho_M = \sup_{x\in[0,1]} \left| \frac{d^{M+3}u(x)}{dx^{M+3}} \right|.$$

Then, the following inequality holds:

$$\|u(x) - u_M(x)\|_2 \leq \frac{\varrho_M}{\sqrt{2M+7}\,(M+3)!}.$$

Proof. Consider the following Taylor polynomial expansion of $u(x)$ around $x = 0$:

$$\mathcal{A}_M(x) = \sum_{i=0}^{M+2} \left(\frac{d^i\, u(x)}{d\,x^i} \right)_{x=0} \frac{x^i}{i!}, \tag{39}$$

with the remainder given by

$$u(x) - \mathcal{A}_M(x) = \frac{x^{M+3}}{(M+3)!} \left(\frac{d^{M+3}\, u(x)}{d\,x^{M+3}} \right)_{x=c}, \quad c \in [0,1].$$

Since $u_M(x)$ is the optimal approximation of $u(x)$, it follows that

$$\begin{aligned} \|u(x) - u_M(x)\|_2^2 &\leq \|u(x) - \mathcal{A}_M(x)\|_2^2 \\ &\leq \int_0^1 \frac{\varrho_M^2\, x^{2(M+3)}}{((M+3)!)^2}\, dx \\ &= \frac{\varrho_M^2}{(2M+7)\,((M+3)!)^2}, \end{aligned} \tag{40}$$

which yields

$$\|u(x) - u_M(x)\|_2 \leq \frac{\varrho_M}{\sqrt{2M+7}(M+3)!}.$$

■

5 Illustrative examples

The nonlinear multi-order fractional issue is illustrated numerically in this section. Due to the non availability of the exact solution for (12), we instead consider the error remainder function

$$RE = \left| D^{\nu}\, u_M - \alpha D^{\lambda}\, u_M + \alpha\, \eta\, (\chi - u_M)\, e^{u_M} \right|, \tag{41}$$

Example 5.1. Consider Equation (12) subject to conditions (13), this equation is solved at different values of ν, λ, η, χ and α as follows:

- At $\nu = 2$, $\lambda = 1$, $\eta = 0.7$, $\chi = 0.8$, $\alpha = 5$. Table 1 presents a comparison of numerical values of u between our method and Laguerre collocation technique (LCT) [45] at $M = 7$. Table 2 presents the approximate solution and RE at different values of x when $M = 24$. Figure 1 shows the RE at different values of M. Also, Figure 2 shows the $Log_{10}RE$ at different values of M. Figure 3 shows the stability $|u_{M+1}(x) - u_M(x)|$ of our method at different values of M. Finally, Figure 4 shows the approximate solution (left) and RE (right) at $M = 24$.

Table 1: Comparison of numerical values of u at $M = 7$.

x	LCT [45]	proposed method	Our CPU time
0.0	0.10171	0.101646	
0.1	0.151679	0.151608	
0.2	0.199758	0.199691	
0.3	0.245711	0.245648	
0.4	0.289255	0.289195	
0.5	0.330028	0.329973	1.203
0.6	0.367514	0.367466	
0.7	0.4009	0.40086	
0.8	0.428821	0.428793	
0.9	0.448906	0.448898	
1.0	0.456985	0.456998	

Table 2: The approximate solution and RE at $M = 24$.

x	Proposed method	RE	CPU time
0.0	0.101646	1.94289×10^{-15}	
0.1	0.151608	1.11022×10^{-15}	
0.2	0.199691	1.63758×10^{-15}	
0.3	0.245648	3.05311×10^{-16}	
0.4	0.289196	4.44089×10^{-16}	
0.5	0.329974	1.11022×10^{-16}	1.299
0.6	0.367467	7.77156×10^{-16}	
0.7	0.400861	3.38618×10^{-15}	
0.8	0.428796	7.66054×10^{-16}	
0.9	0.448903	8.10463×10^{-16}	
1.0	0.457005	1.42109×10^{-14}	

- At $\nu = 1.9$, $\alpha = 0.05$, $\eta = 0.5$, and $\chi = 0.6$. Figure 5 shows the approximate solutions (left) and it's zoom in the interval $[0.2, 0.4]$ (right)at different values of λ when $M = 6$.

- At $\lambda = 0.8$, $\alpha = 2$, $\eta = 0.5$ and $\chi = 0.6$. Figure 6 shows the approximate solutions at different values of ν when $M = 6$.

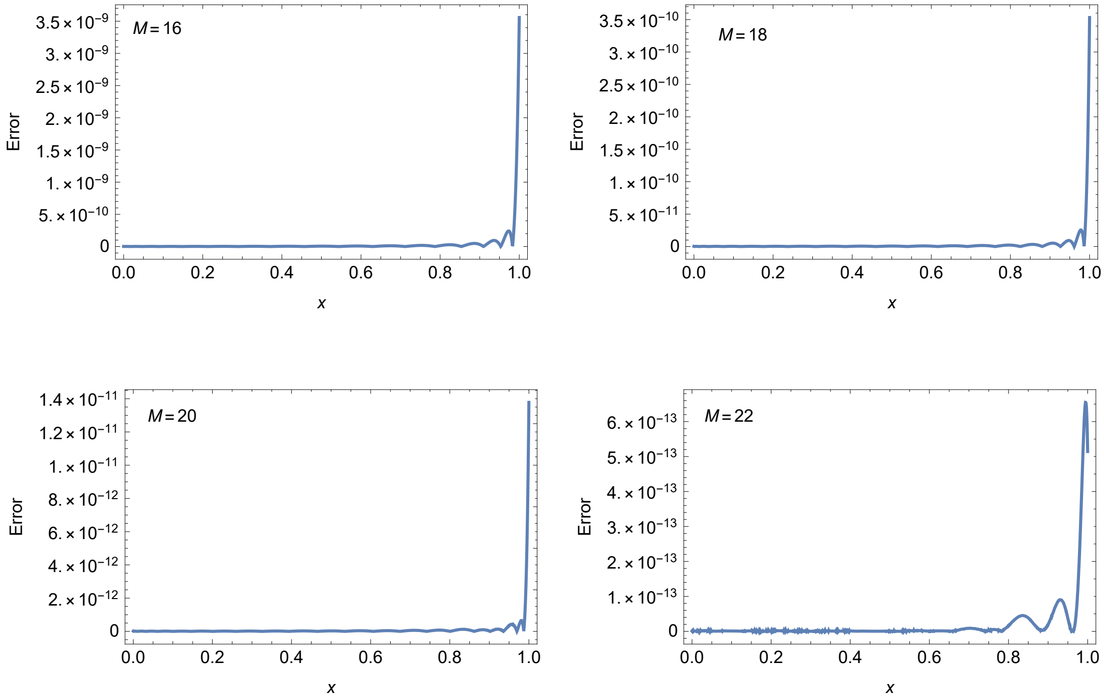


Figure 1: The RE at different values of M.

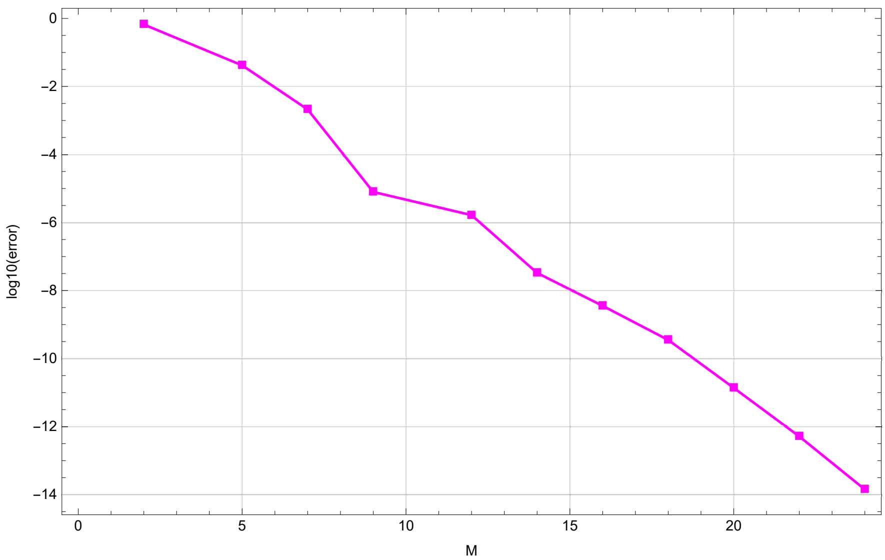


Figure 2: The $Log_{10}RE$ at different values of M.

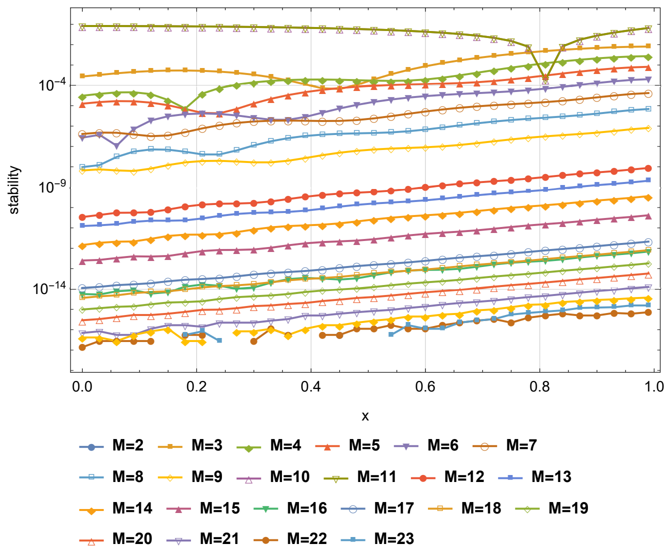


Figure 3: Stability $|u_{M+1}(x) - u_M(x)|$.

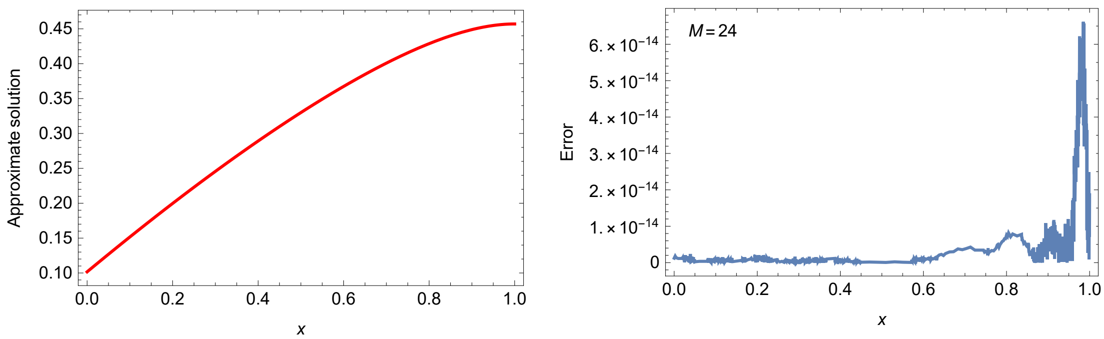


Figure 4: The approximate solution (left) and RE (right) at $M = 24$.

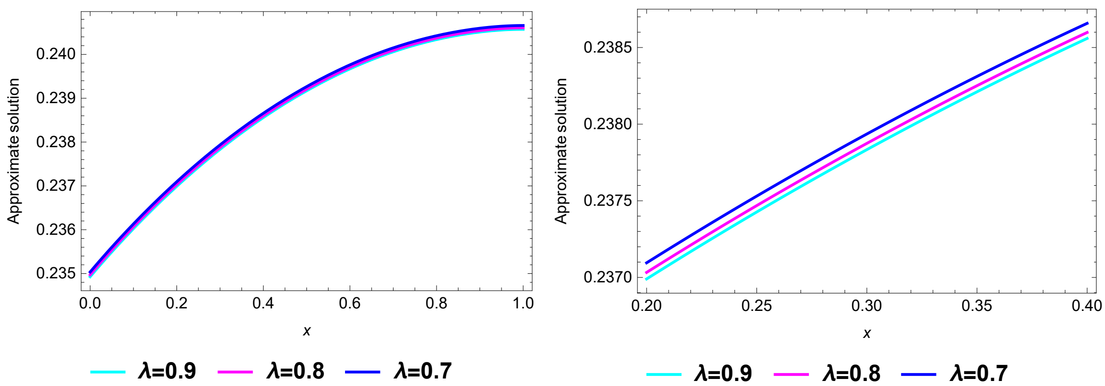


Figure 5: The approximate solutions (left) and it's zoom in the interval $[0.2, 0.4]$ (right)at different values of λ when $M = 6$.

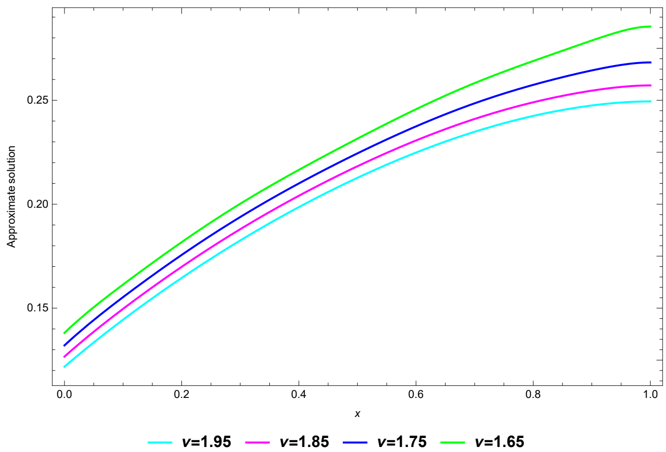


Figure 6: The approximate solutions at different values of ν when $M = 6$.

Example 5.2. Consider the modified nonlinear multi-order fractional version

$$D^{\nu}\, u(x) - \alpha D^{\lambda}\, u(x) + \alpha\, \eta\, \big(\chi - u(x)\big)\, e^{u(x)} = f(x), \qquad 0 < x < 1, \tag{42}$$

subject to

$$u'(0) = \alpha\, u(0), \quad u'(1) = 0, \tag{43}$$

where $f(x)$ is chosen such that the exact solution of this problem is $u(x) = x^5\,(1-x)^2$. Figure 7 shows the comparability of analytic and approximate solution at $M = 5$ when $\nu = 2$, $\lambda = 1$, $\eta = 0.7$, $\chi = 0.8$ and $\alpha = 5$. Figure 8 shows the absolute errors at different values of λ when $\nu = 1.9$, $\alpha = 0.05$, $\eta = 0.5$, $\chi = 0.6$ at $M = 5$.

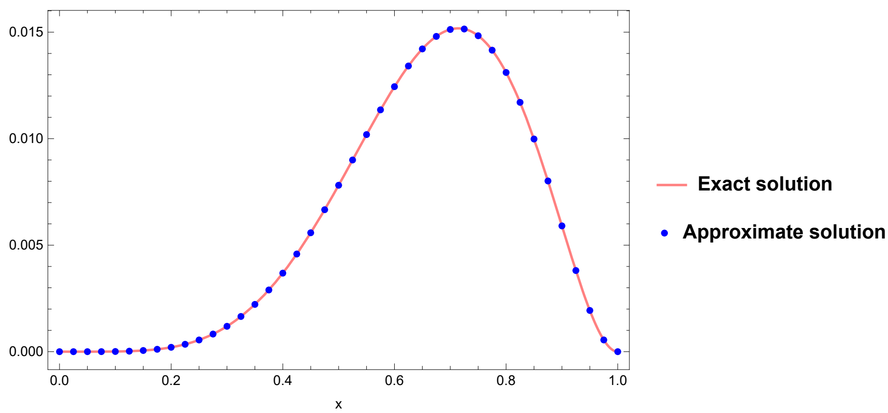
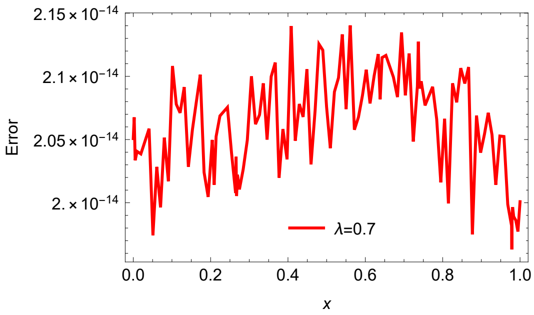


Figure 7: Comparability of analytic and approximate solution at $M = 5$.



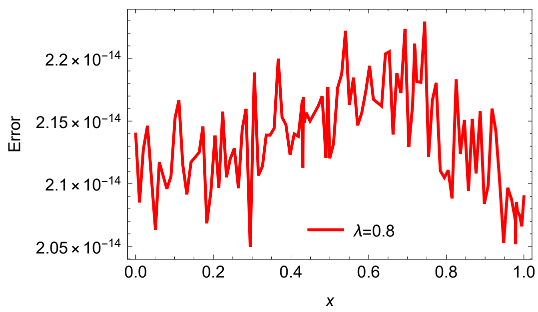


Figure 8: The absolute errors at $M = 5$.

6 Closing remarks

A numerical approach has been proposed and implemented for the treatment of nonlinear multi-order fractional differential equations using a collocation technique based on first-kind Chebyshev polynomials. The suggested method relies on constructing appropriate basis functions that satisfy the boundary conditions and using operational matrices for both integer and fractional derivatives. Two distinct cases have been studied: the classical case ($\nu = 2, \lambda = 1$) and the general fractional-order case ($1 < \nu \leq 2, 0 < \lambda \leq 1$). In both settings, the method demonstrates high accuracy and excellent stability. The use of Chebyshev roots as collocation points enhances the convergence behavior of the numerical scheme. As an expected future work, we aim to employ the developed theoretical results in this paper along with suitable spectral methods to treat some other problems. All codes were written and debugged by *Mathematica* 11 on HP Z420 Workstation, Processor: Intel(R) Xeon(R) CPU E5-1620 v2 - 3.70GHz, 16 GB Ram DDR3, and 512 GB storage.

Moreover, comparisons with existing methods such as the Laguerre collocation technique confirm the superiority of the proposed approach in terms of precision and computational efficiency. Future extensions may include multi-dimensional problems or fractional partial differential equations using the same spectral framework.

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