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Research Paper

Fractional-Order Pell Hybrid Functions for Solving Variable-Order Fractional Differential Equations and their Application in Chemistry

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Abstract

Herein, we introduce a novel computational approach to find a solution of variable-order fractional differential equations (VO-FDE). The desired method is proposed by combining fractional-order Pell functions, the Ritz and collocation methods. First, we define a new set of hybrid functions named fractional-order Pell hybrid functions. Next, to approximate the solution of VO-FDEs, we obtain an extra pseudo-operational matrix of the Caputo variable-order derivative. Then, by using the Ritz method, the operational matrix approach, and the collocation method, the problem is transformed to a system of algebraic equations, which is solved by Newton's iterative method. The error estimation of the proposed method is also examined. Finally, some examples (especially in chemistry) are presented to illustrate the effectiveness of this method.

1 Introduction

Studies in fractional differential equations (FDEs) date back over three hundred years, and many great mathematicians like Liouville, Riemann, Abel, Riesz, Weyl, and Caputo contributed to the development of fractional calculus. Since many real-world phenomena can be better described using fractional operators and their wide range of uses in engineering and science, including dynamical systems [1], disease dynamics [2], biology [3], optimal control [4], and biological systems [5], there has been a substantial increase in research in this field. Although fractional operators can solve certain physical problems, they fall short in representing key types of physical phenomena where the order depends on either dependent or independent variables. Hence,

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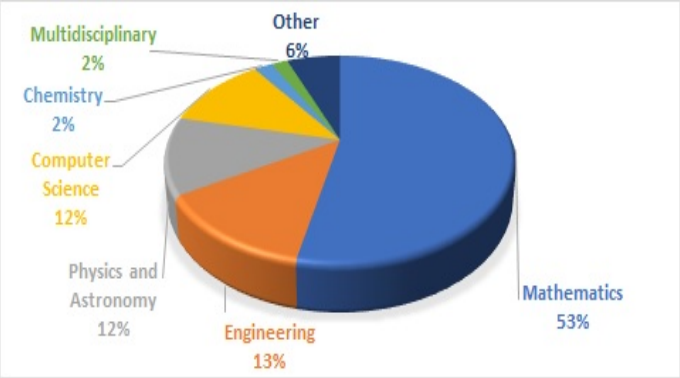

Figure 1: The subject area of the considered problems in Scopus.

there is a class of problems that variable-order (VO) operators would better describe. There are some studies on analytical solutions of VO-FDEs [6]. The important point is that, in most cases, the exact solution of VO-FDEs is unavailable. Therefore, developing accurate numerical methods for solving these equations is an important area of research in engineering science and mathematics. There are numerous approximate methods, each with unique characteristics and varying levels of effectiveness. For example, solving VO-FDEs by Legendre wavelets [7, 8], the finite difference method [9], Chelyshkov wavelets [10], a method based on Lagrange polynomials [11], Gegenbauer wavelets [12], a method based on shifted Legendre cardinal functions [13], generalized shifted Chebyshev polynomial operational matrix [14], spectral element method [15], and so on.

Hybrid basis functions, which involve combining block-pulse functions with certain special polynomials, have been utilized as a powerful mathematical technique for solving a wide range of problems. Over time, these fundamental functions have been used more and more by researchers to address fractional models, and several studies have explored their effectiveness in this area (see, for instance, [16–19]). By changing x to $x^{\alpha}(\alpha > 0)$, fractional order hybrid functions have been introduced.

In this work, we introduce fractional-order Pell hybrid functions (FOPHFs) for solving VO-FDEs. These functions have one more free parameter (α) compared to the classical hybrid functions. We define a new VO extra pseudo-operational matrix for the mentioned functions. The mentioned matrix is computed exactly, which has a direct impact on the method's accuracy. The suggested method in this study is designed based on the mentioned new achievements with the help of Ritz and collocation methods. In recent years, as a result of the wide and diverse applications of VO-FDEs, attention to them has increased significantly. A bibliometric review reveals a growing interest in this field, with a notable rise in publications and citations, reflecting the expanding research efforts surrounding this topic. The data used in this analysis were gathered from documents from the Scopus database covering publications up to 22nd September 2024. We utilize (TITLE ("differential equation")) AND ((TITLE ("variable-order") OR TITLE ("variable order"))), as the main keyword, for searching in the Scopus database. The total number of resulting publications was 238 documents. Figure 1 displays the distribution of publications on VO-FDE across various subject areas in Scopus. As shown, the models are utilized across various fields including mathematics, engineering, computer science, chemistry, and more. Moreover, Figure 2 illustrates the annual number of documents in the considered keyword in Scopus. The distribution of related publications by country is presented

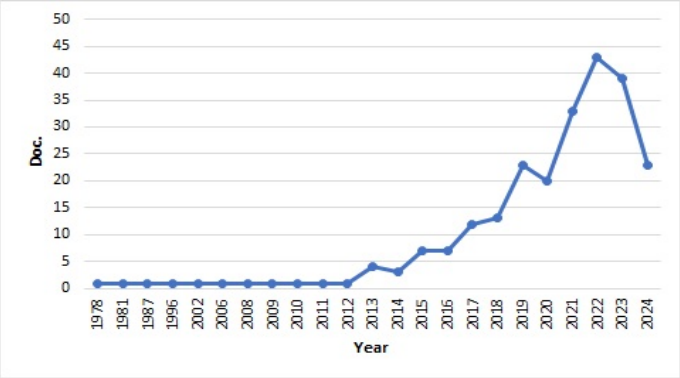

Figure 2: The number of publications each year since 1987.

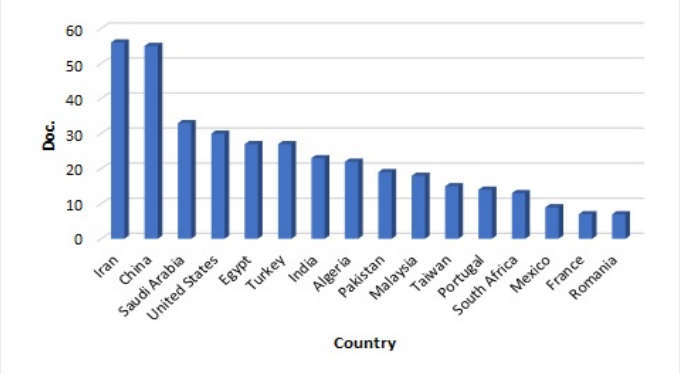

Figure 3: Bar chart showing the distribution of related publications by country in the considered period.

in Figure 3, highlighting the global engagement in this research area.

In this study, the general form of the VO-FDEs is considered as follows [20]:

$$D^{q(x)}y(x) = F(x, y(x), D^{q_1(x)}y(x), D^{q_2(x)}y(x), ..., D^{q_k(x)}y(x)), \quad x \in [0, h], \tag{1}$$

with the following conditions:

$$y^{(i)}(0) = \rho_i, \qquad i = 0, 1, ..., \eta - 1, \quad \eta - 1 \in N,$$

where $\eta - 1 < q(x) \leq \eta, \quad q(x) > q_1(x) > q_2(x) > ... > q_k(x) > 0.$

1.1 Application

Since VO-FDEs can model complex systems with memory effects and varying dynamics, they are utilized across numerous fields for various applications, including chemistry [21–23]. The use of VO-FDEs in chemistry is valuable for explaining phenomena in which the level of differentiation can vary across time, space, or other factors. For example, the relaxation mechanism

in the reaction kinetics of proteins is a temperature-dependent fractional order equation that can model the relaxation mechanism [24]. Therefore, it is justifiable to assume that a differential equation with operators whose order varies with temperature will offer a more precise representation of protein kinetics. Also, reaction-diffusion equations describe processes where chemical substances undergo reactions and diffuse through space, leading to patterns like waves or oscillations [25]. By using VO-FDEs, more complex and non-local temporal and spatial dynamics can be captured in these equations. In [8], Hosseininia and Heydari introduced the Legendre wavelets approach for the numerical solution of the nonlinear variable-order time fractional 2D reaction-diffusion equation. Rajaraman solved integer and fractional order models of reaction-diffusion in enzyme-immobilized systems using the Lucas wavelet method, as described in [26].

There are also some fractional differential equations (or VO-FDEs) that simulate natural phenomena. For example, the Bagley–Torvik equation (BTE) (see Example 7.6), which was introduced by Bagley and Torvik [27] to study the viscoelastically damped structure. BTE plays an important role in numerous applied science and engineering problems. Specifically, any linearly damped fractional oscillator with a damping term that includes a fractional derivative of order 1.5 can be described by the BTE. In particular, equations with half-order or one-and-a-half-order derivatives can model materials with frequency-dependent damping. It can also describe the motion of a rigid plate immersed in a viscous fluid and the motion of a gas in a fluid, respectively [28]. It is worth noting that several methods, including analytical techniques, decomposition methods, and numerical approaches with operational matrices, have been used to solve BTE [29].

Given the significance of fractional differential equations in various applications, the fractional form of Bratu's equation represents another important model. (see Example 7.7). The Bratu equation is a second-order nonlinear ordinary differential equation characterized by its exponential component, which makes it nonlinear and challenging to solve analytically [30]. The fractional Bratu's equation has been widely applied in various fields, including chemical reactor theory and nanotechnology. A specific model incorporating this equation is the solid fuel ignition model for thermal reaction processes [31]. By utilizing fractional derivatives, Bratu's equation enhances the accuracy of these models, particularly in describing complex dynamics such as radiative ignition and chemical reactions [32].

There are many other studies on the application of fractional calculus in chemistry, like addressing reaction mechanisms for potential glycerol hydrogenolysis pathways by modeling the kinetics of catalyzed hydrogenolysis utilizing FDEs [33], and drug release mechanisms [34]. These applications demonstrate how VO-FDEs can effectively represent the complexity of chemical systems better than traditional differential equations.

1.2 Paper outline

The structure of the paper is organized as follows: Section 2 provides essential definitions related to fractional calculus. In Section 3, we present the definition and key properties of Pell polynomials. Following this, we introduce Pell hybrid functions and fractional-order Pell hybrid functions. In Section 4, we construct an extra pseudo-operational matrix of variable order fractional derivative based on FOPHFs. The numerical method for solving the VO-FDEs is expressed in Section 5. We discuss the error analysis for the proposed method in Section 6. In Section 7, we present some test examples to show the efficiency of the proposed method. Finally, we present a brief conclusion.

2 Preliminaries

In this section, we outline the fundamental definitions and prerequisites that are utilized in this study.

Definition 2.1. The Caputo derivative of fractional variable order operator is defined as [35]

$$D^{q(x)}y(x) = \frac{1}{\Gamma(\eta - q(x))}\int_0^x \frac{y^{(\eta)}(s)}{(x-s)^{q(x)-\eta+1}}\,\mathrm{d}s, \quad x > 0, \tag{2}$$

where $\eta - 1 < q(x) \leq \eta$ and $\eta \in \mathbb{N}$. Moreover for $n > 0$, we get the following property [36]

$$D^{q(x)}x^n = \frac{\Gamma(n+1)}{\Gamma(n - q(x) + 1)}x^{n-q(x)}. \tag{3}$$

Definition 2.2. Riemann-Liouville variable-order fractional integral operator is defined as [37]

$${}_0I_x^{q(x)}y(x) = \frac{1}{\Gamma(q(x))}\int_0^x (x-s)^{q(x)-1}y(s)\mathrm{d}s, \qquad q(x) > 0. \tag{4}$$

Definition 2.3. (Generalized Taylor's formula) Suppose that $D^{i\alpha}y \in C(0,1]$ for $i = 0, 1, \ldots, m$. Then we have [38]:

$$y(x) = \sum_{i=0}^{m-1}\frac{x^{i\alpha}}{\Gamma(i\alpha+1)}D^{i\alpha}y\left(0^+\right) + \frac{x^{m\alpha}}{\Gamma(m\alpha+1)}D^{m\alpha}y(\mu), \tag{5}$$

where $0 < \mu \leq x$ for all $x \in [0,1)$. Additionally, we have:

$$|y(x) - \sum_{i=0}^{m-1}\frac{x^{i\alpha}}{\Gamma(i\alpha+1)}D^{i\alpha}y\left(0^+\right)| \leq M_\alpha\frac{x^{m\alpha}}{\Gamma(m\alpha+1)}, \tag{6}$$

with $M_\alpha \geq \sup_{\mu\in(0,1]}|D^{m\alpha}y(\mu)|$. It is important to note that when $\alpha = 1$, the generalized Taylor's formula reduces to the classical Taylor's formula.

3 Fractional-order Pell hybrid function

We introduce FOPHFs by using Pell polynomials and block-pulse functions. This section is divided into Pell polynomials, fractional-order Pell hybrid function, and Ritz-function approximation subsections.

3.1 Pell polynomials

Numerous polynomials are worth consideration, among which are the Pell polynomials. The Pell polynomials are defined by the recurrence relations as [39]

$$\begin{cases} P_m(x) = 2xP_{m-1}(x) + P_{m-2}(x), \\ P_0(x) = 0, \\ P_1(x) = 1, \quad m \geq 2. \end{cases} \tag{7}$$

We have the following property for $m \geq 0$ [40]

$$x^m = \left(\frac{1}{2}\right)^m \sum_{r=0}^{\lfloor\frac{m}{2}\rfloor}(-1)^r\binom{m}{r}\frac{m-2r+1}{m-r+1}P_{m+1-2r}(x). \tag{8}$$

The relationship between Pell polynomials and Fibonacci polynomials can be expressed through the equation

$$P_m(x) = F_m(2x). \tag{9}$$

Based on previous studies [41], Fibonacci polynomials have demonstrated several advantages over other well-known polynomials, particularly in numerical approximations. Since Pell polynomials are a special case of Fibonacci polynomials, as shown in Equation (9), so they inherit the key advantages of Fibonacci polynomials.
The explicit expression for the Pell polynomials is provided [42]

$$P_m(x) = \sum_{n=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-n-1}{n} (2x)^{m-2n-1}. \tag{10}$$

The fractional-order Pell polynomials can be defined by the change of variable x to x^α, $(0 < \alpha \leq 1)$, on the Pell polynomial. The fractional-order Pell polynomials are denoted by $P_m^\alpha(x)$.

3.2 Pell hybrid function

Considering the properties of Pell polynomials, we define Pell hybrid functions $\psi_{n,m}(x)$, $n = 1, 2, \ldots, N$, $m = 1, \ldots, M$ on the interval $[0, 1)$ as follows:

$$\psi_{nm}(x) = \begin{cases} P_m\left(Nx - n + 1\right), & x \in \left[\frac{n-1}{N}, \frac{n}{N}\right), \\ 0, & \text{otherwise}, \end{cases} \tag{11}$$

where n and m are the orders of the block-pulse functions and the Pell polynomials, respectively. To obtain the FOPHFs, we replace x with t^α in Pell hybrid function, where $0 < \alpha \leq 1$ is an additional parameter. So the FOPHF of order α, denoted by $\psi_{n,m}^\alpha(t)$, is defined on $[0, 1)$ as follows

$$\psi_{nm}^\alpha(t) = \begin{cases} P_m\left(Nt^\alpha - n + 1\right), & t^\alpha \in \left[\frac{n-1}{N}, \frac{n}{N}\right), \\ 0, & \text{otherwise}. \end{cases} \tag{12}$$

To extend the FOPHFs to the interval [0, h), we apply a variable transformation and set t by $\frac{x}{h}$ in Equation (12). This yields the following expression:

$$\psi_{nm}^\alpha(x) = \begin{cases} P_m\left(N(\frac{x}{h})^\alpha - n + 1\right), & x \in \left[h\left(\frac{n-1}{N}\right)^{\frac{1}{\alpha}}, h\left(\frac{n}{N}\right)^{\frac{1}{\alpha}}\right), \\ 0, & \text{otherwise}. \end{cases} \tag{13}$$

For instance, when $N = 2, M = 3$ and $\alpha = \frac{1}{2}$, on $[0, 1)$, the corresponding FOPHFs are given by:

$$\psi_{1,1}^{\frac{1}{2}}(x) = \begin{cases} 1, & 0 \leq x < \frac{1}{4} \\ 0, & \text{otherwise} \end{cases}, \quad \psi_{1,2}^{\frac{1}{2}}(x) = \begin{cases} 4\sqrt{x}, & 0 \leq x < \frac{1}{4} \\ 0, & \text{otherwise} \end{cases},$$

$$\psi_{1,3}^{\frac{1}{2}}(x) = \begin{cases} 1 + 16x, & 0 \leq x < \frac{1}{4}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\psi_{2,1}^{\frac{1}{2}}(x) = \begin{cases} 1, & \frac{1}{4} \leq x < 1 \\ 0, & \text{otherwise} \end{cases}, \quad \psi_{2,2}^{\frac{1}{2}}(x) = \begin{cases} -2 + 4\sqrt{x}, & \frac{1}{4} \leq x < 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$\psi_{2,3}^{\frac{1}{2}}(x) = \begin{cases} 1 + 4(-1 + 2\sqrt{x})^2, & \frac{1}{4} \leq x < 1, \\ 0, & \text{otherwise}. \end{cases}$$

Figure 4 shows graph of FOPHFs for $N = 2, M = 3$ and $\alpha = \frac{1}{2}$.

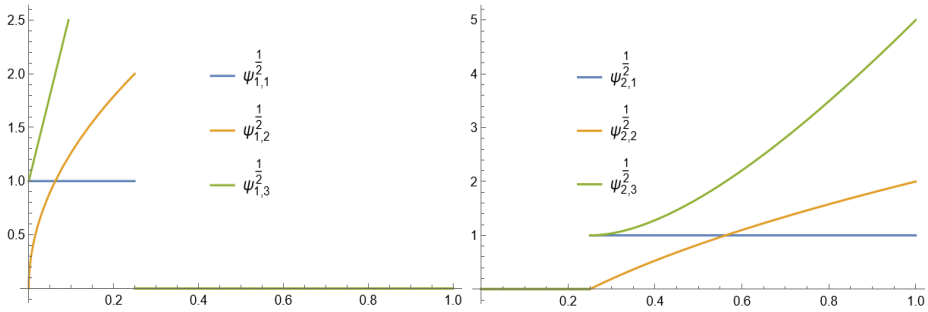

Figure 4: Graph of FOPHFs with $N = 2, M = 3$ and $\alpha = \frac{1}{2}$.

3.3 Ritz-function approximation

An arbitrary square-integral function $y(x)$ can be approximated using FOPHFs and the Ritz method, which is a well-established approach in numerical analysis (see, for example [43, 44]):

$$y(x) \simeq \sum_{n=1}^{N} \sum_{m=1}^{M} \omega(x) c_{n,m} \psi^{\alpha}_{n,m}(x) + \xi(x) = \omega(x) C^T \Psi^{\alpha}(x) + \xi(x), \tag{14}$$

where

$$\Psi^{\alpha}(x) = [\psi^{\alpha}_{1,1}, \ldots, \psi^{\alpha}_{1,M}, \ldots, \psi^{\alpha}_{N,1}, \ldots, \psi^{\alpha}_{N,M}]^T, \tag{15}$$

and

$$C^T = [c_{1,1}, \ldots, c_{1,M}, \ldots, c_{N,1}, \ldots, c_{N,M}]^T,$$

is an unknown coefficient vector. Note that $\xi(x)$ meets the given initial conditions, while $\omega(x)$ satisfies the homogenous initial conditions. These functions are not unique. So due to the considered problem in Equation (1), we choose $\xi(x)$ and $\omega(x)$ as:

$$\omega(x) = x^{\eta}, \qquad \xi(x) = \rho_{\eta-1} \frac{x^{\eta-1}}{(\eta-1)!} + \rho_{\eta-2} \frac{x^{\eta-2}}{(\eta-2)!} + \ldots + \rho_0.$$

4 Extra pseudo-operational matrix of variable order fractional derivative

In this part, we aim to introduce an additional extra pseudo-operational matrix of the variable-order fractional derivative. To achieve this, we first express the FOPHFs in terms of GFPTFs, obtaining the transformation matrix based on this relationship. Using this transformation matrix, we extract the extra pseudo-operational matrix of the variable-order fractional derivative for the GFPTFs and subsequently derive the corresponding matrix for the FOPHFs. Along the way, we outline the necessary prerequisites to finalize the construction of the desired matrix.

4.1 Generalized fractional piecewise Taylor functions

Generalized fractional piecewise Taylor functions (GFPTFs) are defined on the interval $[0, h)$ as the following formula:

$$\varphi^{\alpha}_{n,m}(x) = \begin{cases} x^{\alpha m}, & x \in \left[h(\frac{i-1}{N})^{\frac{1}{\alpha}}, h(\frac{i}{N})^{\frac{1}{\alpha}}\right), \\ 0, & \text{otherwise}, \end{cases} \tag{16}$$

where $n = 1, 2, \ldots, N$, and $m = 0, 1, \ldots, M-1$. Suppose that

$$\begin{aligned} \mathbf{\Phi}^{\alpha}(x) &= \left[\varphi^{\alpha}_{1,0}(x), \ldots, \varphi^{\alpha}_{1,M-1}(x), \ldots, \varphi^{\alpha}_{N,0}(x), \ldots, \varphi^{\alpha}_{N,M-1}(x)\right]^T \\ &= \left[\varphi^{\alpha}_{1}(x), \varphi^{\alpha}_{2}(x), \ldots, \varphi^{\alpha}_{N}(x)\right]^T, \end{aligned} \tag{17}$$

and $\mathrm{T}^{\alpha}(x) = \left[1, x^{\alpha}, \ldots, x^{\alpha(M-1)}\right]^T$. Also, we can obtain

$$\Phi^{\alpha}(x) = [\underbrace{\mathrm{T}^{\alpha}(x), \mathrm{T}^{\alpha}(x), \ldots, \mathrm{T}^{\alpha}(x)}_{N}]^T.$$

4.2 Transformation matrix of FOPHFs to GFPTFs

We can rewrite FOPHFs as GFPTFs by expanding it as follows:

$$\psi^{\alpha}_{i,j}(x) = \sum_{s=1}^{M} \sum_{k=1}^{N} \beta_{k,s} \varphi^{\alpha}_{k,s}(x). \tag{18}$$

So, we have here

$$\Psi^{\alpha}(x) = \mathcal{B}\Phi^{\alpha}(x), \tag{19}$$

which $\mathcal{B}$ is the transformation matrix of FOPHFs to GFPTFs. Also, we can write

$$\Phi^{\alpha}(x) = \mathcal{B}^{-1}\Psi^{\alpha}(x). \tag{20}$$

4.3 Extra pseudo-operational matrix

First, we propose an extra pseudo-operational matrix of variable order fractional derivative of GFPTFs as follows:

$$\begin{aligned} D^{q(x)}(x^{\eta}\mathrm{T}^{\alpha}(x)) &= \left[\frac{\Gamma(\eta+1)}{\Gamma(\eta+1-q(x))}x^{\eta-q(x)}, \ldots, \frac{\Gamma(\eta+\alpha(M-1)+1)}{\Gamma(\eta+\alpha(M-1)+1-q(x))}x^{\eta-q(x)+\alpha(M-1)}\right]^T \\ &= x^{\eta-q(x)} diag\left[\frac{\Gamma(\eta+1)}{\Gamma(\eta+1-q(x))}, \ldots, \frac{\Gamma(\eta+\alpha(M-1)+1)}{\Gamma(\eta+\alpha(M-1)+1-q(x))}\right]\mathrm{T}(x) \\ &= h^{\eta,\alpha}_{q(x)}\mathrm{T}(x). \end{aligned} \tag{21}$$

So for $\Phi^{\alpha}(x)$, we have

$$D^{q(x)}(x^{\eta}\Phi^{\alpha}(x)) = \mathrm{H}^{\eta,\alpha}_{q(x)}\Phi^{\alpha}(x), \tag{22}$$

where

$$\mathrm{H}^{\eta,\alpha}_{q(x)} = diag[\underbrace{h^{\eta,\alpha}_{q(x)}, h^{\eta,\alpha}_{q(x)}, \ldots, h^{\eta,\alpha}_{q(x)}}_{N}]^T.$$

Next, utilizing the previous subsection, we introduce an extra pseudo-operational matrix of variable order fractional derivative of FOPHFs. Using Equations (22), (19) and (20), we have

$$\begin{aligned} D^{q(x)}(x^{\eta}\Psi^{\alpha}(x)) &= D^{q(x)}(x^{\eta}\mathcal{B}\Phi^{\alpha}(x)) = \mathcal{B}D^{q(x)}(x^{\eta}\Phi^{\alpha}(x)) \\ &= \mathcal{B}\mathrm{H}^{\eta,\alpha}_{q(x)}\Phi^{\alpha}(x) = \mathcal{B}\mathrm{H}^{\eta,\alpha}_{q(x)}\mathcal{B}^{-1}\Psi^{\alpha}(x) \\ &= \mathcal{Q}^{\eta,\alpha}_{q(x)}\Psi^{\alpha}(x). \end{aligned} \tag{23}$$

It is worth noting that the fractional and integer derivatives are a special case of variable order derivative, and this extra pseudo-operational matrix is valid for them.

5 Numerical technique

We consider the problem outlined in Equation (1). As the first step, utilizing the Ritz-function approximation (mentioned in Equation (14)), we approximate the unknown function $y(x)$ in the following form:

$$y(x) \simeq \sum_{n=1}^{N}\sum_{m=1}^{M} x^{\eta} c_{n,m}\psi_{n,m}^{\alpha}(x) + \xi(x) = x^{\eta} C^{T}\Psi^{\alpha}(x) + \xi(x), \tag{24}$$

where

$$\xi(x) = \rho_{\eta-1}\frac{x^{\eta-1}}{(\eta-1)!} + \rho_{\eta-2}\frac{x^{\eta-2}}{(\eta-2)!} + ... + \rho_0,$$

and C is an unknown coefficients vector. Now, utilizing Equation (23) allows us to express

$$D^{q(x)}(y(x)) \simeq C^{T}\mathcal{Q}_{q(x)}^{\eta,\alpha}\Psi^{\alpha}(x) + \tilde{\xi}(x), \tag{25}$$

where

$$\tilde{\xi}(x) = D^{q(x)}(\xi(x)) = \sum_{k=1}^{\eta-1}\frac{\Gamma(k+1)}{\Gamma(k-q(x)+1)(k)!}\rho_k x^{k-q(x)}.$$

Then, we substitute Equations (24) and (25) in Equation (1). Next, By implementing the collocation method in the following points

$$x_i = h(\frac{i}{NM+1})^{\frac{1}{\alpha}}, \qquad i = 1, 2, \ldots, NM, \tag{26}$$

we derive a system of algebraic equations that can be resolved using Newton's iterative method, which ultimately determines the values of $c_{i,j}$ in Equation (24). However, applying Newton's method to such systems presents challenges, particularly in complex or high-dimensional problems. One of the primary difficulties is selecting an appropriate initial guess, as convergence significantly depends on this choice. A poor initial guess can lead to divergence or slow convergence of the method.
Additionally, the computational cost, including the number of iterations required, becomes critical as the problem complexity increases.

6 Error estimation

Theorem 6.1. *Suppose that $D^{i\alpha}y \in C(0,h]$ for $i = 0, 1, ..., M$ and $(2M+1)\alpha \geq 1$. Let $\tilde{m} = NM$, $\omega_M = span\{P_1^{\alpha}(x), P_2^{\alpha}(x), ..., P_M^{\alpha}(x)\}$ and $y_M(x) = A^{T}\mathcal{P}^{\alpha}(x)$, in which $\mathcal{P}^{\alpha}(x) = [P_1^{\alpha}(x), .., P_M^{\alpha}(x)]^{T}$ and A is the coefficient vector. If $y_M(x)$ is the best approximation to $y(x)$ out of ω_M on the interval $[\frac{n-1}{N}h, \frac{n}{N}h)$, then the error bound for the approximate solution $y_{\tilde{m}}(x)$ using the FOPHFs on the interval $[0,h)$ can be expressed as follows:*

$$\|y - y_{\tilde{m}}\|_2 \leq \frac{\sqrt{h^{(2M+1)\alpha}}}{\Gamma(M\alpha+1)\sqrt{(2M+1)\alpha}}\sup_{x\in[0,h)}\left|D^{M\alpha}y(x)\right|. \tag{27}$$

Proof. We define,

$$\hat{y}(x) = \sum_{i=0}^{M-1}\frac{x^{i\alpha}}{\Gamma(i\alpha+1)}D^{i\alpha}y\left(0^{+}\right). \tag{28}$$

According to the generalized Taylor's formula provided in Definition 2.3, we obtain:

$$|y(x)-\hat{y}(x)| \leq \frac{x^{M\alpha}}{\Gamma(M\alpha+1)} \sup_{x\in I_n} \left|D^{M\alpha}y(x)\right|, \tag{29}$$

where $I_n = [\frac{n-1}{N}h, \frac{n}{N}h)$. Given that $y_M(x) = A^T\mathcal{P}^\alpha(x)$ represents the best approximation to $y(x)$ out of ω_M on I_n, and $\sum_{i=0}^{M-1} \frac{x^{i\alpha}}{\Gamma(i\alpha+1)} D^{i\alpha}y\,(0^+) \in \omega_M$, we have the following relation:

$$\begin{aligned}
\|y-y_{\tilde{m}}\|^2_{L^2[0,h)} = \left\|y-C^T\Psi^\alpha\right\|^2_{L^2[0,h)} &= \sum_{n=1}^{N} \left\|y-A^T\mathcal{P}^\alpha\right\|^2_{L^2[\frac{n-1}{N}h,\frac{n}{N}h)} \leq \sum_{n=1}^{N} \|y-\hat{y}\|^2_{L^2[\frac{n-1}{N}h,\frac{n}{N}h)} \\
&\leq \sum_{n=1}^{N} \int_{I_n} \left[\frac{x^{M\alpha}}{\Gamma(M\alpha+1)} \sup_{x\in I_n} \left|D^{M\alpha}y(x)\right|\right]^2 x^{\alpha-1}dx \\
&\leq \int_0^h \left[\frac{x^{M\alpha}}{\Gamma(M\alpha+1)} \sup_{x\in[0,h)} \left|D^{M\alpha}y(x)\right|\right]^2 x^{\alpha-1}dx \\
&\leq \frac{h^{(2M+1)\alpha}}{\Gamma(M\alpha+1)^2(2M+1)\alpha} \left(\sup_{x\in[0,h)} \left|D^{M\alpha}y(x)\right|\right)^2.
\end{aligned} \tag{30}$$

The theorem is proved by taking the square roots. ■

7 Numerical examples

To demonstrate the effectiveness and precision of this method for solving variable-order fractional differential equations, we apply it to several examples and compare the numerical results with those from existing methods. The calculations for the tests were carried out using Mathematica 13. 1.

Example 7.1. Consider the VO-FDE given in [7] as:

$$\begin{gathered}
D^{q(x)}y(x) - 10y'(x) + y(x) = h(x), \quad x\in[0,1], \\
h(x) = 10\left(\frac{x^{2-q(x)}}{\Gamma(3-q(x))} + \frac{x^{1-q(x)}}{\Gamma(2-q(x))}\right) + 5x^2 - 90x - 95, \\
q(x) = \frac{x+2e^x}{7}, \\
y(0) = 5.
\end{gathered} \tag{31}$$

The exact solution is $y(x) = 5(1+x)^2$. We apply the present method with $\alpha = 1$, $N = 2$, and $M = 2$ for solving this problem. Then, according to the numerical technique elaborated in Section 5, we have:

$$y(x) \simeq xC^T\Psi^1(x) + 5, \tag{32}$$

By using Equation (25), we obtain

$$D^{q(x)}y(x) \simeq C^T\mathcal{Q}^{1,1}_{q(x)}\Psi^1(x), \tag{33}$$

and

$$y'(x) \simeq C^T\mathcal{Q}^{1,1}_1\Psi^1(x). \tag{34}$$

Table 1: Comparison of the absolute errors at some selected grid points in Example 7.1.

x	SLCF [13] M=2	LWM [7] k=2, M=4	Lagrange polynomials[11] M=2
0.2	1.77636×10^{-15}	8.091305×10^{-12}	2.66454×10^{-15}
0.4	1.77636×10^{-15}	2.024535×10^{-9}	3.55271×10^{-15}
0.6	1.77636×10^{-15}	9.564669×10^{-10}	1.77636×10^{-15}
0.8	0	1.696030×10^{-10}	3.55271×10^{-15}

Therefore, the problem can be transformed into the following linear algebraic equation

$$C^T \mathcal{Q}^{1,1}_{q(x)} \Psi^1(x) - 10 C^T \mathcal{Q}^{1,1}_1 \Psi^1(x) + (x C^T \Psi^1(x) + 5) = h(x). \tag{35}$$

Then, by using the collocation method, we get

$$c_{1,1} = 10, \quad c_{1,2} = \frac{5}{4}, \quad c_{2,1} = \frac{25}{2}, \quad c_{2,2} = \frac{5}{4}.$$

So we obtain $y(x) = 5(1+x)^2$, The exact solution of the problem. The exact solution was not found in the Legendre Wavelet method (LWM) [7], Bernoulli Wavelet method (BWM) [45], SLCF method [13] and method based on Lagrange polynomials[11]. The absolute error these mentioned methods are reported in Table 1.

Example 7.2. Consider the following nonlinear VO-FDE of the form [7]:

$$\begin{gathered} D^{q(x)} y(x) - 7y'(x) + 5y(x) - 6y''(x)y(x) = f(x), \quad x \in [0,1], \\ f(x) = 5\left(\frac{2}{\Gamma(3-q(x))} x^{2-q(x)} + \frac{3}{\Gamma(2-q(x))} x^{1-q(x)}\right) - 275x^2 - 895x - 105, \\ q(x) = \frac{3(\cos(x) + \sin(x))}{5}, \\ y(0) = 0. \end{gathered} \tag{36}$$

The analytic solution is $y(x) = 5(3x + x^2)$. Using the proposed technique, we numerically solve this problem with $\alpha = 1$, $N = 2$ and $M = 2$. Table 2 displays the comparisons between the absolute errors obtained by the proposed method, finite difference scheme (FDS) [7], and LWM [7]. The absolute error of the numerical solutions for this problem is plotted in Figure 5.

Example 7.3. Consider the nonlinear VO-FDE:

$$\begin{gathered} D^{q(x)} y(x) + \sin(x) y^2(x) = f(x), \quad 0 < q(x) \le 1, \\ f(x) = \frac{\Gamma\left(\frac{9}{2}\right) x^{\frac{7}{2} - q(x)}}{\Gamma\left(\frac{9}{2} - q(x)\right)} + \sin(x) x^7, \\ q(x) = 1 - 0.5e^{-x}, y(0) = 0. \end{gathered} \tag{37}$$

The exact solution for this problem is $y(x) = x^{\frac{7}{2}}$. Here we solve the given VO-FDE over $[0,1]$ with $N = 1$, $M = 6$, $\alpha = \frac{1}{2}$ and $N = 1$, $M = 6$, $\alpha = 1$. The graph of absolute error for $N = 1$, $M = 6$, $\alpha = \frac{1}{2}$ is displayed in Figure 7. In Table 3, we compare the absolute errors

Table 2: Comparison of the absolute errors at some selected grid points in Example 7.2.

x	LWM [7] k=2, M=4	FDS [7] M=10	Present method N=2, M=2
0.2	1.199041×10^{-14}	4.7×10^{-4}	0
0.4	1.421085×10^{-14}	5.765×10^{-3}	8.88178×10^{-16}
0.6	2.842171×10^{-14}	1.2471×10^{-2}	5.32907×10^{-15}
0.8	1.669775×10^{-13}	1.8643×10^{-2}	1.06581×10^{-14}
1	2.273737×10^{-13}	2.4094×10^{-2}	7.10543×10^{-15}

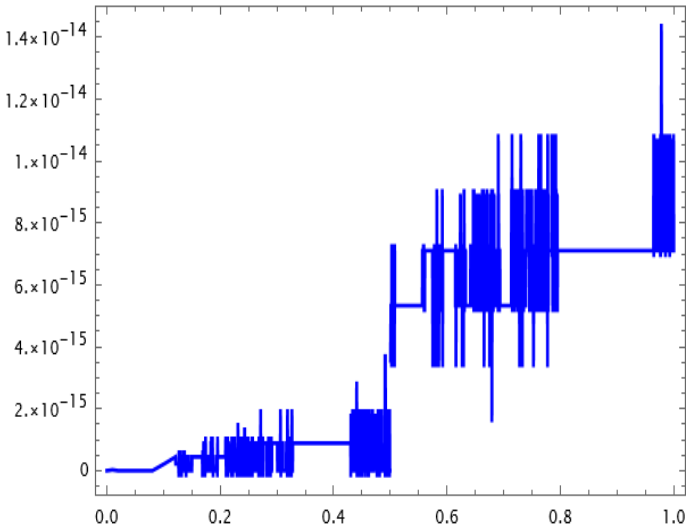

Figure 5: Absolute error of the present method for $N = 2, M = 2$, in Example 7.2.

Table 3: The absolute errors of Example 7.3.

x	Method in [11]		Our method	
	$M=6$	$M=10$	$\alpha=1, M=6$	$\alpha=1/2, M=6$
0.2	3.70929×10^{-6}	2.07663×10^{-7}	9.75393×10^{-6}	1.77809×10^{-17}
0.4	5.79180×10^{-6}	4.14451×10^{-8}	8.02079×10^{-6}	0
0.6	5.22717×10^{-6}	2.67798×10^{-7}	7.03476×10^{-6}	5.55112×10^{-17}
0.8	1.33024×10^{-5}	1.51750×10^{-8}	5.96705×10^{-6}	3.88578×10^{-16}
1.0	3.44242×10^{-5}	1.57873×10^{-5}	2.89527×10^{-5}	1.77636×10^{-15}

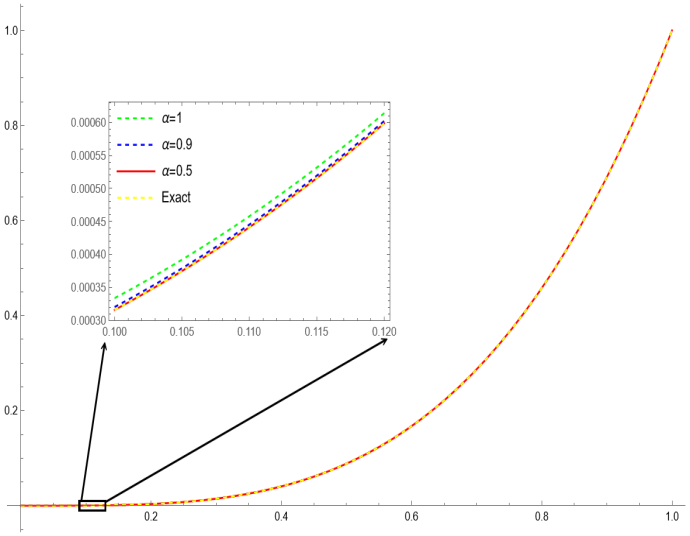

Figure 6: Plot of the exact solution and the approximate solutions with $N=1, M=6$ and different values of α in Example 7.3.

of our scheme and the method in [11]. It is important to mention that the current method produces more accurate approximations using the same number of basis functions compared to the approach proposed in [11]. Also, the approximate solution given by $N=1, M=6$ and $\alpha=0.5, 0.9, 1$, is plotted in Figure 6. The findings presented in this figure indicate that the method yields very close approximations across different values of α.

Example 7.4. Consider the following nonlinear VO-FDE [46]

$$\begin{gathered} D^{q(x)}y(x)+y(x)+\sqrt{x}y^2(x)=f(x), \quad x\in[0,1], \\ f(x)=x^{\lambda}\left(1+x^{\frac{1}{2}+\lambda}+\frac{\Gamma(1+\lambda)x^{-q(x)}}{\Gamma(1+\lambda-q(x))}\right), \\ q(x)=x\cos(x), \\ y(0)=0. \end{gathered} \tag{38}$$

The exact solution is $y(x)=x^{\lambda}$. The absolute errors of our results with $\alpha=1$, $N=1$, $M=6$; $\alpha=0.1$, $N=1$, $M=5$; and $N=1$, $M=6$ for $\lambda=1.2$ are displayed in Table 4. These results

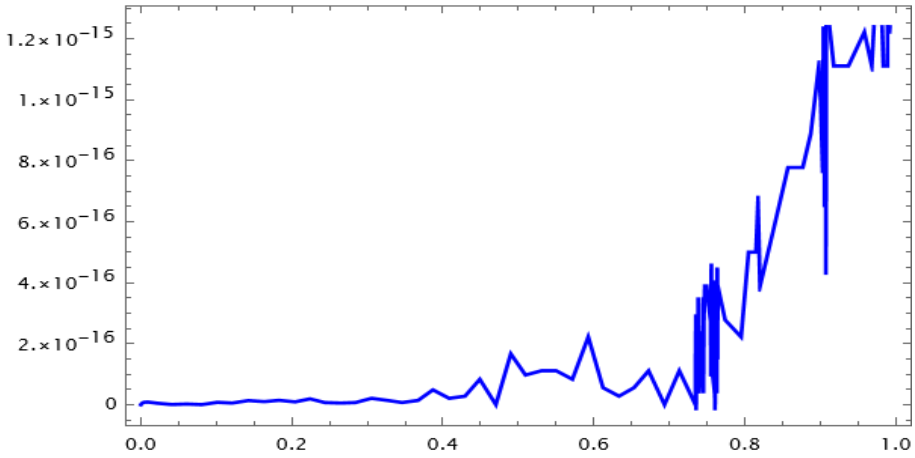

Figure 7: Absolute error of the present method for $N = 1$,$M = 6$, and $\alpha = \frac{1}{2}$ in Example 7.3.

Table 4: Comparison of the absolute error at selected points using the present method for Example 7.4.

x	$M = 6,\ \alpha = 1$	$M = 5,\ \alpha = 0.1$	$M = 6,\ \alpha = 0.1$
0	0	0	0
0.2	5.11506×10^{-5}	5.55112×10^{-17}	0
0.4	8.97153×10^{-5}	3.33067×10^{-16}	0
0.6	4.57737×10^{-5}	7.77156×10^{-16}	1.11022×10^{-16}
0.8	6.98252×10^{-5}	1.33227×10^{-15}	3.33067×10^{-16}

were not obtained in [13, 46, 47]. From Table 4, we can see that the current method provides more precise approximations compared to the other methods mentioned earlier. To examine the effect of varying α, we have used the method with $N = 1$, $M = 5$ and $\alpha = 0.1, 0.5, 0.7, 0.9, 1$. The approximate solutions are plotted in Figure 8.

Example 7.5. We consider the following VO-FDE [48]

$$D^{q(x)}y(x) + 3y'(x) - y(x) = f(x), \qquad x \in [0, 1], \tag{39}$$

where

$$f(x) = e^x \left(3 - \frac{\Gamma(1 - q(x), x)}{\Gamma(1 - q(x))}\right), \qquad q(x) = \frac{1 + \cos^2(x)}{4}, \qquad y(0) = 1.$$

The exact solution of the Equation (39) is $y(x) = e^x$. We applied the present method and the absolute errors of the numerical solution obtained by using $\alpha = 1$ and various values of N and M are shown in Table 5.
We also computed the L_2-norm of the errors for our approximations and compared them with the maximum values of the theoretical upper bound. Table 6 presents the results for different values of N and M.

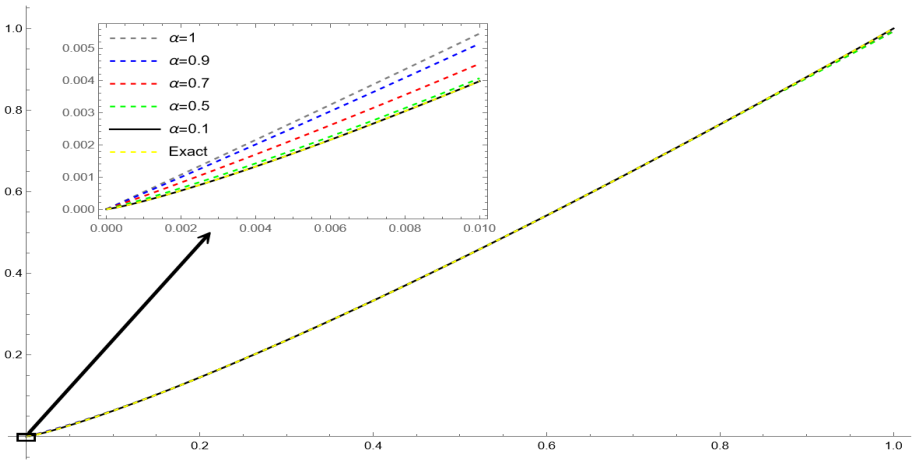

Figure 8: Plot of the approximate solutions with $N = 1$, $M = 5$ and different values of α together with the exact solution in Example 7.4.

Table 5: Comparison of the absolute error at specific points with the current method in Example 7.5.

x	$N = 1, M = 7$	$N = 2, M = 7$	$N = 1, M = 9$
0	0	0	0
0.2	2.53076×10^{-8}	1.36203×10^{-10}	4.05587×10^{-11}
0.4	2.51629×10^{-8}	1.33905×10^{-10}	4.04314×10^{-11}
0.6	2.55203×10^{-8}	3.10016×10^{-6}	4.10096×10^{-11}
0.8	2.6273×10^{-8}	3.1889×10^{-6}	4.20992×10^{-11}

Table 6: Comparison of L_2-norm of errors and theoretical upper bound for Example 7.5.

N	M	L_2-Norm of Error	Upper bound
2	7	2.24541×10^{-6}	1.39257×10^{-4}
1	9	4.00963×10^{-11}	1.71852×10^{-6}
1	12	8.93345×10^{-15}	1.13498×10^{-9}

Table 7: Comparison of the absolute errors of Example 7.6.

N, M	Present method	N	Method in [49]		
			$(\theta,\beta)=(0,1)$	$(\theta,\beta)=(2,4)$	$(\theta,\beta)=(3,6)$
$N=1, M=5$	3.87558×10^{-6}	5	8.318×10^{-3}	2.666×10^{-3}	1.231×10^{-3}
$N=1, M=10$	1.57818×10^{-11}	10	2.515×10^{-3}	3.854×10^{-6}	1.179×10^{-7}
$N=1, M=15$	5.63993×10^{-14}	15	1.771×10^{-4}	2.721×10^{-9}	7.242×10^{-11}

Example 7.6. Here, we present a variable-order (VO) version of the BTE, which, as discussed in the application section of this paper, see Subsection 1.1, plays an important role in modeling the motion of a rigid plate immersed in a viscous fluid, as well as the behavior of a gas in a fluid. Consider the following VO BTE [49]

$$D^{q(x)}y(x) = -y(x) - y''(x) + f(x), \qquad x \in [0,h], \tag{40}$$

with $y(0)=0$, $y'(0)=1$, $q(x)=\frac{9-\sin(x-10)}{5}$ and

$$f(x) = \frac{-1}{\Gamma(\frac{1-\sin(x-10)}{5})}\cdot\int_0^x \sin(t)(x-t)^{-\frac{\sin(x-10)+4}{5}}\,\mathrm{d}t.$$

The analytic solution is $y(x)=\sin(x)$. Here we solve the given VO-FDE over $[0,\frac{\pi}{2}]$ with $\alpha=1$ and $N=2$, $M=6$ and demonstrate the exact and approximate solutions and the graph of absolute errors in Figure 9 and Figure 10, respectively. Moreover in this example for comparing our numerical result with [49], we use ME, maximum absolute error, which is computed using the following formula

$$\max_{x\in[0,h]}|y(x)-\tilde{y}(x)|,$$

where the numerical solution is represented by $\tilde{y}(x)$. ME of the solutions obtained by using the present method and method in [49] are shown in Table 7. In Table 7 we let $\alpha=1$ for different value of N, M, and also θ and β are the parameters utilized in the definition of the Laguerre polynomials in [49].

Example 7.7. Consider the following fractional Bratu-type equation [50]:

$$D^{q(x)}y(x) - 2\exp(y(x)) = 0, \qquad 1 < q(x) \le 2, \quad 0 < x < 1, \tag{41}$$

The fractional Bratu equation is widely applied in chemical reactor theory and nanotechnology. The fractional Bratu equation enhances the solid fuel ignition model by accurately capturing complex dynamics like radiative ignition and chemical reactions (Further details and references can be found in the 1.1. The initial conditions of (41) are:

$$y(0) = y'(0) = 0.$$

For $q(x)=2$, the exact solution is $y(x)=-2\ln(\cos(x))$. By applying the present technique for $\alpha=N=1, M=9$, we solve the Equation (41) for different values of $q(x)$. The exact solution and approximate solutions for $q(x)=2, 1.95, 1.80, 1.65$, are presented in Figure 11.

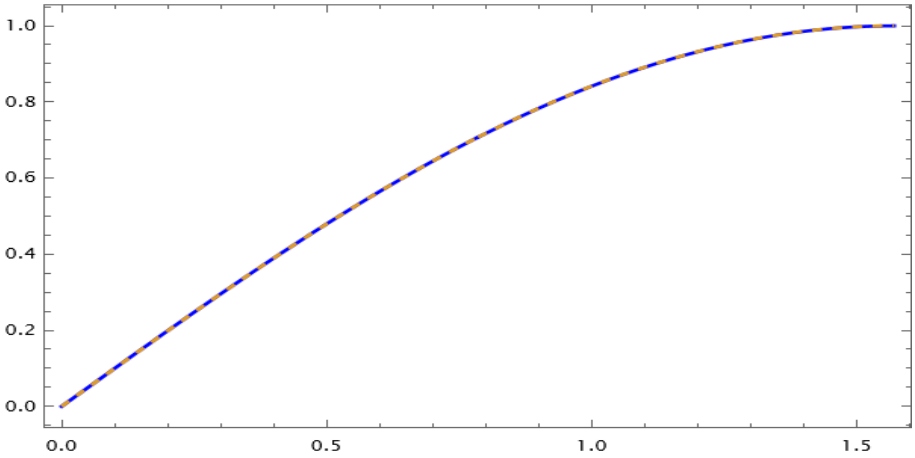

Figure 9: Exact and approximate solution for $N = 2$,$M = 6$, in Example 7.6.

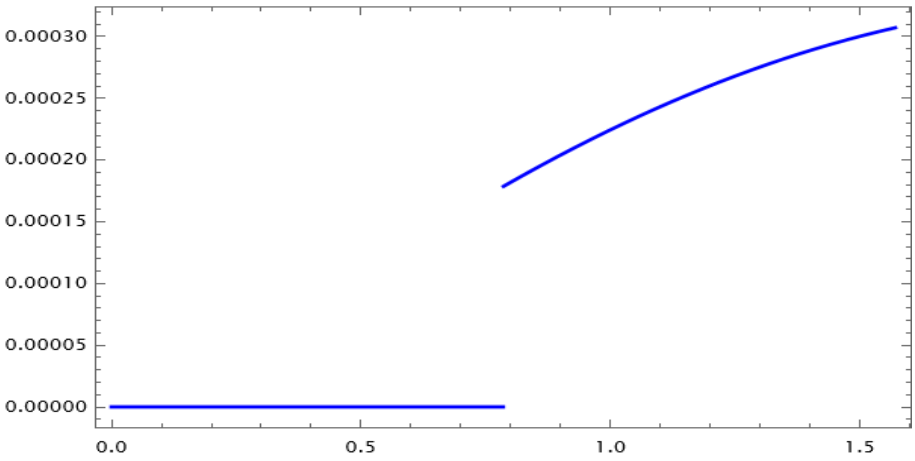

Figure 10: Absolute error for $N = 2$,$M = 6$, in Example 7.6.

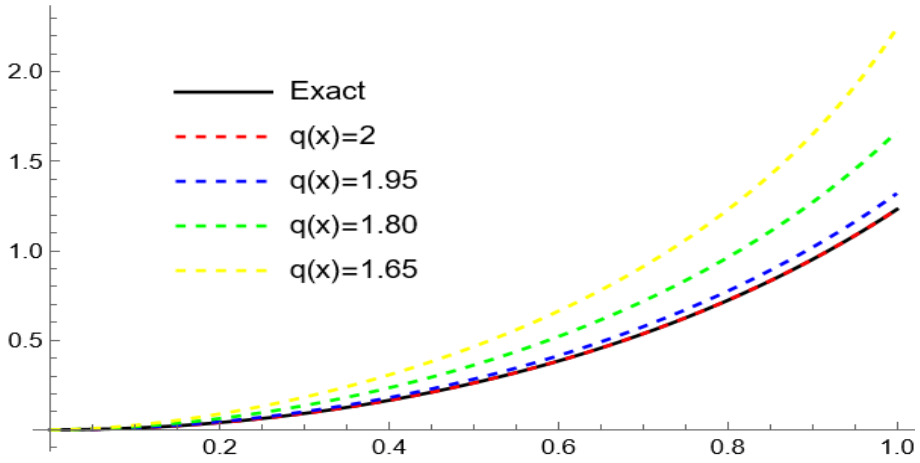

Figure 11: Comparison the exact solution for $q(x) = 2$ with the approximate solutions for the different values $q(x)$ for Example 7.7.

8 Conclusion

This work aims to develop an efficient and accurate method for solving VO-FDEs. To achieve this, we developed a novel method based on FOPHFs. We approximated the function using the Ritz method and obtained the extra pseudo-operational matrix of the variable order fractional derivative for FOPHF. Based on illustrative examples, the presented method was more accurate than the compared methods. Our numerical results show that we could obtain exact solutions for problems and get the solutions not given in some previous papers. The results of this study suggest several potential directions for future research. Extending the current method to fractional partial differential equations and multi-dimensional problems could significantly enhance its applicability. Applying the proposed technique to real-world problems would demonstrate its practical utility and validate its versatility. Addressing these challenges in future studies could further establish the importance of FOPHFs in solving complex fractional differential equations and contribute to advancements in numerical methods for fractional calculus.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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