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Research Paper

Local Metric Dimension of Some $(k, 6)$-Fullerenes

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Abstract

A $(k, 6)$-fullerene graph refers to a planar 3-connected cubic graph whose faces are k-gons and hexagons. The current study involves calculating the local metric dimension for specific $(k, 6)$-fullerene graphs, where k takes values in the set $\{3, 4, 5\}$.

1 Introduction

For a connected graph G, the *distance* $d(u, v)$ between two vertices u and v is defined as the shortest length of the paths connecting u and v in G. A graph G is considered 3-*connected* if the removal of any two distinct vertices u and v keeps the graph connected. In addition, a *planar graph* is a graph that can be represented in a plane without any edges intersecting. Furthermore, a graph that is 3-regular is often termed a *cubic graph*. We now introduce the concept of an $(r, 6)$-*fullerene* graphs.
An $(r, 6)$-fullerene graph is a planar cubic graph that is also 3-connected, with its faces consisting solely of r-gons and hexagons. Previous research has established that the only permissible values of k for which a $(k, 6)$-fullerene can exists are 3, 4, and 5, see [1]. According to Euler's formula, a $(3, 6)$-fullerene graph will have four triangular faces and $(\frac{n}{2} - 2)$ hexagonal faces. For further information regarding terminology and notations used for fullerenes, please consult [2–9].

Given a set $S = \{v_1, \ldots, v_k\} \subseteq V(G)$, the *metric S-code* associated with a vertex $v \in V(G)$ is expressed as the vector $r_S(v) = (d(v_1, v), \ldots, d(v_k, v))$. The set S is said to *distinguishes* the vertices u and v if $r_S(u) \neq r_S(v)$. Moreover, we designate S as a *local metric generator* (LMG) of G when it successfully distinguishes every pair of adjacent vertices in G.

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A *local metric basis* (LMB) represents an LMG with minimum cardinality. The concept of the *Local metric dimension* of G, denoted as $\dim_l(G)$, is defined as the minimum size of S across all LMGs. This concept was first introduced in [10]. In [11, 12] Fernau and Rodríguez-Velázquez showed that the decision version of local metric dimension is NP-complete. Recently, [13] applied certain graph products to ascertain or estimate the local metric dimension for various chemical graph classes; additionally, a novel methodology for coding customers in delivery services was presented utilizing the notion of local metric dimension. For an in-depth exploration of this subject, the reader is directed to [14–16].

To understand the motivation behind this exploration, we recommend examining [17]. This paper aims to investigate the local metric dimension of $(k, 6)$-fullerene graphs where $k \in \{3, 4, 5\}$.

2 Main results

In this section, we begin by presenting a theorem from [10] that pertains to the local metric dimension of bipartite graphs.

Theorem 2.1. *([10]). Let G be a connected graph with at least one vertex and of order n. Then $\dim_l(G) = n - 1$ if and only if G is a complete graph of order n, and $\dim_l(G) = 1$ if and only if G is a bipartite graph.*

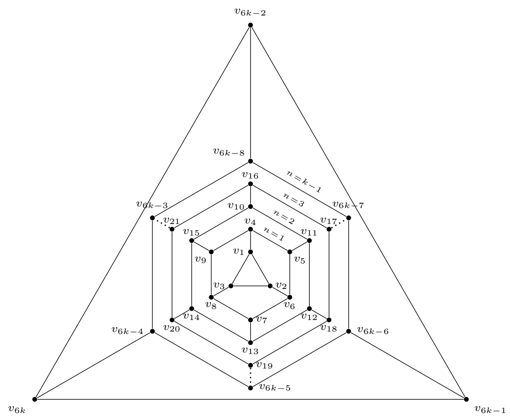

Figure 1: The $(3, 6)$-fullerene graph denoted as F_{6k}.

Next, we determine the local metric dimension of the $(3, 6)$-fullerene graph F_{6k} as illustrated in Figure 1.

Theorem 2.2. *For the $(3, 6)$-fullerene F_{6k}, the local metric dimension is 2, provided that k is an odd integer greater than 4.*

Proof. According to Theorem 2.1, it follows that $\dim_l(F_{6k}) \geq 2$. Thus, it suffices to show that $\dim_l(F_{6k}) \leq 2$. Let us define the set $S = \{v_{6k-3}, v_{6k-5}\}$. We will demonstrate that S is an serves as a LMG for F_{6k}. To do so, we will analyze the S-codes of vertices in F_{6k} as follows: The S-codes of the triangular faces in F_{6k} are specified as:

$$r_S(v_1) = (2k-3, 2k-2), \qquad r_S(v_2) = (2k-2, 2k-3), \qquad r_S(v_3) = (2k-3, 2k-3),$$
$$r_S(v_{6k}) = (2, 2), \qquad r_S(v_{6k-1}) = (3, 2), \qquad r_S(v_{6k-2}) = (2, 3).$$

Additionally, for the hexagonal faces, labelled by $n = 1, \ldots, k-1$, the S-codes are determined as follows:

Case 1. For n is odd and $n < k-3$:
$r_S(v_i) = (2k-(2n+3), 2k-(2n+3))$ for $i \in \{6n-1, 6n+1, 6n+3\}$ and
$r_S(v_i) = (2k-(2n+2), 2k-(2n+2))$ for $i \in \{6n-2, 6n, 6n+2\}$.

Case 2. n is even and $n < k-3$. Then
$r_S(v_i) = (2k-(2n+3), 2k-(2n+3))$ for $i \in \{6n-2, 6n, 6n+2\}$ and
$r_S(v_i) = (2k-(2n+2), 2k-(2n+2))$ for $i \in \{6n-1, 6n+1, 6n+3\}$.

Case 3. When $n = k-3$:

$$r_S(v_{6k-20}) = (3,5), \quad r_S(v_{6k-19}) = (4,4), \quad r_S(v_{6k-18}) = (5,3),$$
$$r_S(v_{6k-17}) = (4,4), \quad r_S(v_{6k-16}) = (3,3), \quad r_S(v_{6k-15}) = (4,4).$$

Case 4. When $n = k-2$:

$$r_S(v_{6k-14}) = (2,4), \quad r_S(v_{6k-13}) = (3,3), \quad r_S(v_{6k-12}) = (4,2),$$
$$r_S(v_{6k-11}) = (3,1), \quad r_S(v_{6k-10}) = (2,2), \quad r_S(v_{6k-9}) = (1,3).$$

Case 5. When $n = k-1$:
$r_S(v_{6k-8}) = (1,3),\ r_S(v_{6k-7}) = (2,2),\ r_S(v_{6k-6}) = (3,1),\ r_S(v_{6k-4}) = (1,1)$.

As a result, we observe that $r_S(u) \neq r_S(v)$ for every edge $uv \in E(F_{6k})$, confirming that S is indeed an LMG for F_{6k}. ∎

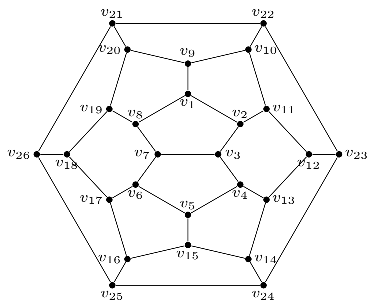

Figure 2: The $(5,6)$-fullerene graph represented as $C_{26}(D_{3h})$ with 3 hexagonal faces.

The subsequent theorem provides a precise assessment of the local metric dimension for the $(5,6)$-fullerene $C_{26}(D_{3h})$ illustrated in Figure 2.

Theorem 2.3. *The local metric dimension of* $C_{26}(D_{3h})$ *is equal to* 2.

Proof. According to Theorem 2.1, the local metric dimension of $C_{26}(D_{3h})$ cannot be lower than 2. We define the set $S = \{v_2, v_8\}$. Our objective is to prove that S functions an LMG for $C_{26}(D_{3h})$. To achieve this, we will enumerate the S-codes for the vertices of $C_{26}(D_{3h})$ as follows:

$$\begin{array}{llll} r_S(v_1) = (1,1), & r_S(v_3) = (1,2), & r_S(v_4) = (2,3), & r_S(v_5) = (3,3), \\ r_S(v_6) = (3,2), & r_S(v_7) = (2,1), & r_S(v_9) = (2,2), & r_S(v_{10}) = (2,3), \\ r_S(v_{11}) = (1,3), & r_S(v_{12}) = (2,4), & r_S(v_{13}) = (3,4), & r_S(v_{14}) = (4,5), \\ r_S(v_{15}) = (4,4), & r_S(v_{16}) = (5,4), & r_S(v_{17}) = (4,3), & r_S(v_{18}) = (4,2), \\ r_S(v_{19}) = (3,1), & r_S(v_{20}) = (3,2), & r_S(v_{21}) = (4,3), & r_S(v_{22}) = (3,4), \\ r_S(v_{23}) = (3,5), & r_S(v_{24}) = (5,6), & r_S(v_{25}) = (6,5), & r_S(v_{26}) = (5,3). \end{array}$$

This confirms that S qualifies as an LMG for $C_{26}(D_{3h})$, leading to the conclusion that $\dim_l(C_{26}(D_{3h})) = 2$. ■

Let $F_1[n]$ represent the $(3,6)$-fullerene illustrated in Figure 3, which has an order $8n+4$. We will use the notation established in this figure moving forward.

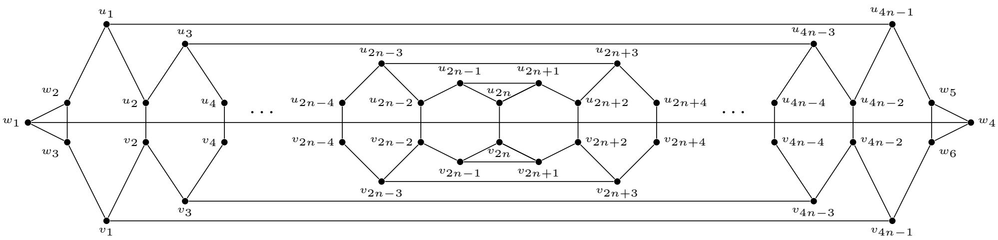

Figure 3: The graph $F_1[n]$.

Theorem 2.4. *The local metric dimension of $F_1[n]$ is equal to 2.*

Proof. By referencing Theorem 2.1, we conclude that $\dim_l(F_1[n]) \geq 2$. Next, we aim to prove that $\dim_l(F_1[n]) \leq 2$. We will consider $S = \{u_{2n-1}, v_{2n-1}\} \subset V(F_1[n])$ as our candidate set.

To verify that S acts as an LMG for $F_1[n]$, we will compute the S-code for the vertices of $F_1[n]$ as follows:

$$\begin{array}{lll} r_S(w_1) = (2n, 2n), & r_S(w_3) = (2n, 2n-1), & r_S(w_5) = (2n, 2n+1), \\ r_S(w_2) = (2n-1, 2n), & r_S(w_4) = (2n+1, 2n+1), & r_S(w_6) = (2n+1, 2n). \end{array}$$

The S-codes for the upper half of $F_1[n]$ are:

$$r_S(u_i) = \begin{cases} (2n-i-1, 2n-i), & \text{if } 1 \leq i \leq 2n-1, \\ (1,2), & \text{if } i = 2n, \\ (i-2n, i-2n+1), & \text{if } 2n+1 < i \leq 4n-1, \\ (1,3), & \text{if } i = 2n+1. \end{cases}$$

For the lower half of $F_1[n]$, the S-codes are given by:

$$r_S(v_i) = \begin{cases} (2n-i, 2n-i-1), & \text{if } 1 \leq i \leq 2n-1, \\ (2,1), & \text{if } i = 2n, \\ (i-2n+1, i-2n), & \text{if } 2n+1 < i \leq 4n-1, \\ (3,1), & \text{if } i = 2n+1. \end{cases}$$

It is evident that all the adjacent vertices of $F_1[n]$ possess distinct S-codes. This guarantees that S serves as an LMG for $F_1[n]$, leading us to conclude that $\dim_l(F_1[n]) \le 2$. As a result, we can state that the local metric dimension of $F_1[n]$ is 2, thus completing the proof. ■

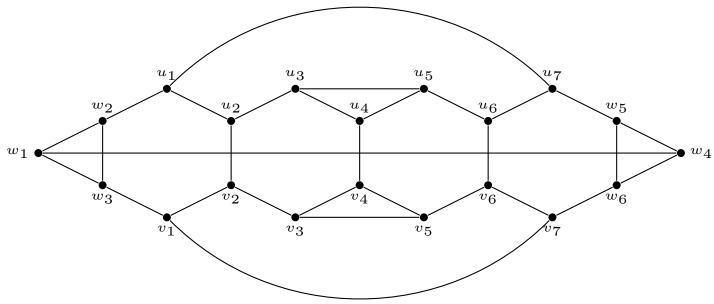

Figure 4: The graph $F_1[2]$.

For further clarification, we will demonstrate the proof of Theorem 2.4 using the graph $F_1[2]$. Consider $F_1[2]$, as illustrated in Figure 4, and two vertex groups $\{w_1, w_2, w_3\}$ and $\{w_4, w_5, w_6\}$ corresponding to the outer triangles of $F_1[2]$. Let us take $S = \{u_3, v_3\} \subset V(F_1[2])$. We will now prove that S acts an LMG for $F_1[2]$. To accomplish this, we present the S-codes of vertices in $V(F_1[2])$ as follows:

$$\begin{array}{lll}
r_S(w_1) = (4,4), & r_S(u_1) = (2,3), & r_S(v_1) = (3,2), \\
r_S(w_2) = (3,4), & r_S(u_2) = (1,2), & r_S(v_2) = (2,1), \\
r_S(w_3) = (4,3), & r_S(u_3) = (0,3), & r_S(v_3) = (3,0), \\
r_S(w_4) = (5,5), & r_S(u_4) = (1,2), & r_S(v_4) = (2,1), \\
r_S(w_5) = (4,5), & r_S(u_5) = (1,3), & r_S(v_5) = (3,1), \\
r_S(w_6) = (5,4), & r_S(u_6) = (2,3), & r_S(v_6) = (3,2), \\
 & r_S(u_7) = (3,4), & r_S(v_7) = (4,3).
\end{array}$$

Consequently, since all adjacent vertices of this graph have unique S-codes, we conclude that S is indeed an LMG for $F_1[2]$.

Theorem 2.5. $\dim_l(F_2[n]) = 2$.

Proof. Given that $F_2[n]$ is not bipartite, Theorem 2.1 assures us that $\dim_l(F_2[n]) \ge 2$.
Let $\{w_1, w_2, w_{12}\}$ and $\{w_6, w_7, w_{11}\}$ represent the vertex sets in outer triangles of $F_2[n]$. Assume $S = \{w_9, w_4\} \subset V(F_2[n])$. Our objective is to show that S functions as an LMG for $F_2[n]$. We will start by computing the S-codes of vertices in $V(F_2[n]) \setminus S$. The codes for the outer vertices of the graph $F_2[n]$ are listed below:

$$\begin{array}{lll}
r_S(w_1) = (2,3), & r_S(w_5) = (4,1), & r_S(w_9) = (0,3), \\
r_S(w_2) = (3,2), & r_S(w_6) = (3,2), & r_S(w_{10}) = (1,4), \\
r_S(w_3) = (2,1), & r_S(w_7) = (2,3), & r_S(w_{11}) = (3,3), \\
r_S(w_4) = (3,0), & r_S(w_8) = (1,2), & r_S(w_{12}) = (3,3).
\end{array}$$

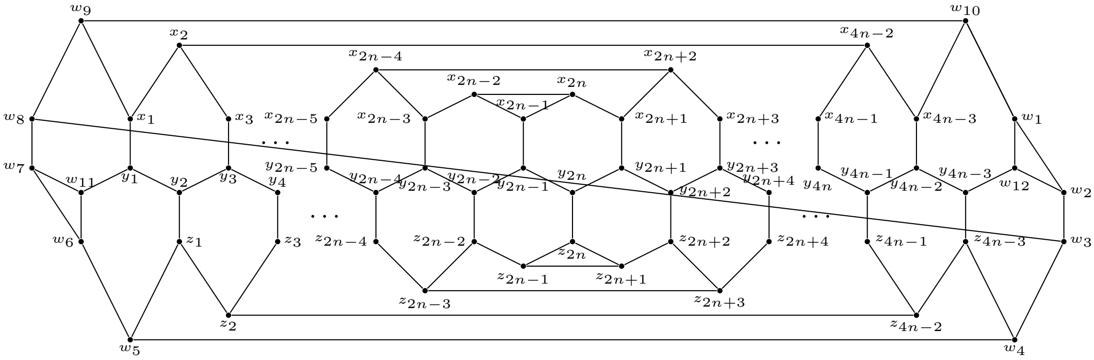

Figure 5: The graph $F_2[n]$.

Next, we detail the S-codes for the vertices in the upper half of the Fullerene graph $F_2[n]$:

$$r_S(x_i) = \begin{cases} (i, i+3), & \text{if } 1 \le i \le 2n-2, \\ (2n-1, 2n+2), & \text{if } i = 2n-1, \\ (4n-1-i, 4n+1-i), & \text{if } 2n \le i \le 4n-3. \end{cases}$$

We also specify the S-code for the middle vertices of $F_2[n]$:

$$r_S(y_i) = \begin{cases} (2,4), & \text{if } 1 = i, \\ (i+1, i+1), & \text{if } 2 \le i \le 2n-1, \\ (4n-i, 4n-i), & \text{if } 2n \le i \le 4n-3, \\ (4,2), & i = 4n-2. \end{cases}$$

Furthermore, the S-codes for the lower half of $F_2[n]$ are as follows:

$$r_S(z_i) = \begin{cases} (i+3, i+1), & \text{if } 1 \le i \le 2n-2, \\ (4n+1-i, 4n-2-i), & \text{if } 2n-1 \le i \le 4n-3. \end{cases}$$

Thus, we see that any adjacent vertex pairs can be distinctly resolved using the set S. This verifies that the set S acts as an LMG for $F_2[n]$ and $\dim_L(F_2[n]) \le 2$. Hence, $\dim_l(F_2[n]) = 2$. ∎

Theorem 2.6. $\dim_l(F_3[n]) = 2$.

Proof. Since the cycle $z_1z_2z_3z_1$ is odd in the graph $F_3[n]$, by invoking Theorem 2.1, we can establish that $\dim_l(F_3[n]) \ge 2$. Now, consider the vertex groups $\{z_1, z_2, z_3\}$ and $\{z_4, z_5, z_6\}$ representing the outer triangles, along with the vertices $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ of the outer hexagon in $F_3[n]$. Let us denote $S = \{a_1, v_{4n-1}\} \subset V(F_3[n])$. We aim to demonstrate that S serves as an LMG for $F_3[n]$.

To begin, we provide the S-codes for the vertices in $V(F_3[n]) \setminus S$. The S-codes for the outer vertices of $F_3[n]$ are as follows:

$$r_S(a_1) = (0,6), \quad r_S(a_3) = (2,4), \quad r_S(a_5) = (2,4),$$
$$r_S(a_2) = (1,5), \quad r_S(a_4) = (3,3), \quad r_S(a_6) = (1,5).$$

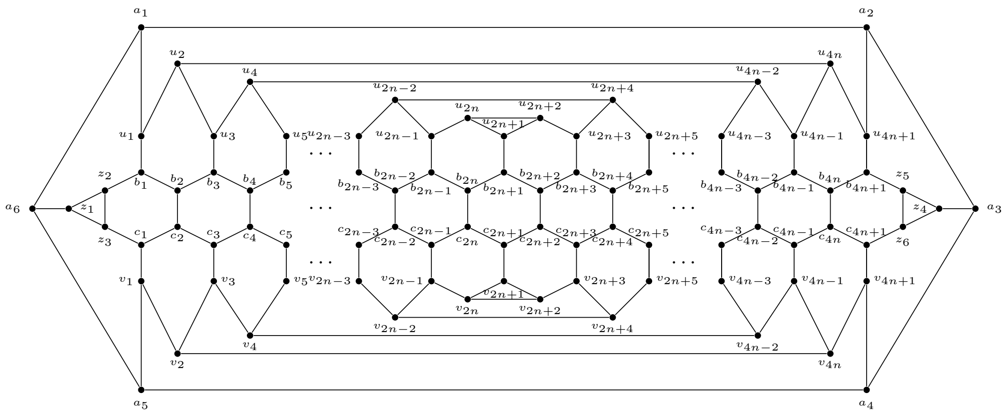

Figure 6: The graph $F_3[n]$.

Next, we present the S-codes of the vertices corresponding to the outer triangles in $F_3[n]$:

$$r_S(z_1) = (2,6), \qquad r_S(z_3) = (3,5), \qquad r_S(z_5) = (4,5),$$
$$r_S(z_2) = (3,6), \qquad r_S(z_4) = (3,5), \qquad r_S(z_6) = (4,4).$$

The coding for the vertices in the upper half of $F_3[n]$ is defined as:

$$r_S(u_i) = \begin{cases} (i,v), & \text{if } 1 \le i \le 2, \\ (i, n+1), & \text{if } 3 \le i \le 2n, \\ (2n+1, 2n+1), & \text{if } i = 2n+1, \\ (4n+3-i, 4n+2-i), & \text{if } 2n+2 \le i \le 4n-3, \\ (4n+3-i, 5), & \text{if } i \in \{4n-1, 4n+1\}, \\ (4n+2-i, 6), & \text{if } i \in \{4n-2, 4n\}. \end{cases}$$

For the middle vertices in $F_3[n]$ (valid for $n \ge 2$), the S-codes are expressed as:

$$r_S(b_i) = \begin{cases} (i+1, 7), & \text{if } 2k-1 \quad k \in \mathbb{N} \quad 1 \le i \le 2n-1, \quad i \le 7, \\ (i+1, 6), & \text{if } 2k \quad k \in \mathbb{N} \quad 1 \le i \le 2n-1, \quad i \le 7, \\ (i+1, i), & \text{if } 1 \le i \le 2n-1, \quad i > 7, \\ (i+1, i+1), & \text{if } i = 2n, \\ (i+1, i-1), & \text{if } i = 2n+1, \\ (i+1, 4n+4-i), & \text{if } 2n+2 \le i \le 4n-1, \\ (4,3), & \text{if } i = 4n, \\ (3,4), & \text{if } i = 4n+1. \end{cases}$$

For another set of middle vertices in $F_3[n]$, the codes are indicated as:

$$r_S(c_i) = \begin{cases} (4,4), & \text{if } i = 1, \\ (i+2,4), & \text{if } 2k-1 \quad k \in \mathbb{N} \quad 2 \leq i \leq 2n-1, \quad i \leq 7, \\ (i+2,5), & \text{if } 2k \quad k \in \mathbb{N} \quad 2 \leq i \leq 2n-1, \quad i \leq 7, \\ (i+2,i-1), & \text{if } 1 \leq i \leq 2n-1, \quad i > 7, \\ (i+2,i), & \text{if } i = 2n, \\ (i+2,4n-i), & \text{if } i = 2n+1, \\ (4n+5-i,4n-i), & \text{if } 2n+2 \leq i \leq 4n-2, \\ (6,1), & \text{if } i = 4n-1, \\ (5,2), & \text{if } i = 4n, \\ (5,3), & \text{if } i = 4n+1. \end{cases}$$

Finally, the S-codes for the vertices located in the lower half of $F_3[n]$ are specified as:

$$r_S(v_i) = \begin{cases} (i+2,3), & \text{if } i = 1,3, \\ (i+2,2), & \text{if } i = 2,4, \\ (i+2,i-2), & \text{if } 5 \leq i \leq 2n, \\ (i+2,2n-2), & \text{if } i = 2n+1, \\ (4n+5-i,4n-1-i), & \text{if } 2n+2 \leq i \leq 4n-1, \\ (1,5), & \text{if } i = 4n, \\ (2,4), & \text{if } i = 4n+1. \end{cases}$$

As a consequence, we see that the set S-codes can uniquely identify all pairs of adjacent vertices. Thus, it follows that S is an LMG for $F_2[n]$, and consequently, $\dim_L(F_3[n]) \leq 2$. We can conclude that the local metric dimension of $F_3[n]$ is indeed 2. ■

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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