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Research Paper

A Study of Vertex-Degree Function Indices via Branching Operations on Trees

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Abstract

Let G be a graph with vertex set $V(G)$. The vertex-degree function index $H_f(G)$ is defined on G as:

$$H_f(G) = \sum_{u \in V(G)} f(d_u),$$

where $f(x)$ is a function defined on positive real numbers. Our main concern in this paper is to study H_f over the set $\mathcal{T}_n$ of trees with n vertices, over the set $\mathcal{T}_{n,k}$ of trees with n vertices and k branching vertices, and over the set $\mathcal{T}_n^p$ of trees with n vertices and p pendant vertices. Namely, we will show in each of these sets of trees that it is possible via branching operations to construct a strictly monotone sequence of trees that reaches the extremal values of H_f, when $f(x+1) - f(x)$ is a strictly increasing function.

1 Introduction

Let G be a simple connected graph with vertex set $V(G)$. The degree of a vertex $u \in V(G)$ will be denoted by $d_u = d_u(G)$. We say that the vertex $u \in V(G)$ is a branching vertex if $d_u \geq 3$, while it is a pendant vertex if $d_u = 1$. The vertex-degree function index $H_f(G)$ is defined on G as:

$$H_f(G) = \sum_{u \in V(G)} f(d_u),$$

where $f(x)$ is a function defined on positive real numbers [1]. For example, the first Zagreb index $\mathcal{M}_1(G) = \sum_{u \in V(G)} d_u^2$ is a special case when $f(x) = x^2$ [2], the forgotten index $F(G) = \sum_{u \in V(G)} d_u^3$ is a special case when $f(x) = x^3$ [3]. More generally, the zeroth-order general

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Randić index ${}^{0}\mathcal{R}_{\alpha}(G) = \sum_{u \in V(G)} d_u^{\alpha}$, where $\alpha \notin \{0, 1\}$ is a real number, is a particular case when $f(x) = x^{\alpha}$ [4, 5]. For recent results on the degree function index of graphs we refer to [6–9].

We are particularly interested in the vertex-degree function index over trees, i.e., connected graphs with no cycles. Let T be a tree. A branch at $u \in V(T)$ is a maximal subtree containing u as an end vertex. Hence, the number of branches at u is d_u. We say that tree U is obtained from tree T by a branching operation, denoted as $U = \beta(T)$, when U is obtained from T by moving one branch of T at $u \in V(T)$ to another vertex $v \in V(T)$ (see Figure 1).

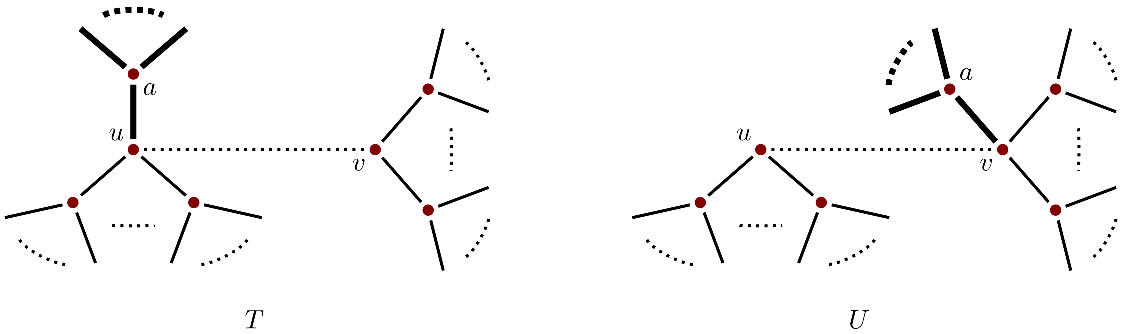


Figure 1: U is obtained from T by a branching operation.

Let us denote by $\mathcal{T}_n$ the set of trees with n vertices and let $\mathcal{F} \subseteq \mathcal{T}_n$. We define the following relation on $\mathcal{F}$: if $S, T \in \mathcal{F}$ we write $S \succ T$ in $\mathcal{F}$ if and only if there exists a sequence $\{U_j\}_{j=1}^{k} \subseteq \mathcal{F}$ such that $U_0 = S$, $U_k = T$, $U_j = \beta(U_{j-1})$ for all $1 \leq j \leq k$, and

$$H_f(S) = H_f(U_0) > H_f(U_1) > \cdots > H_f(U_k) = H_f(T).$$

In this case we say that $\{U_j\}_{j=1}^{k}$ is a strictly monotone sequence of trees.

Our main concern in this paper is to study the relation $\succ$ in $\mathcal{F}$ on three significant classes:

1. $\mathcal{F} = \mathcal{T}_n$, the set of trees with n vertices;
2. $\mathcal{F} = \mathcal{T}_{n,k}$, the set of trees with n vertices and k branching vertices;
3. $\mathcal{F} = \mathcal{T}_n^{p}$, the set of trees with n vertices and p pendant vertices.

We will show that given a tree in $\mathcal{F}$, it is possible via branching operations to construct a strictly monotone sequence of trees in $\mathcal{F}$ that reach the extremal values of H_f, when $f(x+1) - f(x)$ is a strictly increasing function, a property satisfied by strictly convex functions. Examples of such functions are $f(x) = x^{\alpha}$, when $\alpha > 0$, which induce the general zeroth-order Randić indices ${}^{0}\mathcal{R}_{\alpha}$. From this general approach, it is possible to deduce several well-known results on the extremal value problem of H_f over the classes of trees mentioned above [10–12].

2 Variation of H_f under branching operations on trees

If $T \in \mathcal{T}_n$, then the degree sequence of T is expressed in the form $(d_1, d_2, \ldots, d_n)$, where $d_1 \geq d_2 \geq \cdots \geq d_n$ are the degrees of the vertices of T in descending order. Note that $\sum_{i=1}^{n} d_i = 2(n-1)$. Moreover, any non-increasing sequence $(e_1, e_2, \ldots, e_n)$ of positive integers such that $\sum_{i=1}^{n} e_i = 2(n-1)$ is the degree sequence of some tree in $\mathcal{T}_n$.

In this section we study the variation of H_f when a special branching operation is applied to a tree T. Let $i, j \in \{1, \ldots, n\}$ such that

$$i < j, \; d_i > d_{i+1}, \; d_{j-1} > d_j, \; \text{and } d_i > d_j + 1. \tag{1}$$

Consider **Operation I**:

$$(d_1, \ldots, d_i, \ldots, d_j, \ldots, d_n) \rightsquigarrow (d_1, \ldots, d_i - 1, \ldots, d_j + 1, \ldots, d_n), \tag{2}$$

where only the positions i, j are modified. By conditions given in (1), the sequence on the right of (2) is non-increasing. In fact, the transformation given in (2) corresponds to a branching operation on T, by moving a branch of T at vertex i to the vertex j.

In the other direction, let $j \in \{1, \ldots, n\}$ such that

$$j > 1 \; \text{ and } d_j > d_{j+1}. \tag{3}$$

Consider **Operation II**:

$$(d_1, \ldots, d_j, \ldots, d_n) \rightsquigarrow (d_1 + 1, \ldots, d_j - 1, \ldots, d_n), \tag{4}$$

where only the positions $1, j$ are modified. By condition (3), the sequence on the right of (4) is non-increasing. The transformation given in (4) corresponds to a branching operation on T, by moving a branch of T at vertex j to the vertex 1.

We will assume throughout this paper that $f(x+1) - f(x)$ is a strictly increasing function. Clearly, every strictly convex function $f(x)$ satisfies this property.

Example 2.1. Consider the function $f(x) = x^2 + \lfloor x \rfloor$. Then $f(x)$ is not convex since it is discontinuous at each positive integer. However, $f(x+1) - f(x) = 2x + 2$ is a strictly increasing function.

With our next result, we show that H_f is strictly monotone with respect to the operations defined above.

Theorem 2.2. *Let T be a tree with degree sequence $(d_1, d_2, \ldots, d_n)$.*

1. *Assume that i, j satisfy conditions given in (1). If U is the tree obtained from T by operation I, then $H_f(T) > H_f(U)$;*
2. *Assume that j satisfies condition (3). If V is the tree obtained from T by operation II, then $H_f(T) < H_f(V)$.*

Proof. Let $h(x) = f(x+1) - f(x)$.

1. If U is obtained from T by operation I, then it follows from (2) that
$$\begin{aligned} H_f(T) - H_f(U) &= f(d_i) - f(d_i - 1) + f(d_j) - f(d_j + 1) \\ &= h(d_i - 1) - h(d_j) > 0, \end{aligned}$$
since $d_i > d_j + 1$ and $h(x)$ is strictly increasing.
2. If V is obtained from T by operation II, then by (4),
$$\begin{aligned} H_f(T) - H_f(V) &= f(d_1) - f(d_1 + 1) + f(d_j) - f(d_j - 1) \\ &= h(d_j - 1) - h(d_1) < 0, \end{aligned}$$
since $d_1 \geq d_j > d_j - 1$ and $h(x)$ is strictly increasing.

■

3 Trees with a fixed number of vertices

We first show that it is possible to reach the path P_n from any tree $T \neq P_n$, by a sequence of branching tree operations which have strictly decreasing value of H_f.

Theorem 3.1. *If $T \in \mathcal{T}_n$ and $T \neq P_n$, then $T \succ P_n$ in $\mathcal{T}_n$.*

Proof. Let $(d_1, d_2, \ldots, d_n)$ be the degree sequence of T. Since $T \neq P_n$, then $\Delta(T) \geq 3$. Choose i such that $d_i = \Delta$ and $d_{i+1} < \Delta$. On the other hand, choose j such that $d_j = 1$ and $d_{j-1} > 1$. Then $d_i \geq 3 > 2 = d_j + 1$. Consequently, $i < j$ satisfy conditions given in (1), so by Theorem 2.2, after we apply operation I to T we obtain a tree $U_1 \in \mathcal{T}_n$ such that $H_f(T) > H_f(U_1)$. If $U_1 = P_n$ we are done. Otherwise, we repeat the previous argument to construct a tree $U_2 \in \mathcal{T}_n$, such that $H_f(U_1) > H_f(U_2)$. Eventually, after a finite number of steps we arrive at $U_k = P_n$, where $U_j = \beta(U_{j-1})$ for all $1 \leq j \leq k$ and

$$H_f(T) = H_f(U_0) > H_f(U_1) > H_f(U_2) > \cdots > H_f(U_k) = H_f(P_n).$$

■

In the other direction, we can obtain the star S_n from any tree $T \neq S_n$, by a sequence of trees which have strictly increasing value of H_f.

Theorem 3.2. *If $T \in \mathcal{T}_n$ and $T \neq S_n$, then $S_n \succ T$ in $\mathcal{T}_n$.*

Proof. Since $T \neq S_n$, the degree sequence of T has the form $(d_1, \ldots, d_j, 1, \ldots, 1)$, where $j > 1$ satisfies $d_1 \geq d_j > 1 = d_{j+1}$. Hence, j satisfies condition (3), so by Theorem 2.2, after applying operation II to the tree T we obtain a tree $V_1 \in \mathcal{T}_n$ such that $H_f(V_1) > H_f(T)$. If $V_1 = S_n$ we are done. Otherwise, we repeat the previous argument to construct a tree $V_2 \in \mathcal{T}_n$ such that $H_f(V_2) > H_f(V_1)$. After a finite number of steps we arrive at a tree $V_k \in \mathcal{T}_n$ such that $V_k = S_n$, where $V_j = \beta(V_{j-1})$ for all $1 \leq j \leq k$, and

$$H_f(S_n) = H_f(V_k) > \cdots > H_f(V_2) > H_f(V_1) > H_f(V_0) = H_f(T).$$

■

Example 3.3. In Table 1 we illustrate the sequences of trees given in Theorems 3.1 and 3.2. Note that in each step we 'move' the maximal subtree at u which contains the vertex a, to the vertex v.

Remark 1. Note that Theorems 3.1 and 3.2 are stronger results than [10, Theorems 4 and 8], since they not only present the extremal trees, but also state the existence of a strictly monotone sequence of trees that reach extreme values of H_f, starting from any tree in $\mathcal{T}_n$.

Recall that the zeroth-order general Randić index is defined as ${}^0\mathcal{R}_\alpha(T) = H_f(G)$ where $f(x) = x^\alpha$. In the next result, we affirm that from any tree in $\mathcal{T}_n$ it is possible to construct a strictly monotone sequence of trees that reach maximum and minimum trees with respect to the zeroth-order general Randić index, a result obtained in [11].

Corollary 3.4. *Let $T \in \mathcal{T}_n$.*

1. *If $T \neq P_n$, then $T \succ P_n$ in $\mathcal{T}_n$ if $\alpha \in (-\infty, 0) \cup (1, +\infty)$ and, $P_n \succ T$ in $\mathcal{T}_n$ if $\alpha \in (0, 1)$.*
2. *If $T \neq S_n$, then $S_n \succ T$ in $\mathcal{T}_n$ if $\alpha \in (-\infty, 0) \cup (1, +\infty)$ and, $T \succ S_n$ in $\mathcal{T}_n$ if $\alpha \in (0, 1)$.*

Proof. For $\alpha \in (-\infty, 0) \cup (1, +\infty)$ it holds that $f(x+1) - f(x)$ is strictly increasing function. Statements 1 and 2 follow from Theorems 3.1 and 3.2, respectively.

On the other hand, since $f(x+1) - f(x)$ is strictly decreasing function if $\alpha \in (0, 1)$, we apply Theorems 3.1 and 3.2 to $H_{\overline{f}}(G)$ with $\overline{f}(x) = -x^\alpha$. ■

4 Trees with fixed number of vertices and branching vertices

Let us denote by $\mathcal{T}_{n,k}$ the set of trees with n vertices and k branching vertices.

Lemma 4.1. *The set* $\mathcal{T}_{n,k}$ *is nonempty if and only if* $n \geq 2k+2$. *If* $n = 2k+2$, *then any tree* $T \in \mathcal{T}_{2k+2,k}$ *has degree sequence* $\big(\underbrace{3,\dots,3}_{k},\underbrace{1,\dots,1}_{k+2}\big)$.

Proof. Recall that if a tree $T \in \mathcal{T}_{n,k}$ has p pendant vertices and $\mathcal{X}$ is the set of branching vertices, then

$$p - 2 = \sum_{v \in \mathcal{X}} (d_v - 2) \geq k. \tag{5}$$

If n_2 is the number of vertices of degree 2 in T, then using (5) we have:

$$n = k + n_2 + p \geq 2k + n_2 + 2 \geq 2k + 2. \tag{6}$$

On the other hand, from (6),

$$0 \leq n_2 \leq n - 2k - 2.$$

If $n = 2k+2$, then for any tree $T \in \mathcal{T}_{2k+2,k}$, $n_2 = 0$ and $p = k+2$. From (5)

$$k = \sum_{v \in \mathcal{X}} (d_v - 2),$$

which implies that $d_v = 3$ for each $v \in \mathcal{X}$. Then, any tree $T \in \mathcal{T}_{2k+2,k}$ has degree sequence $\big(\underbrace{3,\dots,3}_{k},\underbrace{1,\dots,1}_{k+2}\big)$. ■

Example 4.2. The trees in Figure 2 belong to $\mathcal{T}_{2k+2,k}$.

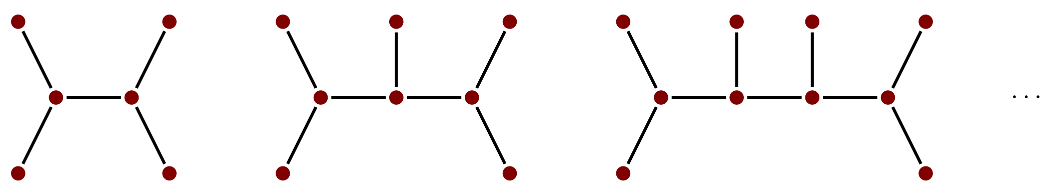


Figure 2: Trees in $\mathcal{T}_{2k+2,k}$.

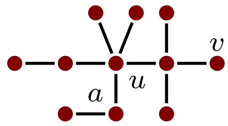
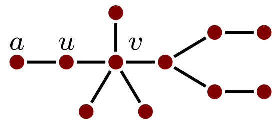
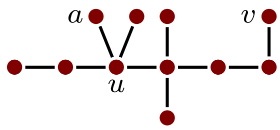
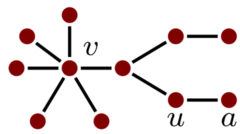
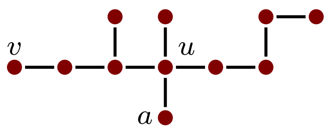
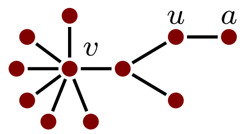
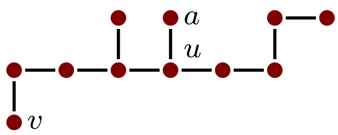
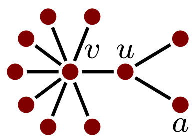
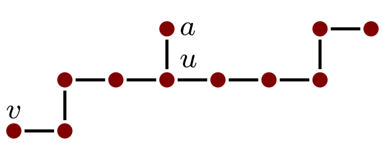
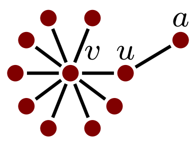
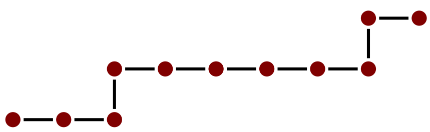
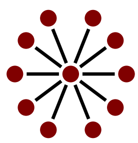
In what follows in this section we consider the set $\mathcal{T}_{n,k}$ with $n > 2k+2$. We shall see that the trees in

$$\mathcal{A} = \left\{ T \in \mathcal{T}_{n,k} : T \text{ has degree sequence} \big(\underbrace{3,\dots,3}_{k},\underbrace{2,\dots,2}_{n-2k-2},\underbrace{1,\dots,1}_{k+2}\big) \right\},$$

have the minimal value of H_f among all trees in $\mathcal{T}_{n,k}$.

Lemma 4.3. *Let* $A \in \mathcal{T}_{n,k}$. *Then* $A \in \mathcal{A}$ *if and only if* $\Delta(A) = 3$.

Table 1: Decreasing and increasing sequences of trees in $\mathcal{T}_{11}$.

Sequence in Theorem 3.1	Sequence in Theorem 3.2
 U_0	 V_0
 U_1	 V_1
 U_2	 V_2
 U_3	 V_3
 U_4	 V_4
 U_5	 V_5

Proof. Clearly, $A \in \mathcal{A}$ implies $\Delta(A) = 3$. Conversely, assume that $A \in \mathcal{T}_{n,k}$ and $\Delta(A) = 3$. Then the degree sequence of A is of the form $\big(\underbrace{3,\ldots,3}_{k},\underbrace{2,\ldots,2}_{m},\underbrace{1,\ldots,1}_{p}\big)$. By relation (5), $p = k+2$, and since $k+m+p = n$, we deduce that $m = n-2k-2$. Consequently, $A \in \mathcal{A}$. ■

Theorem 4.4. *If $T \in \mathcal{T}_{n,k}$ and $T \notin \mathcal{A}$, then there exists $A \in \mathcal{A}$ such that $T \succ A$ in $\mathcal{T}_{n,k}$.*

Proof. Let $(d_1, d_2, \ldots, d_n)$ be the degree sequence of T. Since $T \in \mathcal{T}_{n,k}$ and $T \notin \mathcal{A}$, then by Lemma 4.3, $\Delta \geq 4$. Let i such that $d_i = \Delta > d_{i+1}$, and $j > 1$ such that $d_j = 1$ but $d_{j-1} > 1$. Note that $d_i \geq 4 > 2 = d_j + 1$. Hence conditions given in (1) hold and so by Theorem 2.2, after applying operation I to the tree T we find a tree $U_1 \in \mathcal{T}_n$ such that $H_f(T) > H_f(U_1)$. Note that since $d_i \geq 4$ and $d_j = 1$ in T, then $U_1 \in \mathcal{T}_{n,k}$. If $U_1 \in \mathcal{A}$ then we are done. Otherwise, using a similar argument as before, we construct a tree $U_2 \in \mathcal{T}_{n,k}$ such that $H_f(U_2) > H_f(U_1)$. After a finite number of steps we arrive at a tree $U_s \in \mathcal{T}_{n,k}$ such that $U_s = A \in \mathcal{A}$, $U_j = \beta(U_{j-1})$ for all $j = 1, \ldots, s$ and

$$H_f(T) = H_f(U_0) > H_f(U_1) > \cdots > H_f(U_s) = H_f(A).$$

■

Now we consider the set

$$\mathcal{B} = \left\{ T \in \mathcal{T}_{n,k} : T \text{ has degree sequence } \big(n-2k+1, \underbrace{3,\ldots,3}_{k-1}, \underbrace{1,\ldots,1}_{n-k}\big) \right\}.$$

Theorem 4.5. *If $T \in \mathcal{T}_{n,k}$ and $T \notin \mathcal{B}$, then there exists $B \in \mathcal{B}$ such that $B \succ T$ in $\mathcal{T}_{n,k}$.*

Proof. Let $(d_1, d_2, \ldots, d_n)$ be the degree sequence of T. Recall that $\Delta(T) \geq 3$ and assume that T has a vertex of degree 2. Choose $j > 1$ such that $d_j = 2$ and $d_{j+1} = 1$. Hence, by Theorem 2.2, after applying operation II to the tree T we obtain a tree $V_1 \in \mathcal{T}_n$ such that $H_f(V_1) > H_f(T)$. Since $d_1 = \Delta(T) \geq 3$ and $d_j = 2$, it is clear that $V_1 \in \mathcal{T}_{n,k}$. So repeating this procedure as many times as necessary, we arrive at a tree $V_r \in \mathcal{T}_{n,k}$ without vertices of degree 2, such that $V_j = \beta(V_{j-1})$, for all $1 \leq j \leq r$ and

$$H_f(V_r) > \cdots > H_f(V_1) > H_f(V_0) = H_f(T).$$

Since $n_2(V_r) = 0$, the number of pendant vertices is $n-k$. Let $(e_1, e_2, \ldots, e_n)$ be the degree sequence of V_r. If $e_2 = 3$ then

$$e_1 = 2(n-1) - 3(k-1) - (n-k) = n - 2k + 1,$$

and $V_r \in \mathcal{B}$. If $e_2 \geq 4$ then choose $j > 1$ such that $e_j \geq 4$ and $e_j > e_{j+1}$. It follows from Theorem 2.2 that the tree V_{r+1} obtained from V_r by operation II satisfies $H_f(V_{r+1}) > H_f(V_r)$. Moreover, since $e_j \geq 4$ then $V_{r+1} \in \mathcal{T}_{n,k}$ and has no vertices of degree 2. Repeating this procedure as many times as necessary we arrive at a tree $V_s \in \mathcal{T}_{n,k}$ with degree sequence of the form $\big(a, \underbrace{3,\ldots,3}_{k-1}, \underbrace{1,\ldots,1}_{n-k}\big)$, where $V_{r+j} = \beta(V_{r+j-1})$ for all $1 \leq j \leq s-r$ and

$$H_f(V_s) > \cdots > H_f(V_{r+1}) > H_f(V_r) > \cdots > H_f(V_1) > H_f(V_0) = H_f(T).$$

Finally, since the sum of all degrees of V_s is equal to $2(n-1)$, we deduce that $a = n - 2k + 1$. Hence, $V_s \in \mathcal{B}$. ■

Example 4.6. In Table 2 we illustrate the sequences of trees given in Theorems 4.4 and 4.5. Note that in each step we 'move' the maximal subtree at u which contains the vertex a, to the vertex v.

The next result states the existence of a strictly monotone sequence of trees that reach maximum and minimum trees with respect to the zeroth-order general Randić index in $\mathcal{T}_{n,k}$. This result implies a solution of the extremal problem solved in [12] with respect to the zeroth-order Randić index over the class $\mathcal{T}_{n,k}$.

Corollary 4.7. *Let $T \in \mathcal{T}_{n,k}$ with $n > 2k+2$.*

1. *If $T \notin \mathcal{A}$, then there exists $A \in \mathcal{A}$ such that $T \succ A$ in $\mathcal{T}_{n,k}$ if $\alpha \in (-\infty, 0) \cup (1, +\infty)$ and, $A \succ T$ in $\mathcal{T}_{n,k}$ if $\alpha \in (0,1)$.*
2. *If $T \notin \mathcal{B}$, then there exists $B \in \mathcal{B}$ such that $B \succ T$ in $\mathcal{T}_{n,k}$ if $\alpha \in (-\infty, 0) \cup (1, +\infty)$ and, $T \succ B$ in $\mathcal{T}_{n,k}$ if $\alpha \in (0,1)$.*

5 Trees with fixed number of vertices and pendant vertices

Let us denote by $\mathcal{T}_n^p$ the set of trees on n vertices and p pendant vertices. If $p = n-1$ then $\mathcal{T}_n^p = \{S_n\}$, and if $p = 2$, then $\mathcal{T}_n^p = \{P_n\}$.

If $p = 3$ and $T \in \mathcal{T}_n^3$, by (5), T has exactly one branching vertex of degree 3 and $n_2 = n-4$ vertices of degree 2. It means that any tree $T \in \mathcal{T}_n^3$ has degree sequence

$$\big(3, \underbrace{2, \dots, 2}_{n-4}, 1, 1, 1\big)$$

with $H_f(T) = f(3) + (n-4)f(2) + 3f(1)$.

We assume throughout this section that $4 \le p \le n-2$. Let

$$\mathcal{C} = \left\{ T \in \mathcal{T}_n^p \colon T \text{ has degree sequence } \big(\underbrace{a+1, \dots, a+1}_{r}, \underbrace{a, \dots, a}_{s}, \underbrace{1, \dots, 1}_{p}\big) \right\},$$

where $a = \lfloor \frac{n-2}{n-p} \rfloor + 1$, $r = n - 2 - (n-p)\lfloor \frac{n-2}{n-p} \rfloor$, and $s = (n-p)\lfloor \frac{n-2}{n-p} \rfloor - p + 2$.

Theorem 5.1. *Let $T \in \mathcal{T}_n^p$ such that $T \notin \mathcal{C}$. Then there exists $C \in \mathcal{C}$ such that $T \succ C$ in $\mathcal{T}_n^p$.*

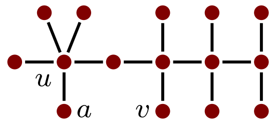
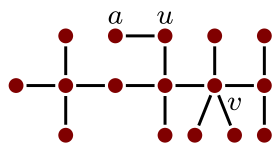
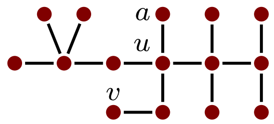
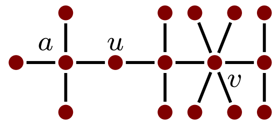
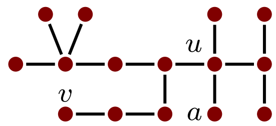
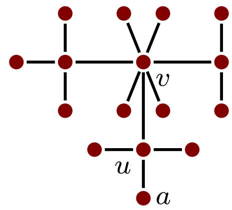
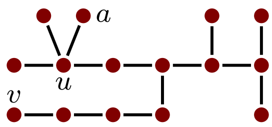
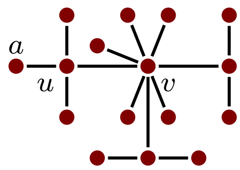
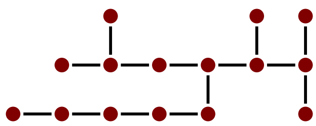
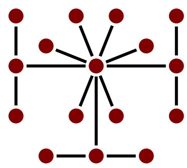
Proof. Let $(d_1, d_2, \dots, d_n)$ be the degree sequence of T. Let j such that $d_j \ge 2$ and $d_{j+1} = 1$. Let i such that $d_1 = d_i > d_{i+1}$. Note that $i < j$, otherwise, $T = S_n$ which contradicts the fact that $p \le n-2$. If $d_i - d_j \ge 2$ then, by Theorem 2.2, there exists $U_1 = \beta(T) \in \mathcal{T}_n^p$ such that $H_f(U_1) > H_f(T)$. Assume that U_1 has degree sequence $(e_1, e_2, \dots, e_n)$. Let j such that $e_j \ge 2$ and $e_{j+1} = 1$. Let i such that $e_1 = e_i > e_{i+1}$. If $e_i - e_j \ge 2$, then as before, there exists $U_2 = \beta(U_1) \in \mathcal{T}_n^p$ such that $H_f(U_2) > H_f(U_1)$. Repeating this process we arrive after a finite number of steps to a tree $U_k \in \mathcal{T}_n^p$ such that

$$H_f(T) > H_f(U_1) > \cdots > H_f(U_k),$$

where $U_j = \beta(U_{j-1})$, for all $1 \le j \le k$, and U_k has degree sequence of the form

$$\big(\underbrace{a+1, \dots, a+1}_{r}, \underbrace{a, \dots, a}_{s}, \underbrace{1, \dots, 1}_{p}\big).$$

Table 2: Decreasing and increasing sequences of trees in $\mathcal{T}_{15,4}$ and $\mathcal{T}_{16,4}$, respectively.

Sequence in Theorem 4.4	Sequence in Theorem 4.5
 U_0	 V_0
 U_1	 V_1
 U_2	 V_2
 U_3	 V_3
 U_4	 V_4

From the relations

$$r+s+p=n,$$

and

$$r(a+1)+sa+p=2(n-1),$$

it follows that $a=\lfloor\frac{n-2}{n-p}\rfloor+1$, $r=n-2-(n-p)\lfloor\frac{n-2}{n-p}\rfloor$, and $s=(n-p)\lfloor\frac{n-2}{n-p}\rfloor-p+2$. In particular, $U_k\in\mathcal{C}$, so the proof is complete. ■

Now consider the set

$$\mathcal{D}=\left\{T\in\mathcal{T}_n^p : T \text{ has degree sequence } (p,\underbrace{2,\ldots,2}_{n-p-1},\underbrace{1,\ldots,1}_{p})\right\}.$$

Theorem 5.2. *Let $T\in\mathcal{T}_n^p$ and $T\notin\mathcal{D}$. Then there exists $D\in\mathcal{D}$ such that $D\succ T$ in $\mathcal{T}_n^p$.*

Proof. Since $p\geq 4$ then $T\neq P_n$, so $\Delta(T)\geq 3$. Assume that T has degree sequence $(d_1,d_2,\ldots,d_n)$. Choose j such that $d_j\geq 3$ but $1\leq d_{j+1}\leq 2$. If $j=1$, then the degree sequence of T is of the form $(a,\underbrace{2,\ldots,2}_{n-p-1},\underbrace{1,\ldots,1}_{p})$. Then from the relation

$$a+2(n-p-1)+p=2(n-1),$$

we deduce that $a=p$, which implies that $T\in\mathcal{D}$, a contradiction. Hence $j>1$. Now by Theorem 2.2, after we apply operation II to the tree T we obtain a tree $V_1\in\mathcal{T}_n^p$ such that $H_f(V_1)>H_f(T)$. Again $\Delta(V_1)\geq 3$. Then as before, either $V_1\in\mathcal{D}$ or there exists $V_2\in\mathcal{T}_n^p$ such that $H_f(V_2)>H_f(V_1)$. Continuing this process, after a finite number of steps we arrive at a tree $V_s\in\mathcal{D}$ such that

$$H_f(D)=H_f(V_s)>\cdots>H_f(V_1)>H_f(V_0)=H_f(T).$$

■

Example 5.3. In Table 3 we illustrate the sequences of trees given in Theorems 5.1 and 5.2. Note that in each step we 'move' the maximal subtree at u which contains the vertex a, to the vertex v.

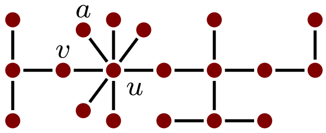
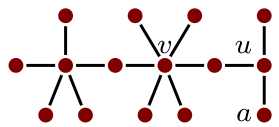
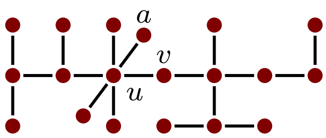
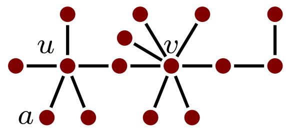
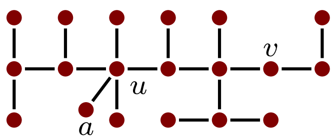
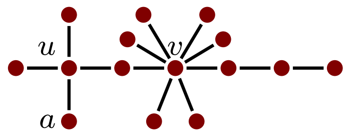
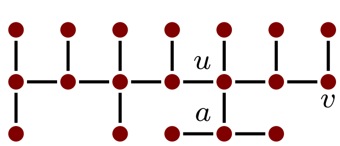
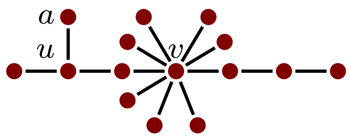
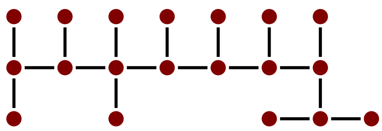
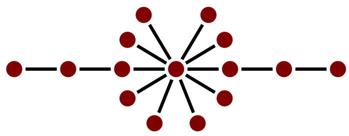
Next we provide a constructive solution to the problem of finding extremal trees in $T\in\mathcal{T}_n^p$ with respect to ${}^0\mathcal{R}_\alpha$. This problem was originally solved by Khalid and Ali in [12].

Corollary 5.4. *Let $T\in\mathcal{T}_n^p$ with $4\leq p\leq n-2$.*

1. *If $T\notin\mathcal{C}$, then there exists $C\in\mathcal{C}$ such that $T\succ C$ in $\mathcal{T}_n^p$ if $\alpha\in(-\infty,0)\cup(1,+\infty)$ and, $C\succ T$ in $\mathcal{T}_n^p$ if $\alpha\in(0,1)$.*
2. *If $T\notin\mathcal{D}$, then there exists $D\in\mathcal{D}$ such that $D\succ T$ in $\mathcal{T}_n^p$ if $\alpha\in(-\infty,0)\cup(1,+\infty)$ and, $T\succ D$ in $\mathcal{T}_n^p$ if $\alpha\in(0,1)$.*

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

Table 3: Decreasing and increasing sequences of trees in $\mathcal{T}_{19}^{11}$ and $\mathcal{T}_{15}^{10}$, respectively.

Sequence in Theorem 5.1	Sequence in Theorem 5.2
 U_0	 V_0
 U_1	 V_1
 U_2	 V_2
 U_3	 V_3
 U_4	 V_4

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