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Research Paper

On the Reduced and Increased Sombor Indices of Trees with Given Order and Maximum Degree

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Abstract

The Sombor index is a newly introduced vertex-degree-based graph invariant with the ability to predict the enthalpy of vaporization and entropy of octane isomers. Recently, two new variants of the Sombor index namely the reduced and increased Sombor indices were put forward. The reduced and increased Sombor indices are respectively defined for graph Γ as

$$SO_{red}(\Gamma) = \sum_{\mathcal{F}\mathcal{G}\in E(\Gamma)} \sqrt{(d_\Gamma(\mathcal{F})-1)^2 + (d_\Gamma(\mathcal{G})-1)^2},$$

and

$$SO^{\ddagger}(\Gamma) = \sum_{\mathcal{F}\mathcal{G}\in E(\Gamma)} \sqrt{(d_\Gamma(\mathcal{F})+1)^2 + (d_\Gamma(\mathcal{G})+1)^2},$$

in which $d_\Gamma(\mathcal{F})$ is the degree of the vertex $\mathcal{F}$ in Γ. Our purpose is to establish sharp lower bounds on the reduced and increased Sombor indices of trees in terms of their order and maximum vertex degree. Moreover, the extremal trees that attain the bounds are characterized.

1 Introduction

Consider a simple connected graph Γ where $V(\Gamma)$ and $E(\Gamma)$ are its vertex and edge sets, respectively. For $\mathcal{F} \in V(\Gamma)$, the set $N_\Gamma(\mathcal{F}) = \{\mathcal{G} \in V(\Gamma) : \mathcal{F}\mathcal{G} \in E(\Gamma)\}$ is called the open neighborhood of the vertex $\mathcal{F}$ in Γ and the degree $d_\Gamma(\mathcal{F})$ of $\mathcal{F}$ in Γ is the order of $N_\Gamma(\mathcal{F})$. Let $\mathcal{D}_{max} = \mathcal{D}_{max}(\Gamma) = \max\{d_\Gamma(\mathcal{F}) : \mathcal{F} \in V(\Gamma)\}$ be the maximum vertex degree of Γ. The distance $d_\Gamma(\mathcal{F}, \mathcal{G})$ is the number of edges in the shortest path connecting the vertices $\mathcal{F}$ and $\mathcal{G}$ in Γ.

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A pendent vertex in a tree is called a leaf and a vertex incident to a leaf is said to be a support vertex. A support vertex incident to more than one leaf is a strong support vertex. A rooted tree is a tree with a vertex recognized as the root. If $\mathcal{G}$ is a non-root vertex in a tree, the vertex adjacent to $\mathcal{G}$ on the path joining $\mathcal{G}$ and the root vertex is known as the parent of $\mathcal{G}$.

A tree with at most a vertex $\mathcal{F}$ of degree greater than 2 is a spider and $\mathcal{F}$ is called its center. If all vertices of a spider are of degree at most 2, then every vertex can be considered as its center. A path connecting the center of a spider to one of its pendent vertices is called a leg of the spider. By this definition, an n-vertex star can be seen as a spider containing $n-1$ legs and an n-vertex path is a spider having 1 or 2 legs.

Graph invariants are real numbers associated with a graph that are invariants under all graph isomorphisms. One of the most important categories of graph invariants is vertex-degree-based invariants. Zagreb indices [1, 2] are the oldest members of this category which are defined as:

$$M_1(\Gamma) = \sum_{\mathcal{F}\in V(\Gamma)} d_\Gamma^2(\mathcal{F}), \quad M_2(\Gamma) = \sum_{\mathcal{F}\mathcal{G}\in E(\Gamma)} d_\Gamma(\mathcal{F}) d_\Gamma(\mathcal{G}).$$

Further information on these indices can be found in [3–5].

In the last decade, some variants of the Zagreb indices such as Zagreb coindices [6–10], reformulated Zagreb indices [11, 12], multiplicative Zagreb indices [13–16], Lanzhou index [17–19] and entire Zagreb indices [20, 21] have been considered. One of such variants is the *Sombor index* which was suggested by Gutman [22] in 2021. Its definition for a graph Γ is

$$SO(\Gamma) = \sum_{\mathcal{F}\mathcal{G}\in E(\Gamma)} \sqrt{d_\Gamma^2(\mathcal{F}) + d_\Gamma^2(\mathcal{G})}.$$

Gutman [22] also put forward a modification of the Sombor index as:

$$SO_{red}(\Gamma) = \sum_{\mathcal{F}\mathcal{G}\in E(\Gamma)} \sqrt{(d_\Gamma(\mathcal{F})-1)^2 + (d_\Gamma(\mathcal{G})-1)^2},$$

and named it the *reduced Sombor index*. Another modification of the Sombor index entitled the *increased Sombor index* was proposed by Das *et al.* [23] as:

$$SO^{\ddagger}(\Gamma) = \sum_{\mathcal{F}\mathcal{G}\in E(\Gamma)} \sqrt{(d_\Gamma(\mathcal{F})+1)^2 + (d_\Gamma(\mathcal{G})+1)^2}.$$

It is interesting to note that, the Sombor index, reduced Sombor index, and increased Sombor index are all special cases of the (p,q)-*Sombor index* proposed by Milovanović *et al.* [24] as:

$$SO_{p,q}(\Gamma) = \sum_{\mathcal{F}\mathcal{G}\in E(\Gamma)} \Big((d_\Gamma(\mathcal{F})+q)^p + (d_\Gamma(\mathcal{G})+q)^p\Big)^{\frac{1}{p}},$$

where p,q are real numbers and $p \neq 0$. Das *et al.* [25, 26] presented upper and lower bounds on $SO(\Gamma)$ in terms of certain parameters of Γ. Wang *et al.* [27] considered the relationships between $SO(\Gamma)$ and some other degree-based invariants of Γ. Réti *et al.* [28] computed the maximum values of $SO(\Gamma)$ among all r-cyclic connected graphs Γ with n vertices, for $1 \leq r \leq n-2$. For more information about variants of the Sombor index, see [24, 29–38] and the references therein.

Here, we give sharp lower bounds on $SO_{red}(\tau)$ and $SO^{\ddagger}(\tau)$ where τ is a tree with a given order and maximum vertex degree. Moreover, we determine the extremal tree τ which attains the bounds.

The following observation are immediately achieved from the definitions of SO_{red} and $SO^{\ddagger}$ indices.

Observation 1.1. If $\mathcal{E} \notin E(\Gamma)$, then

$$SO_{red}(\Gamma + \mathcal{E}) > SO_{red}(\Gamma),$$

and

$$SO^{\ddagger}(\Gamma + \mathcal{E}) > SO^{\ddagger}(\Gamma).$$

2 Main results

Throughout this section, let $\mathcal{T}(n, \mathcal{D}_{max})$ be the set of trees with order n and maximum degree $\mathcal{D}_{max}$. Assume that $\tau \in \mathcal{T}(n, \mathcal{D}_{max})$ is a rooted tree in which a vertex x of degree $\mathcal{D}_{max}$ is considered to be its root. Also let $N_\tau(x) = \{x_1, x_2, \ldots, x_{\mathcal{D}_{max}}\}$. We begin by proving three useful lemmas.

Lemma 2.1. *If τ has a non-root strong support vertex of degree greater than or equal to 3, then there exists a tree $\tau' \in \mathcal{T}(n, \mathcal{D}_{max})$ with $SO_{red}(\tau) > SO_{red}(\tau')$ and $SO^{\ddagger}(\tau) > SO^{\ddagger}(\tau')$.*

Proof. Suppose that $y \neq x$ is a strong support vertex and $d_T(y) = \alpha \geq 3$ where $d_\tau(x, y)$ is as large as possible and let $N_\tau(y) = \{y_1, y_2, \ldots, y_\alpha\}$. Without loss of generality, let y_α be the parent of y and $d_\tau(y_1) = d_\tau(y_2) = 1$. Denote by τ' the tree achieved by attaching the path y_1y_2y to $\tau - \{y_1, y_2\}$. Clearly, $\tau' \in \mathcal{T}(n, \mathcal{D}_{max})$. Since $\alpha \geq 3$, we get

$$\begin{aligned} SO_{red}(\tau) - SO_{red}(\tau') &= \sum_{rs \in E(\tau)} \sqrt{(d_\tau(r) - 1)^2 + (d_\tau(s) - 1)^2} \\ - & \sum_{rs \in E(\tau')} \sqrt{(d_{\tau'}(r) - 1)^2 + (d_{\tau'}(s) - 1)^2} = \sqrt{(d_\tau(y_1) - 1)^2 + (d_\tau(y) - 1)^2} \\ + & \sqrt{(d_\tau(y_2) - 1)^2 + (d_\tau(y) - 1)^2} + \sum_{i=3}^{\alpha} \sqrt{(d_\tau(y) - 1)^2 + (d_\tau(y_i) - 1)^2} \\ - & \sqrt{(d_{\tau'}(y_1) - 1)^2 + (d_{\tau'}(y_2) - 1)^2} - \sqrt{(d_{\tau'}(y_2) - 1)^2 + (d_{\tau'}(y) - 1)^2} \\ - & \sum_{i=3}^{\alpha} \sqrt{(d_\tau(y) - 2)^2 + (d_\tau(y_i) - 1)^2} = 2(\alpha - 1) + \sum_{i=3}^{\alpha} \sqrt{(\alpha - 1)^2 + (d_\tau(y_i) - 1)^2} \\ - & 1 - \sqrt{(\alpha - 2)^2 + 1} - \sum_{i=3}^{\alpha} \sqrt{(\alpha - 2)^2 + (d_\tau(y_i) - 1)^2} > 2(\alpha - 1) - 1 - \sqrt{(\alpha - 2)^2 + 1} > 0, \end{aligned}$$

and

$$\begin{aligned} SO^{\ddagger}(\tau) - SO^{\ddagger}(\tau') &= \sum_{rs\in E(\tau)} \sqrt{(d_\tau(r)+1)^2+(d_\tau(s)+1)^2} \\ - &\sum_{rs\in E(\tau')} \sqrt{(d_{\tau'}(r)+1)^2+(d_{\tau'}(s)+1)^2} = \sqrt{(d_\tau(y_1)+1)^2+(d_\tau(y)+1)^2} \\ + &\sqrt{(d_\tau(y_2)+1)^2+(d_\tau(y)+1)^2} + \sum_{i=3}^{\alpha}\sqrt{(d_\tau(y)+1)^2+(d_\tau(y_i)+1)^2} \\ - &\sqrt{(d_{\tau'}(y_1)+1)^2+(d_{\tau'}(y_2)+1)^2} - \sqrt{(d_{\tau'}(y_2)+1)^2+(d_{\tau'}(y)+1)^2} \\ - &\sum_{i=3}^{\alpha}\sqrt{d_\tau^2(y)+(d_\tau(y_i)+1)^2} = 2\sqrt{(\alpha+1)^2+4} + \sum_{i=3}^{\alpha}\sqrt{(\alpha+1)^2+(d_T(y_i)+1)^2} \\ - &\sqrt{13} - \sqrt{\alpha^2+9} - \sum_{i=3}^{\alpha}\sqrt{\alpha^2+(d_T(y_i)+1)^2} > 2\sqrt{(\alpha+1)^2+4} - \sqrt{13} - \sqrt{\alpha^2+9} > 0. \end{aligned}$$

Hence the desired results hold. ■

Lemma 2.2. *If τ has a non-root support vertex of degree greater than or equal to 3, then there exists a tree $\tau' \in \mathcal{T}(n, \mathcal{D}_{max})$ with $SO_{red}(\tau) > SO_{red}(\tau')$ and $SO^{\ddagger}(\tau) > SO^{\ddagger}(\tau')$.*

Proof. Assume that $y \neq x$ is a support vertex of degree $d_\tau(y) = \alpha \geq 3$ where $d_\tau(x, y)$ is as large as possible and suppose $N_\tau(y) = \{y_1, y_2, \ldots, y_\alpha\}$. Let y_α be the parent of y. Since y is a support vertex, it may be assumed that $d_\tau(y_1) = 1$ and by Lemma 2.1, $d_\tau(y_i) = 2$ for $2 \leq i \leq \alpha - 1$. Let $yz_1z_2 \ldots z_t$ be a path in τ where $t \geq 2$ and $y_2 = z_1$. Assume that τ' is the tree derived from $\tau - \{y_1\}$ by attaching the path z_ty_1. Since $\alpha \geq 3$, we obtain:

$$\begin{aligned} SO_{red}(\tau) - SO_{red}(\tau') &= \sum_{rs\in E(\tau)} \sqrt{(d_\tau(r)-1)^2+(d_\tau(s)-1)^2} \\ - &\sum_{rs\in E(\tau')} \sqrt{(d_{\tau'}(r)-1)^2+(d_{\tau'}(s)-1)^2} = \sqrt{(d_\tau(y_1)-1)^2+(d_\tau(y)-1)^2} \\ + &\sqrt{(d_\tau(z_t)-1)^2+(d_\tau(z_{t-1})-1)^2} + \sum_{i=2}^{\alpha}\sqrt{(d_\tau(y)-1)^2+(d_\tau(y_i)-1)^2} \\ - &\sqrt{(d_{\tau'}(y_1)-1)^2+(d_{\tau'}(z_t)-1)^2} - \sqrt{(d_{\tau'}(z_t)-1)^2+(d_{\tau'}(z_{t-1})-1)^2} \\ - &\sum_{i=2}^{\alpha}\sqrt{(d_\tau(y)-2)^2+(d_\tau(y_i)-1)^2} = \alpha - 1 + 1 + \sum_{i=2}^{\alpha}\sqrt{(\alpha-1)^2+(d_\tau(y_i)-1)^2} \\ - &1 - \sqrt{2} - \sum_{i=2}^{\alpha}\sqrt{(\alpha-2)^2+(d_\tau(y_i)-1)^2} > \alpha - 1 - \sqrt{2} > 0, \end{aligned}$$

and

$$SO^{\ddagger}(\tau)-SO^{\ddagger}(\tau')=\sum_{rs\in E(\tau)}\sqrt{(d_\tau(r)+1)^2+(d_\tau(s)+1)^2}$$
$$-\sum_{rs\in E(\tau')}\sqrt{(d_{\tau'}(r)+1)^2+(d_{\tau'}(s)+1)^2}=\sqrt{(d_\tau(y_1)+1)^2+(d_\tau(y)+1)^2}$$
$$+\sqrt{(d_\tau(z_t)+1)^2+(d_\tau(z_{t-1})+1)^2}+\sum_{i=2}^{\alpha}\sqrt{(d_\tau(y)+1)^2+(d_\tau(y_i)+1)^2}$$
$$-\sqrt{(d_{\tau'}(y_1)+1)^2+(d_{\tau'}(z_t)+1)^2}-\sqrt{(d_{\tau'}(z_t)+1)^2+(d_{\tau'}(z_{t-1})+1)^2}$$
$$-\sum_{i=2}^{\alpha}\sqrt{d_\tau^2(y)+(d_\tau(y_i)+1)^2}=\sqrt{(\alpha+1)^2+4}+\sqrt{13}+\sum_{i=2}^{\alpha}\sqrt{(\alpha+1)^2+(d_\tau(y_i)+1)^2}$$
$$-\sqrt{13}-\sqrt{18}-\sum_{i=2}^{\alpha}\sqrt{\alpha^2+(d_\tau(y_i)+1)^2}>\sqrt{(\alpha+1)^2+4}-\sqrt{18}>0,$$

and the desired results hold. ■

Lemma 2.3. *If τ has a non-root vertex of degree greater than or equal to 3, then there exists a tree $\tau'\in\mathcal{T}(n,\mathcal{D}_{max})$ with $SO_{red}(\tau)>SO_{red}(\tau')$ and $SO^{\ddagger}(\tau)>SO^{\ddagger}(\tau')$.*

Proof. Assume that $y\neq x$ is a support vertex and $d_\tau(y)=\alpha\geq 3$ where $d_\tau(x,y)$ is as large as possible and suppose $N_\tau(y)=\{y_1,y_2,\ldots,y_\alpha\}$. Let y_α be the parent of y. By Lemmas 2.1 and 2.2, $d_\tau(y_i)=2$ for $1\leq i\leq\alpha-1$. Let $yz_1z_2\ldots z_t$ and $yw_1w_2\ldots w_k$ be two paths in τ for $t,k\geq 2$ with $y_1=w_1$ and $y_2=z_1$. Let τ' be the tree deduced from τ by removing the edge xy_1 and adding the edge z_ty_1. Since $\alpha\geq 3$, we get:

$$\begin{aligned}SO_{red}(\tau)-SO_{red}(\tau') &= \sum_{rs\in E(\tau)}\sqrt{(d_\tau(r)-1)^2+(d_\tau(s)-1)^2}\\ &- \sum_{rs\in E(\tau')}\sqrt{(d_{\tau'}(r)-1)^2+(d_{\tau'}(s)-1)^2}\\ &= \sqrt{(d_\tau(y_1)-1)^2+(d_\tau(y)-1)^2}\\ &+ \sqrt{(d_\tau(z_t)-1)^2+(d_\tau(z_{t-1})-1)^2}\\ &+ \sum_{i=2}^{\alpha}\sqrt{(d_\tau(y)-1)^2+(d_\tau(y_i)-1)^2}\\ &- \sqrt{(d_{\tau'}(y_1)-1)^2+(d_{\tau'}(z_t)-1)^2}\\ &- \sqrt{(d_{\tau'}(z_t)-1)^2+(d_{\tau'}(z_{t-1})-1)^2}\\ &- \sum_{i=2}^{\alpha}\sqrt{(d_\tau(y)-2)^2+(d_\tau(y_i)-1)^2}\\ &= \sqrt{(\alpha-1)^2+1}+1+\sum_{i=2}^{\alpha}\sqrt{(\alpha-1)^2+(d_\tau(y_i)-1)^2}\\ &- 2\sqrt{2}-\sum_{i=2}^{\alpha}\sqrt{(\alpha-2)^2+(d_\tau(y_i)-1)^2}\\ &> \sqrt{(\alpha-1)^2+1}+1-2\sqrt{2}>0,\end{aligned}$$

and

$$
\begin{aligned}
SO^{\ddagger}(\tau)-SO^{\ddagger}(\tau') &= \sum_{rs\in E(\tau)} \sqrt{(d_{\tau}(r)+1)^2+(d_{\tau}(s)+1)^2} \\
&- \sum_{rs\in E(\tau')} \sqrt{(d_{\tau'}(r)+1)^2+(d_{\tau'}(s)+1)^2} \\
&= \sqrt{(d_{\tau}(y_1)+1)^2+(d_{\tau}(y)+1)^2} \\
&+ \sqrt{(d_{\tau}(z_t)+1)^2+(d_{\tau}(z_{t-1})+1)^2} \\
&+ \sum_{i=2}^{\alpha} \sqrt{(d_{\tau}(y)+1)^2+(d_{\tau}(y_i)+1)^2} \\
&- \sqrt{(d_{\tau'}(y_1)+1)^2+(d_{\tau'}(z_t)+1)^2} \\
&- \sqrt{(d_{\tau'}(z_t)+1)^2+(d_{\tau'}(z_{t-1})+1)^2} \\
&- \sum_{i=2}^{\alpha} \sqrt{d_{\tau}^2(y)+(d_{\tau}(y_i)+1)^2} \\
&= \sqrt{(\alpha+1)^2+9}+\sqrt{13}+\sum_{i=2}^{\alpha} \sqrt{(\alpha+1)^2+(d_{\tau}(y_i)+1)^2} \\
&- 2\sqrt{18}-\sum_{i=2}^{\alpha} \sqrt{\alpha^2+(d_{\tau}(y_i)+1)^2} \\
&> \sqrt{(\alpha+1)^2+9}+\sqrt{13}-2\sqrt{18}>0,
\end{aligned}
$$

and the proof is completed. ∎

Proposition 2.4. *Consider a spider τ with n vertices and $l \geq 3$ legs. If τ contains a leg with length 1 and another leg with a length of at least 3, then there exists a spider τ' with n vertices and l legs for which $SO_{red}(\tau) > SO_{red}(\tau')$ and $SO^{\ddagger}(\tau) > SO^{\ddagger}(\tau')$.*

Proof. Denote by x the center of τ and assume that $N_{\tau}(x) = \{x_1, \ldots, x_l\}$. Root τ at x. One may assume that $d_{\tau}(x_1) = 1$. Let $x_2y_1y_2\ldots y_t$, $t \geq 2$ be a leg of τ with greatest length. Denote by τ' the tree derived from τ by removing the edge y_ty_{t-1} and adding the pendent edge x_1y_t. From the definition, we have:

$$
\begin{aligned}
SO_{red}(\tau)-SO_{red}(\tau') &= \sum_{rs\in E(\tau)} \sqrt{(d_{\tau}(r)-1)^2+(d_{\tau}(s)-1)^2} \\
&- \sum_{rs\in E(\tau')} \sqrt{(d_{\tau'}(r)-1)^2+(d_{\tau'}(s)-1)^2} \\
&= \sqrt{(\mathcal{D}_{max}-1)^2}+\sqrt{2}-\sqrt{(\mathcal{D}_{max}-1)^2+1}-1>0,
\end{aligned}
$$

and

$$
\begin{aligned}
SO^{\ddagger}(\tau)-SO^{\ddagger}(\tau') &= \sum_{rs\in E(\tau)} \sqrt{(d_{\tau}(r)+1)^2+(d_{\tau}(s)+1)^2} \\
&- \sum_{rs\in E(\tau')} \sqrt{(d_{\tau'}(r)+1)^2+(d_{\tau'}(s)+1)^2} \\
&= \sqrt{(\mathcal{D}_{max}+1)^2+4}+\sqrt{18}-\sqrt{(\mathcal{D}_{max}+1)^2+9}-\sqrt{13}>0.
\end{aligned}
$$

This complete the proof. ∎

Here, we state our main results.

Theorem 2.5. *Let* $\tau \in \mathcal{T}(n, \mathcal{D}_{max})$ *and* $n \geq 3$*. If* $\mathcal{D}_{max} \leq \frac{n-1}{2}$*, then*

$$SO_{red}(\tau) \geq \mathcal{D}_{max}\sqrt{(\mathcal{D}_{max}-1)^2+1} + \sqrt{2}(n-2\mathcal{D}_{max}-1) + \mathcal{D}_{max},$$

otherwise,

$$SO_{red}(\tau) \geq (2\mathcal{D}_{max}+1-n)(\mathcal{D}_{max}-1) + (n-\mathcal{D}_{max}-1)(\sqrt{(\mathcal{D}_{max}-1)^2+1}+1).$$

The equality holds if and only if τ *is a spider with all legs are of length less than 3 or all legs are of length more than 1.*

Proof. Assume that $\tau^* \in \mathcal{T}(n, \mathcal{D}_{max})$ is a tree of order at least 3 such that

$$SO_{red}(\tau^*) = \min\{SO_{red}(\tau) : \tau \in \mathcal{T}(n, \mathcal{D}_{max})\}.$$

Select a vertex x with $d_{\tau^*}(x) = \mathcal{D}_{max}$ as the root vertex of τ^*. If $\mathcal{D}_{max} = 2$, then τ is an n-vertex path and $SO_{red}(P_n) = \sqrt{2}(n-3)+2$. If $\mathcal{D}_{max} \geq 3$, then by the selection of τ^*, it is clear from Lemmas 2.1 to 2.3, that τ^* must be a spider centered at x. By Proposition 2.4 and the selection of τ^*, all legs of τ^* are of length less than 3 or all are of length more than 1. First, consider the case that all legs of τ^* are of length more than 1. It is obvious that $\mathcal{D}_{max} \leq \frac{n-1}{2}$ and

$$SO_{red}(\tau) \geq \mathcal{D}_{max}\sqrt{(\mathcal{D}_{max}-1)^2+1} + \sqrt{2}(n-2\mathcal{D}_{max}-1) + \mathcal{D}_{max}.$$

Now let all legs of τ^* be of length less than 3. Considering the previous case, it might be assumed that τ^* contains a leg of length 1. In case $\tau^* = S_n$, then there is nothing to prove, otherwise there are $2\mathcal{D}_{max}+1-n$ leaves adjacent to x and we get

$$SO_{red}(\tau) \geq (2\mathcal{D}_{max}+1-n)(\mathcal{D}_{max}-1) + (n-\mathcal{D}_{max}-1)(\sqrt{(\mathcal{D}_{max}-1)^2+1}+1),$$

and the desired result holds. ∎

Using the same argument as given in Theorem 2.5, we arrive at:

Theorem 2.6. *Let* $\tau \in \mathcal{T}(n, \mathcal{D}_{max})$ *and* $n \geq 3$*. If* $\mathcal{D}_{max} \leq \frac{n-1}{2}$*, then*

$$SO^{\ddagger}(\tau) \geq \mathcal{D}_{max}\sqrt{(\mathcal{D}_{max}+1)^2+9} + \sqrt{18}(n-2\mathcal{D}_{max}-1) + \sqrt{13}\mathcal{D}_{max},$$

otherwise,

$$SO^{\ddagger}(\tau) \geq (2\mathcal{D}_{max}+1-n)\sqrt{(\mathcal{D}_{max}+1)^2+4} + (n-\mathcal{D}_{max}-1)(\sqrt{(\mathcal{D}_{max}+1)^2+9}+\sqrt{13}).$$

The equality holds if and only if τ *is a spider with all legs are of length less than 3 or all legs are of length more than 1.*

In Figure 1, three trees of orders $n = 8, 9, 10$ with maximum degree $\mathcal{D}_{max} = 4$ and minimum SO_{red} and $SO^{\ddagger}$ indices are depicted.

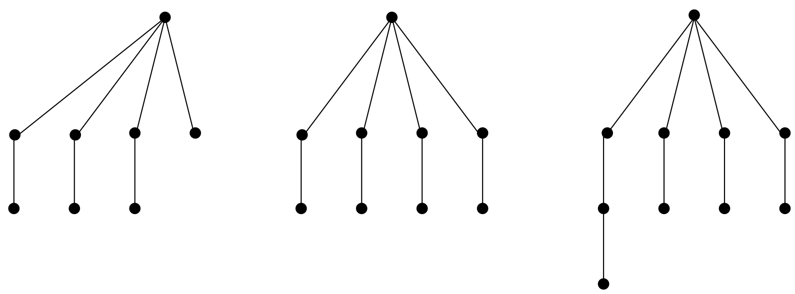

Figure 1. Trees with $n = 8, 9, 10$ and $\mathcal{D}_{max} = 4$.

By Observation 1.1, we arrive at the following corollaries.

Corollary 2.7. *Let* Γ *be a graph with* n *vertices and maximum degree* $\mathcal{D}_{max}$. *If* $\mathcal{D}_{max} \leq \frac{n-1}{2}$, *then*

$$SO_{red}(\Gamma) \geq \mathcal{D}_{max}\sqrt{(\mathcal{D}_{max}-1)^2+1} + \sqrt{2}(n-2\mathcal{D}_{max}-1) + \mathcal{D}_{max},$$

otherwise,

$$SO_{red}(\Gamma) \geq (2\mathcal{D}_{max}+1-n)(\mathcal{D}_{max}-1) + (n-\mathcal{D}_{max}-1)(\sqrt{(\mathcal{D}_{max}-1)^2+1}+1),$$

with equality if and only if Γ *is a spider with all legs are of length less than 3 or all legs are of length more than 1.*

Corollary 2.8. *Let* Γ *be a graph of order* n *and maximum degree* $\mathcal{D}_{max}$. *If* $\mathcal{D}_{max} \leq \frac{n-1}{2}$, *then*

$$SO^{\ddagger}(\Gamma) \geq \mathcal{D}_{max}\sqrt{(\mathcal{D}_{max}+1)^2+9} + \sqrt{18}(n-2\mathcal{D}_{max}-1) + \sqrt{13}\mathcal{D}_{max},$$

otherwise,

$$SO^{\ddagger}(\Gamma) \geq (2\mathcal{D}_{max}+1-n)\sqrt{(\mathcal{D}_{max}+1)^2+4} + (n-\mathcal{D}_{max}-1)(\sqrt{(\mathcal{D}_{max}+1)^2+9}+\sqrt{13}),$$

with equality if and only if Γ *is a spider with all legs are of length less than 3 or all legs are of length more than 1.*

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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