Iranian Journal of Mathematical Chemistry

IJMC

DOI: 10.22052/IJMC.2023.253969.1794
Vol. 15, No. 1, 2024, pp. 27-37
Research Paper

Expected Value of Zagreb Indices of Random Bipartite Graphs

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(Dedicated to the memory of Professor Ali Reza Ashrafi.)

Keywords:

Random bipartite graphs,
Random symmetric $(0, 1)$-matrix,
Zagreb indices,
Expected value

AMS Subject Classification (2020):

05C07; 05C80; 60B20

Article History:

Received: 8 December 2023
Accepted: 23 December 2023

Abstract

In this paper, we calculate the expected values of the first and second Zagreb indices, denoted as $\mathbf{E}(M_1)$ and $\mathbf{E}(M_2)$ respectively, as well as the expected value of the forgotten index, $\mathbf{E}(F)$, for two models of random bipartite graphs. To evaluate our findings, we establish the growth rate by demonstrating that for a random bipartite graph G of order n in either model, the expected value of $M_1(G)$ is $O(n^3)$. Furthermore, we prove that the expected values of $M_2(G)$ and $F(G)$ are both $O(n^4)$.

1 Introduction

The study of random graphs is a critical area in theoretical graph theory with numerous applications in physics. Currently, they are commonly utilized as standard null models in simulating a variety of physical processes on graphs and networks, as discussed in [1]. A random graph involves a set of isolated vertices and starting with a random manner by adding edges between them. Various random graph models generate distinct probability distributions for the resulting graphs. One of the frequently studied models is $G(n, p)$, encompassing all labeled graphs with n vertices. In this model, each possible edge appears independently with a probability of $0 < p < 1$, as explained in references [2, 3]. Another natural model of random graphs is $G(n, m)$, which represents the probability space of all graphs with n labeled vertices and m edges. In [4, 5], Erdös and Rényi considered these random graphs.

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Academic Editor: Gholam Hossein Fath-Tabar

Here we only consider bipartite graphs with no multiple edges and no loops. In a random bipartite graph $G(n_1, n_2, p)$, there are n_1 labeled vertices of one color (e.g., red) and n_2 labeled vertices of another color (e.g., blue) and let each of $n_1 n_2$ possible edges connecting a red vertex with a blue one, occurs with a prescribed probability p, $(0 < p < 1)$, independent of all other edges [6, 7]. A random bipartite graph in $G(n_1, n_2, m)$ is a random bipartite graph with vertex partition sizes (n_1, n_2) and m number of edges [7]. The notation of random bipartite graph models is consistent with that used for the general random graph models considered in [8].

Assume that G is a graph with the sets $V(G) = \{v_1, \ldots, v_n\}$ and $E(G)$ as the vertex set and edge set, respectively. For a vertex v in G, its degree is denoted by $d(v)$. The $n \times n$ matrix $A = [a_{ij}]$ is the *adjacency matrix* of the graph G, where $a_{ij} = 1$ if $v_i v_j \in E(G)$ and $a_{ij} = 0$, otherwise. Suppose G is a bipartite graph with vertex set partitioned as $V(G) = (U, W)$, where $U = \{u_1, \ldots, u_{n_1}\}$ and $W = \{w_1, \ldots, w_{n_2}\}$. The adjacency matrix of G has the block form $A = \begin{pmatrix} 0_{n_1 n_1} & B \\ B^{\mathsf{T}} & 0_{n_2 n_2} \end{pmatrix}$, where B is a $n_1 \times n_2$ matrix with $b_{ij} = 1$ if $u_i w_j \in E(G)$, and $b_{ij} = 0$ otherwise. Additionally, $0_{n_1 n_1}$ and $0_{n_2 n_2}$ are zero matrices of orders $n_1 \times n_1$ and $n_2 \times n_2$, respectively. The matrix B uniquely represents the bipartite graph G, rendering the remaining parts of A redundant. As a result, B is commonly referred to as the *biadjacency matrix* of G. It is noteworthy that each random bipartite graph in $G(n_1, n_2, p)$ corresponds to a random $(0, 1)$-matrix B of order $n_1 \times n_2$, where each entry is equal to 1 with probability p, and vice versa. Similarly, a random bipartite graph in $G(n_1, n_2, m)$ can be represented by a random $(0, 1)$-matrix B of order $n_1 \times n_2$ that contains exactly m entries equal to 1, and vice versa. Consequently, investigating bipartite random graphs are interchangeable with studying the associated random $(0, 1)$-matrices.

By a *graph invariant* or a *topological index*, we mean a numerical quantity that can be determined for a graph, uniquely. Also, it remains unchanged under graph isomorphism. In chemistry, graph invariants are widely used as molecular descriptors. Thus various graph invariants have been studied in chemical graph theory and applied in research. The *first Zagreb* (M_1) and the *second Zagreb index* (M_2) are among the oldest and most extensively investigated invariants, for further details on degree-based graph invariants refer to [9–15]. These indices are defined as follows for a graph G:

$$M_1(G) = \sum_{ab \in E(G)} d(a) + d(b) = \sum_{u \in V(G)} d^2(u) \quad \text{and} \quad M_2(G) = \sum_{ab \in E(G)} d(a)d(b).$$

One can arises the *forgotten topological index* as a measure of a graph's structural properties, by substituting the cube of the vertex degrees in the first Zagreb index instead of the square. This index was introduced in [16], denoted by $F(G)$. The applications of $F(G)$ were demonstrated in [16].

$$F(G) = \sum_{ab \in E(G)} d^2(a) + d^2(b) = \sum_{u \in V(G)} d^3(u).$$

Consider a random bipartite graph G with bipartition sizes n_1 and n_2, and corresponding biadjacency matrix B. Alternatively, one can consider a random $(0, 1)$-matrix B. In either case, the following relationship holds:

$$M_1(G) = D_1 D_1^T + D_2^T D_2,$$

where $D_1 = j_{n_1}^T B$ and $D_2 = B j_{n_2}$ are two vectors containing the degrees of vertices in two parts of G and j_n is an $n \times 1$ matrix consisting of all ones. Furthermore, let $R = (r_1, \ldots, r_{n_1})^T$ and $S = (s_1, \ldots, s_{n_2})^T$ are two real-valued column vectors, and $C = [c_{ij}]$ is an $n_1 \times n_2$ real-valued matrix. It is apparent that

$$R^T CS = \sum_{1\le p\le n_1, 1\le q\le n_2} c_{pq} r_p s_q.$$

Therefore, for a random bipartite graph G whose biadjacency matrix is B (alternatively, for a random $(0,1)$-matrix B), the following correlation can be derived:

$$M_2(G) = D_1 B D_2^T.$$

Consider a random bipartite graph G with parts $U = \{u_1, \dots, u_{n_1}\}$ and $W = \{w_1, \dots, w_{n_2}\}$. The *indicator random variables* X_{ij}, $1 \le i \le n_1$ and $1 \le j \le n_2$ are defined as follows:

$$X_{ij} = \begin{cases} 1, & \text{if } u_i \text{ is adjacent to } w_j, \\ 0, & \text{otherwise.} \end{cases}$$

It is important to observe that for a random bipartite graph G with edge probability p, denoted by $G(n_1, n_2, p)$, the indicator random variables X_{ij} and X_{rs} are independent when $1 \le i, r \le n_1$, $1 \le j, s \le n_2$, and $i, j \ne r, s$. Also if G is a random bipartite graph with a fixed number of edges m, denoted by $G(n_1, n_2, m)$, then the indicator random variables X_{ij} are not independent, where $1 \le i \le n_1$ and $1 \le j \le n_2$. The expected value of a random variable such as X is its average value, denoted by $\mathbf{E}(X)$. When two random variables A and B are independent, it follows that $\mathbf{E}(AB) = \mathbf{E}(A)\mathbf{E}(B)$. Furthermore, in the case in which A is an indicator random variable, then $\mathbf{E}(A^k) = \mathbf{E}(A)$ for every $k > 0$.

The authors in [17] calculated the expected values of generalized Zagreb indices of graphs in $G(n, p)$ and $G(n, m)$. Motivated by this, here we continue the process of exploring the expected values of the first and second Zagreb indices, along with the forgotten index, for graphs in $G(n_1, n_2, p)$ and $G(n_1, n_2, m)$. We also analyze the growth rates of these indices' expected values.

2 Random bipartite graphs $G(n_1, n_2, p)$

This section derives the expected values of certain degree-based graph invariants for random bipartite graphs $G(n_1, n_2, p)$. The main theorem of this section is presented below.

Theorem 2.1. *Let $G \in G(n_1, n_2, p)$ and $n_1, n_2 \ge 2$. Then*

(a) $\boldsymbol{E}(M_1(G)) = pn_1n_2(p(n_1 + n_2 - 2) + 2)$,

(b) $\boldsymbol{E}(M_2(G)) = n_1n_2\left(p + (n_1 + n_2 - 2)p^2 + (n_1 - 1)(n_2 - 1)p^3\right)$.

Also if $n_1, n_2 \ge 3$, then

(c) $\boldsymbol{E}(F(G)) = pn_1n_2(p^2(n_2 - 1)(n_2 - 2) + p^2(n_1 - 1)(n_1 - 2) + 3p(n_1 + n_2 - 2) + 2$.

Proof. Let U and W be the bipartition parts of $V(G)$ such that $U = \{u_1, \dots, u_{n_1}\}$ and $W = \{w_1, \dots, w_{n_2}\}$. Consider a random variable D_i for the degree of u_i, $i = 1, \dots, n_1$, and similarly, consider D_j as a random variable associating to the degree of w_j, $j = 1, \dots, n_2$. So, for each $1 \le i \le n_1$, $D_i = \sum_{k=1}^{n_2} X_{ik}$, where each X_{ik} is an indicator random variable corresponding to the edge u_iw_k and similarly, for every $1 \le j \le n_2$, D_j can be computed as $D_j = \sum_{r=1}^{n_1} X_{jr}$.

Now, we have

$$\begin{aligned}
\mathbf{E}\left(M_1(G)\right) &= \mathbf{E}\left(\sum_{i=1}^{n_1} D_i^2 + \sum_{j=1}^{n_2} D_j^2\right) = \sum_{i=1}^{n_1}\mathbf{E}\left(D_i^2\right) + \sum_{j=1}^{n_2}\mathbf{E}\left(D_j^2\right) \\
&= \sum_{i=1}^{n_1}\mathbf{E}\left((\sum_{k=1}^{n_2} X_{ik})^2\right) + \sum_{j=1}^{n_2}\mathbf{E}\left((\sum_{r=1}^{n_1} X_{jr})^2\right) \\
&= \sum_{i=1}^{n_1}\mathbf{E}\left(\sum_{k=1}^{n_2} X_{ik}^2\right) + \sum_{i=1}^{n_1}\mathbf{E}\left(\sum_{k=1}^{n_2}\sum_{\substack{t=1\\t\neq k}}^{n_2} X_{ik}X_{it}\right) \\
&\quad + \sum_{j=1}^{n_2}\mathbf{E}\left(\sum_{r=1}^{n_1} X_{jr}^2\right) + \sum_{j=1}^{n_2}\mathbf{E}\left(\sum_{r=1}^{n_1}\sum_{\substack{s=1\\s\neq r}}^{n_1} X_{jr}X_{js}\right),
\end{aligned}$$

and so,

$$\begin{aligned}
\mathbf{E}\left(M_1(G)\right) &= \sum_{i=1}^{n_1}\sum_{k=1}^{n_2}\mathbf{E}\left(X_{ik}^2\right) + \sum_{i=1}^{n_1}\sum_{k=1}^{n_2}\sum_{\substack{t=1\\t\neq k}}^{n_2}\mathbf{E}\left(X_{ik}X_{it}\right) \\
&\quad + \sum_{i=1}^{n_2}\sum_{r=1}^{n_1}\mathbf{E}\left(X_{jr}^2\right) + \sum_{j=1}^{n_2}\sum_{r=1}^{n_1}\sum_{\substack{s=1\\s\neq r}}^{n_1}\mathbf{E}\left(X_{jr}X_{js}\right) \\
&= n_1n_2p + n_1n_2(n_2-1)p^2 + n_2n_1p + n_2n_1(n_1-1)p^2 \\
&= 2n_1n_2p + n_1n_2(n_1+n_2-2)p^2,
\end{aligned}$$

as desired. Now, continue with the exploring of the second Zagreb index's expected value, as follows:

$$\begin{aligned}
\mathbf{E}\left(M_2(G)\right) &= \mathbf{E}\left(\sum_{u_iw_j\in E(G)} D_iD_j\right) = \mathbf{E}\left(\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} D_iD_jX_{ij}\right) \qquad (1) \\
&= \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\mathbf{E}\left(D_iD_jX_{ij}\right).
\end{aligned}$$

Consider $\mathbf{E}\left(D_iD_jX_{ij}\right)$ in the last summation in Equation (1) for fixed positive integers i and j. Due to X_{st}'s are independent variables, $s = 1,\ldots,n_1$ and $t = 1,\ldots,n_2$, the following holds.

$$\begin{aligned}
\mathbf{E}\left(D_iD_jX_{ij}\right) &= \sum_{k=1}^{n_2}\sum_{r=1}^{n_1}\mathbf{E}\left(X_{ik}X_{jr}X_{ij}\right) \qquad (2) \\
&= \mathbf{E}\left(X_{ij}X_{ji}X_{ij}\right) + \sum_{\substack{k=1\\k\neq j}}^{n_2}\mathbf{E}\left(X_{ik}\right)\mathbf{E}\left(X_{ji}X_{ij}\right) \\
&\quad + \sum_{\substack{r=1\\r\neq i}}^{n_1}\mathbf{E}\left(X_{jr}\right)\mathbf{E}\left(X_{ij}X_{ij}\right) + \sum_{\substack{k=1\\k\neq j}}^{n_2}\sum_{\substack{r=1\\r\neq i}}^{n_1}\mathbf{E}(X_{ik})\mathbf{E}(X_{jr})\mathbf{E}(X_{ij}) \\
&= p + (n_2-1)p^2 + (n_1-1)p^2 + (n_1-1)(n_2-1)p^3.
\end{aligned}$$

The equations in (1) and (2) yield the next relations:

$$\begin{aligned}
\mathbf{E}\left(M_2(G)\right) &= \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\mathbf{E}\left(D_iD_jX_{ij}\right)\\
&= \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\left(p+(n_1+n_2-2)p^2+(n_1-1)(n_2-1)p^3\right)\\
&= n_1n_2\left(p+(n_1+n_2-2)p^2+(n_1-1)(n_2-1)p^3\right),
\end{aligned}$$

which gives the Part (b).

In the sequel, we obtain the expected value of the forgotten index of G.

$$\begin{aligned}
\mathbf{E}\left(F(G)\right) &= \mathbf{E}\left(\sum_{u_i\in U(G)}D_i^3+\sum_{w_j\in W(G)}D_j^3\right)=\sum_{i=1}^{n_1}\mathbf{E}\left(D_i^3\right)+\sum_{j=1}^{n_2}\mathbf{E}\left(D_j^3\right)\\
&= \sum_{i=1}^{n_1}\mathbf{E}\left((\sum_{k=1}^{n_2}X_{ik})^3\right)+\sum_{j=1}^{n_2}\mathbf{E}\left((\sum_{r=1}^{n_1}X_{jr})^3\right),
\end{aligned}$$

and so,

$$\begin{aligned}
\mathbf{E}\left(F(G)\right) &= \sum_{i=1}^{n_1}\mathbf{E}\left(\sum_{k=1}^{n_2}X_{ik}^3\right)+3\sum_{i=1}^{n_1}\sum_{k=1}^{n_2}\mathbf{E}\left(X_{ik}^2\left(\sum_{\substack{t=1\\t\neq k}}^{n_2}X_{it}\right)\right)\\
&+\sum_{i=1}^{n_1}\mathbf{E}\left(\sum_{k=1}^{n_2}\sum_{\substack{t=1\\t\neq k}}^{n_2}\sum_{\substack{s=1\\s\neq k,t}}^{n_2}X_{ik}X_{it}X_{is}\right)+\sum_{j=1}^{n_2}\mathbf{E}\left(\sum_{r=1}^{n_1}X_{jr}^3\right)\\
&+3\sum_{j=1}^{n_2}\sum_{r=1}^{n_1}\mathbf{E}\left(X_{jr}^2\left(\sum_{\substack{l=1\\l\neq r}}^{n_1}X_{jl}\right)\right)+\sum_{j=1}^{n_2}\mathbf{E}\left(\sum_{r=1}^{n_1}\sum_{\substack{l=1\\l\neq r}}^{n_1}\sum_{\substack{m=1\\m\neq r,l}}^{n_1}X_{jr}X_{jl}X_{jm}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbf{E}\left(F(G)\right) &= \sum_{i=1}^{n_1}\sum_{k=1}^{n_2}\mathbf{E}\left(X_{ik}^3\right)+3\sum_{i=1}^{n_1}\sum_{k=1}^{n_2}\mathbf{E}\left(X_{ik}^2\right)\left(\sum_{\substack{t=1\\t\neq k}}^{n_2}\mathbf{E}\left(X_{it}\right)\right)\\
&+\sum_{i=1}^{n_1}\sum_{k=1}^{n_2}\sum_{\substack{t=1\\t\neq k}}^{n_2}\sum_{\substack{s=1\\s\neq k,t}}^{n_2}\mathbf{E}\left(X_{ik}\right)\mathbf{E}\left(X_{it}\right)\mathbf{E}\left(X_{is}\right)\\
&+\sum_{j=1}^{n_2}\sum_{r=1}^{n_1}\mathbf{E}\left(X_{jr}^3\right)+3\sum_{j=1}^{n_2}\sum_{r=1}^{n_1}\mathbf{E}\left(X_{jr}^2\right)\left(\sum_{\substack{l=1\\l\neq r}}^{n_1}\mathbf{E}\left(X_{jl}\right)\right)\\
&+\sum_{j=1}^{n_2}\sum_{r=1}^{n_1}\sum_{\substack{l=1\\l\neq r}}^{n_1}\sum_{\substack{m=1\\m\neq r,l}}^{n_1}\mathbf{E}\left(X_{jr}\right)\mathbf{E}\left(X_{jl}\right)\mathbf{E}\left(X_{jm}\right)\\
&= n_1n_2p+3n_1n_2(n_2-1)p^2+n_1n_2(n_2-1)(n_2-2)p^3+n_1n_2p\\
&+3n_1n_2(n_1-1)p^2+n_1n_2(n_1-1)(n_1-2)p^3\\
&= pn_1n_2(p^2(n_2-1)(n_2-2)+p^2(n_1-1)(n_1-2)\\
&+3p(n_1+n_2-2)+2),
\end{aligned}$$

and we are done. ■

Now, using an algorithmic analysis approach to assess growth rate, we present the following corollary.

Corollary 2.2. *Assume that $G \in G(n_1, n_2, p)$ is a graph of order n, then $\boldsymbol{E}(M_1(G)) = O\left(n^3\right)$, $\boldsymbol{E}(M_2(G)) = O\left(n^4\right)$ and $\boldsymbol{E}(F(G)) = O\left(n^4\right)$.*

Proof. Since $n_1 + n_2 = n$, it is easy to see that $n_1 n_2 \leq \frac{n^2}{4}$. Hence Theorem 2.1 yields the result. ■

3 Random bipartite graphs $G(n_1, n_2, m)$

This section presents explicit formulas for the first Zagreb, and second Zagreb, along with the forgotten index of a random bipartite graph in $G(n_1, n_2, m)$. We consider the following parameters p_i's, $i = 1, 2, 3$, for three positive integers n_1, n_2, and m, as follow [17]:

$$p_i = \frac{\binom{n_1 n_2 - i}{m-i}}{\binom{n_1 n_2}{m}} = \frac{m(m-1)\cdots(m-i+1)}{(n_1 n_2)((n_1 n_2)-1)\cdots((n_1 n_2)-i+1)}.$$

Theorem 3.1. *If $G \in G(n_1, n_2, m)$ and $n_1, n_2 \geq 2$, then*

(a) $\boldsymbol{E}(M_1(G)) = n_1 n_2 (2p_1 + p_2(n_1 + n_2 - 2))$,

(b) $\boldsymbol{E}(M_2(G)) = n_1 n_2 \left(p_1 + (n_1 + n_2 - 2)p_2 + (n_1 - 1)(n_2 - 1)p_3\right)$.

Furthermore, if $n_1, n_2 \geq 3$, then

(c) $\boldsymbol{E}(F(G)) = n_1 n_2((n_2 - 1)(n_2 - 2)p_3 + (n_1 - 1)(n_1 - 2)p_3 + 3(n_1 + n_2 - 2)p_2 + 2p_1$.

Proof. Due to bipartition parts U and W of $V(G)$, where $U = \{u_1, \ldots, u_{n_1}\}$ and $W = \{w_1, \ldots, w_{n_2}\}$, define the random variables D_i's and D_j's, ($i = 1, \ldots, n_1$ and $j = 1, \ldots, n_2$) corresponding to the degree of u_i's and w_j's, respectively. Hence for each $1 \leq i \leq n_1$ and $1 \leq j \leq n_2$, one can see that $D_i = \sum_{k=1}^{n_2} X_{ik}$ and $D_j = \sum_{r=1}^{n_1} X_{jr}$, where X_{st} is the indicator random variable corresponding to the edge $u_s w_t$. Now, we start with the proof of Part (a).

$$\begin{aligned}
\mathbf{E}(M_1(G)) &= \mathbf{E}\left(\sum_{u_i \in U} D_i^2 + \sum_{w_j \in W} D_j^2\right) = \sum_{i=1}^{n_1} \mathbf{E}\left(D_i^2\right) + \sum_{j=1}^{n_2} \mathbf{E}\left(D_j^2\right) \\
&= \sum_{i=1}^{n_1} \mathbf{E}\left((\sum_{k=1}^{n_2} X_{ik})^2\right) + \sum_{j=1}^{n_2} \mathbf{E}\left((\sum_{r=1}^{n_1} X_{jr})^2\right) \\
&= \sum_{i=1}^{n_1} \mathbf{E}\left(\sum_{k=1}^{n_2} X_{ik}^2\right) + \sum_{i=1}^{n_1} \mathbf{E}\left(\sum_{k=1}^{n_2} \sum_{\substack{t=1 \\ t \neq k}}^{n_2} X_{ik} X_{it}\right) \\
&\quad + \sum_{j=1}^{n_2} \mathbf{E}\left(\sum_{r=1}^{n_1} X_{jr}^2\right) + \sum_{j=1}^{n_2} \mathbf{E}\left(\sum_{r=1}^{n_1} \sum_{\substack{s=1 \\ s \neq r}}^{n_1} X_{jr} X_{js}\right),
\end{aligned}$$

and so,

$$\begin{aligned}\mathbf{E}\left(M_1(G)\right) &= \sum_{i=1}^{n_1}\sum_{k=1}^{n_2}\mathbf{E}\left(X_{ik}^2\right)+\sum_{i=1}^{n_1}\sum_{k=1}^{n_2}\sum_{\substack{t=1\\t\neq k}}^{n_2}\mathbf{E}\left(X_{ik}X_{it}\right)\\ &\quad+\sum_{i=1}^{n_2}\sum_{r=1}^{n_1}\mathbf{E}\left(X_{jr}^2\right)+\sum_{j=1}^{n_2}\sum_{r=1}^{n_1}\sum_{\substack{s=1\\s\neq r}}^{n_1}\mathbf{E}\left(X_{jr}X_{js}\right)\\ &= 2n_1n_2p_1+n_1n_2(n_1+n_2-2)p_2.\end{aligned}$$

Now, we are ready to prove the Part (b).

$$\begin{aligned}\mathbf{E}\left(M_2(G)\right) &= \mathbf{E}\left(\sum_{u_iw_j\in E(G)} D_iD_j\right)=\mathbf{E}\left(\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}D_iD_jX_{ij}\right)\\ &= \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\mathbf{E}\left(D_iD_jX_{ij}\right).\end{aligned}\tag{3}$$

Now, for the fixed positive integers i,j, we focus on the term $\mathbf{E}\left(D_iD_jX_{ij}\right)$ which appeared in the last equation.

$$\begin{aligned}\mathbf{E}\left(D_iD_jX_{ij}\right) &= \sum_{k=1}^{n_2}\sum_{r=1}^{n_1}\mathbf{E}(X_{ik}X_{jr}X_{ij})\\ &= \mathbf{E}(X_{ij}X_{ji}X_{ij})+\sum_{\substack{k=1\\k\neq j}}^{n_2}\mathbf{E}(X_{ik}X_{ji}X_{ij})\\ &\quad+\sum_{\substack{r=1\\r\neq i}}^{n_1}\mathbf{E}(X_{ij}X_{jr}X_{ij})+\sum_{\substack{k=1\\k\neq j}}^{n_2}\sum_{\substack{r=1\\r\neq i}}^{n_1}\mathbf{E}(X_{ik}X_{jr}X_{ij})\\ &= p_1+(n_2-1)p_2+(n_1-1)p_2+(n_1-1)(n_2-1)p_3\\ &= p_1+(n_1+n_2-2)p_2+(n_1-1)(n_2-1)p_3.\end{aligned}\tag{4}$$

Due to (3) and (4), the following relations can be obtained:

$$\begin{aligned}\mathbf{E}\left(M_2(G)\right) &= \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\mathbf{E}\left(D_iD_jX_{ij}\right)\\ &= \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\left(p_1+(n_1+n_2-2)p_2+(n_1-1)(n_2-1)p_3\right)\\ &= n_1n_2\left(p_1+(n_1+n_2-2)p_2+(n_1-1)(n_2-1)p_3\right),\end{aligned}$$

as desired. Finally, we prove the Part (c).

$$\begin{aligned}
\mathbf{E}(F(G)) &= \mathbf{E}\left(\sum_{u_i\in U(G)} D_i^3 + \sum_{w_j\in W(G)} D_j^3\right) = \sum_{i=1}^{n_1}\mathbf{E}\left(D_i^3\right) + \sum_{j=1}^{n_2}\mathbf{E}\left(D_j^3\right) \\
&= \sum_{i=1}^{n_1}\mathbf{E}\left((\sum_{k=1}^{n_2} X_{ik})^3\right) + \sum_{j=1}^{n_2}\mathbf{E}\left((\sum_{r=1}^{n_1} X_{jr})^3\right) \\
&= \sum_{i=1}^{n_1}\mathbf{E}\left(\sum_{k=1}^{n_2} X_{ik}^3\right) + 3\sum_{i=1}^{n_1}\sum_{k=1}^{n_2}\mathbf{E}\left(X_{ik}^2\left(\sum_{\substack{t=1\\t\neq k}}^{n_2} X_{it}\right)\right) \\
&\quad + \sum_{i=1}^{n_1}\mathbf{E}\left(\sum_{k=1}^{n_2}\sum_{\substack{t=1\\t\neq k}}^{n_2}\sum_{\substack{s=1\\s\neq k,t}}^{n_2} X_{ik}X_{it}X_{is}\right) + \sum_{j=1}^{n_2}\mathbf{E}\left(\sum_{r=1}^{n_1} X_{jr}^3\right) \\
&\quad +3\sum_{j=1}^{n_2}\sum_{r=1}^{n_1}\mathbf{E}\left(X_{jr}^2\left(\sum_{\substack{l=1\\l\neq r}}^{n_1} X_{jl}\right)\right) \\
&\quad + \sum_{j=1}^{n_2}\mathbf{E}\left(\sum_{r=1}^{n_1}\sum_{\substack{l=1\\l\neq r}}^{n_1}\sum_{\substack{m=1\\m\neq r,l}}^{n_1} X_{jr}X_{jl}X_{jm}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbf{E}(F(G)) &= \sum_{i=1}^{n_1}\sum_{k=1}^{n_2}\mathbf{E}\left(X_{ik}^3\right) + 3\sum_{i=1}^{n_1}\sum_{k=1}^{n_2}\sum_{\substack{t=1\\t\neq k}}^{n_2}\mathbf{E}(X_{ik}^2X_{it}) \\
&\quad + \sum_{i=1}^{n_1}\sum_{k=1}^{n_2}\sum_{\substack{t=1\\t\neq k}}^{n_2}\sum_{\substack{s=1\\s\neq k,t}}^{n_2}\mathbf{E}(X_{ik}X_{it}X_{is}) + \sum_{j=1}^{n_2}\sum_{r=1}^{n_1}\mathbf{E}\left(X_{jr}^3\right) \\
&\quad +3\sum_{j=1}^{n_2}\sum_{r=1}^{n_1}\sum_{\substack{l=1\\l\neq r}}^{n_1}\mathbf{E}(X_{jr}^2X_{jl}) + \sum_{j=1}^{n_2}\sum_{r=1}^{n_1}\sum_{\substack{l=1\\l\neq r}}^{n_1}\sum_{\substack{m=1\\m\neq r,l}}^{n_1}\mathbf{E}(X_{jr}X_{jl}X_{jm}) \\
&= n_1n_2p_1 + 3n_1n_2(n_2-1)p_2 + n_1n_2(n_2-1)(n_2-2)p_3 \\
&\quad + n_1n_2p_1 + 3n_1n_2(n_1-1)p_2 + n_1n_2(n_1-1)(n_1-2)p_3 \\
&= n_1n_2((n_2-1)(n_2-2)p_3 + (n_1-1)(n_1-2)p_3 \\
&\quad +3(n_1+n_2-2)p_2 + 2p_1),
\end{aligned}$$

and the proof is completed. ■

Based on Theorem 3.1, we can derive the following corollary that provides the time complexity of the results obtained in this section.

Corollary 3.2. *Let* $G \in G(n_1, n_2, p)$ *and* $n = n_1+n_2$. *Then* $\boldsymbol{E}(M_1(G)) = O\left(n^3\right)$, $\boldsymbol{E}(M_2(G)) = O\left(n^4\right)$ *and* $\boldsymbol{E}(F(G)) = O\left(n^4\right)$.

4 Computationally results

In this section, we compare the experimental results obtained from a computer search using Sage Mathematics Software System [18] with the results presented in this paper. Let n_1, n_2, m be positive integers and $0 < p < 1$. We generate 10000 random bipartite graphs for each of $G(n_1, n_2, p)$ and $G(n_1, n_2, m)$, where n_1, n_2, m and p are chosen as specific values. We then obtain the mean exact values of the first Zagreb, and second Zagreb, along with the forgotten index, for the generated random bipartite graphs, and in the tables denoted by "mean()".

We proceed by comparing these computed values with the expected values of the corresponding topological indices computed in Theorems 2.1 and 3.1, denoted by "**E**( )" in the tables. The comparison results for random graphs in $G(10, 20, p)$ are presented in Table 1. Here, p is taken as $k/10$ with $k = 1, \dots, 9$. For random graphs in $G(10, 20, m)$ with $m = 25k$, the comparison results are presented in Table 2, where k is taken as $k = 1, \dots, 7$.

Table 1: Comparison of Theorem 2.1 and the experimental results for random graphs in $G(n_1, n_2, p)$.

$G = G(n_1, n_2, p)$	mean$(M_1(G))$	$\mathbf{E}(M_1(G))$
$G(10, 20, 0.1)$	95.79440	96
$G(10, 20, 0.2)$	304.09140	304
$G(10, 20, 0.3)$	624.34080	624
$G(10, 20, 0.4)$	1052.95520	1056
$G(10, 20, 0.5)$	1602.00180	1600
$G(10, 20, 0.6)$	2258.90600	2256
$G(10, 20, 0.7)$	3023.25920	3024
$G(10, 20, 0.8)$	3905.62880	3904
$G(10, 20, 0.9)$	4895.47280	4896

$G = G(n_1, n_2, p)$	mean$(M_2(G))$	$\mathbf{E}(M_2(G))$
$G(10, 20, 0.1)$	109.45380	110.20
$G(10, 20, 0.2)$	540.51570	537.60
$G(10, 20, 0.3)$	1482.76640	1487.40
$G(10, 20, 0.4)$	3175.80170	3164.80
$G(10, 20, 0.5)$	5774.57150	5775.00
$G(10, 20, 0.6)$	9545.88270	9523.20
$G(10, 20, 0.7)$	14598.41170	14614.60
$G(10, 20, 0.8)$	21259.30540	21254.40
$G(10, 20, 0.9)$	29662.95900	29647.80

$G = G(n_1, n_2, p)$	mean$(F(G))$	$\mathbf{E}(F(G))$
$G(10, 20, 0.1)$	291.01540	290.80
$G(10, 20, 0.2)$	1411.64780	1414.40
$G(10, 20, 0.3)$	3877.79180	3867.60
$G(10, 20, 0.4)$	8154.52580	8147.20
$G(10, 20, 0.5)$	14748.82880	14750.00
$G(10, 20, 0.6)$	24120.25860	24172.80
$G(10, 20, 0.7)$	36965.39440	36912.40
$G(10, 20, 0.8)$	53497.87100	53465.60
$G(10, 20, 0.9)$	74367.69320	74329.20

Table 2: Comparison of Theorem 3.1 and the experimental results for random graphs in $G(n_1, n_2, m)$.

$G = G(n_1, n_2, m)$	mean$(M_1(G))$	$\mathbf{E}(M_1(G))$
$G(10, 20, 25)$	134.41360	134.42211
$G(10, 20, 50)$	444.61480	444.72362
$G(10, 20, 75)$	930.48960	930.90452
$G(10, 20, 100)$	1592.80200	1592.96482
$G(10, 20, 125)$	2430.77160	2430.90452
$G(10, 20, 150)$	3444.51440	3444.72362
$G(10, 20, 175)$	4634.52800	4634.42211

$G = G(n_1, n_2, m)$	mean$(M_2(G))$	$\mathbf{E}(M_2(G))$
$G(10, 20, 25)$	169.49390	169.31247
$G(10, 20, 50)$	903.88970	905.09365
$G(10, 20, 75)$	2612.59220	2614.20740
$G(10, 20, 100)$	5700.93710	5703.51759
$G(10, 20, 125)$	10579.45350	10579.88808
$G(10, 20, 150)$	17650.61520	17650.18273
$G(10, 20, 175)$	27320.51790	27321.26542

$G = G(n_1, n_2, m)$	mean$(F(G))$	$\mathbf{E}(F(G))$
$G(10, 20, 25)$	447.52940	448.26404
$G(10, 20, 50)$	2370.35560	2369.80356
$G(10, 20, 75)$	6746.51820	6749.65738
$G(10, 20, 100)$	14582.81120	14572.86432
$G(10, 20, 125)$	26830.17820	26824.46322
$G(10, 20, 150)$	44497.94940	44489.49292
$G(10, 20, 175)$	68559.42260	68552.99223

Conclusion

In this paper, we used combinatorial methods and probabilistic techniques to analyze the expected values of the Zagreb indices for two types of well-known random bipartite graphs $G(n_1, n_2, p)$ and $G(n_1, n_2, m)$. Our analytical results showed that the expected value of the Zagreb indices of a random bipartite graph is proportional to the vertex partition sizes. Specifically, we showed that these expected values can be expressed as polynomials in terms of vertices part sizes, number of vertices and edges and the probability p. We also compared our theoretical predictions with numerical simulations on randomly generated bipartite graphs which confirmed the validity of our theoretical predictions. Our results shed light on the average degree of bipartite networks and can be used to estimate their properties in practical scenarios. Future work may focus on extending our analysis to more general classes of bipartite graphs or exploring other topological measures of interest.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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