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Research Paper

Shifted Second-Kind Chebyshev Spectral Collocation -Based Technique for Time-Fractional KdV-Burgers' Equation

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Abstract

The main goal of this research work is to provide a numerical technique based on choosing a set of basis functions for handling the third-order time-fractional Korteweg–De Vries Burgers' equation. The trial functions are selected for the shifted second-kind Chebyshev polynomials (S2KCPs) compatible with the problem's governing initial and boundary conditions. The spectral tau method transforms the equation and its underlying conditions into a nonlinear system of algebraic equations that can be efficiently numerically inverted with the standard Newton's iterative procedures after the approximate solutions have been expressed as a double expansion of the two chosen basis functions. The truncation error is estimated. Various numerical examples are displayed together with comparisons to other approaches in the literature to show the applicability and accuracy of the provided methodology. Different numerical models are displayed and compared to other methods in the literature.

1 Introduction

KdV equation serves as a mathematical representation of waves on shallow water surfaces. It is noteworthy because it is the archetypal illustration of an entirely solvable model or a nonlinear partial differential equation whose solutions can be precisely described. It was first presented by Boussinesq in 1877 [1], Diederik Korteweg and Gustav de Vries in 1895 [2] rediscovered the KdV equation, which is a nonlinear, dispersive partial differential equation for the function u of two dimensionless real variables, x and t, that are proportional to space and time, respectively [3]. The traditional but insignificant constant 6 is placed before the final term. By multiplying t, x, and u by constants, any of the three terms' coefficients can equal any non-zero constants.

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Numerous applications use equations of the KdV type and their modified versions. The Korteweg-de Vries-Burgers' equation appears as a model equation integrating the effects of dispersion, dissipation, and nonlinearity in a variety of physical situations. Examples include the propagation of waves in a viscous fluid inside an elastic tube, the movement of gas bubble-containing liquids, and turbulence [4]. The Petrov-Galerkin finite element method handled a modified third-order KdV problem in [5]. Youssri and Atta in [6] offered a double Chebyshev Spectral Tau Algorithm for Solving the standard KdV Equation.

Spectral methods are one of the most popular numerical techniques for resolving different kinds of differential equations. When compared to other numerical methods, these methods have some advantages. For example, these methods are global methods, unlike finite element methods. The trial and test functions families of basis functions are the two that comprise the spectral method's main characteristic. Standard orthogonal polynomials or combinations are used to express these two families of parts. We must decide which trial and test functions to use depending on the approach we select. It is commonly known that spectral methods fall into three primary categories: Galerkin, collocation, and tau approaches.

The boundary/initial conditions specified by the supplied differential equation are satisfied by each member of the test functions, which are identical to the trial functions in the Galerkin technique. Unlike the Galerkin method, there are no limitations on selecting the basis functions with the tau technique. Applying it to many differential equation types is more straightforward than using the Galerkin approach. It is therefore utilized to resolve several varieties of differential equations. To address multi-term fractional differential equations, the authors of the two articles [7] and [8] used Chebyshev polynomials of the fifth and sixth kinds, respectively, the authors in [9] used the sixth-kind Chebyshev polynomials to handle the time-fractional heat equation. In [10], Abd-Elhameed derived formulae for the high-order derivatives of sixth-kind Chebyshev polynomials, and he applied the tau method together with specific unique derivative procedures to treat the nonlinear Burgers' problem numerically. The authors in [11] suggested a spectral tau algorithm based on choosing generalized Fibonacci polynomials as basis functions for solving fractional Bagley-Torvik equation. Any sort of differential equation can be solved using the collocation method; for an example, see [12–14]. We mention here that the spectral methods are flexible in the sense that, the basis functions are not necessary to be orthogonal polynomials, for nonorthogonal spectral methods, the interested reader is referred to [15, 16].

In recent decades, Chebyshev polynomials have attracted much interest from theoretical and practical perspectives. The Chebyshev polynomials, which are special Jacobi polynomials, come in four well-known varieties. The first and second types of polynomials, which are symmetric, are frequently employed to numerically treat the various varieties of differential equations. For instance, in [17], the spectral Galerkin method and Chebyshev polynomials of the first kind were used to treat the linear and non-linear hyperbolic telegraph-type hyperbolic problems numerically. The second kind of Chebyshev polynomials was also employed in several works, and they were offered in [18] as numerical solutions to systems of several FDEs. For instance, the authors in [19] used the collocation method and second-kind Chebyshev polynomials to solve the space fractional advection-dispersion problem. The variable order fractional differential-integral problem was solved using an operational matrix technique in [20] based on the second-kind Chebyshev polynomials. To get spectral solutions of the linear hyperbolic first-order partial differential equations, the authors in [21] also utilized the shifted fifth-kind Chebyshev polynomials. Additionally, the Adomian decomposition approach was used in [22] to treat certain different forms of differential equations, along with the Laguerre polynomials and the second kind of Chebyshev polynomials. The third and fourth forms of Chebyshev polynomials were also used to solve various sorts of differential equations. For more studies see [23–27].

The spectral collocation method ends up with a system of nonlinear algebraic equations.

The computational cost required to solve a nonlinear system of algebraic equations can vary depending on several factors, such as the number of equations, the complexity of the equations, the desired level of accuracy, and the specific numerical method used for solving the system. In general, solving a nonlinear system of algebraic equations is a challenging task, and there is no universal method that works efficiently for all types of systems. Different numerical techniques can be employed, including iterative methods like Newton's method, the Broyden method, or the secant method, as well as optimization algorithms or symbolic methods. The computational cost of solving a nonlinear system is typically measured in terms of the number of arithmetic operations required, such as additions, subtractions, multiplications, and divisions. The number of operations can vary depending on the specific method used, the convergence behavior of the system, and the desired level of accuracy. for more details about computational complexity, see [28].

Numerous authors have put forth numerical solutions for time partial differential equations with second-order partial derivatives; see, for instance, [29–31]. However, numerical investigations for time partial differential equations with third-order partial derivatives are insufficient. This inspires our desire to look into such issues. The authors of [32] suggested a Petrov-Galerkin spectral approach to handle the linearized time fractional KdV equation, while in [33] the authors used the tau approach to handle the same model. Some fractional KdV equations may be treated using certain techniques; for example, see [34, 35].

As we know, fractional integral and differential calculus have attracted the attention of a large number of authors due to their great importance in numerous scientific and engineering disciplines, see for example [36–40]. So, in this work, we are concerned with building and implementing a robust second-kind explicit collocation approach for handling the third-order time fractional KdV Burgers' equation subject to initial and boundary conditions with nonlinear product term; this work generalizes the approach in [33], an explicit formula for the first, second and third order derivatives of a particular choice of shifted second kind Chebyshev basis are derived, and proved based on the third-order derivative formula of the third-order product obtained by Abd-Elhameed and Youssri in [33].

This article encompasses: Section 2 is devoted to essential properties of fractional calculus and second-kind Chebyshev polynomials properties. Section 3 is the main section where we structure the collocation algorithm for handling the KdV equation. Section 4 is devoted to a note on the computational cost of solving the nonlinear system of equations. Section 5 for estimating the truncation error. Numerical examples with some comparisons are exhibited in Section 6. Some concluding remarks are reported in Section 7.

2 Preliminaries and essential relations

2.1 The fractional derivative in the Caputo sense

Definition 2.1. ([41]). The Caputo fractional derivative of order s is defined as:

$$D_x^s u(x) = \frac{1}{\Gamma(m-s)} \int_0^x (x-y)^{m-s-1} u^{(m)}(y) dy, \quad s > 0, \quad x > 0, \tag{1}$$

where $m-1 \leqslant s < m, \quad m \in \mathbb{N}$.

The following properties are satisfied by the operator D_x^s for $\quad m-1 \leqslant s < m, \quad m \in \mathbb{N}$,

$$D_x^s c = 0, \text{ (c is a constant)}. \tag{2}$$

$$D_x^s\, x^m = \begin{cases} 0, & \text{if } m \in \mathbb{N}_0 \quad and \quad m < \lceil s \rceil, \\ \frac{\Gamma(m+1)}{\Gamma(m-s+1)}\, x^{m-s}, & \text{if } m \in \mathbb{N}_0 \quad and \quad m \geq \lceil s \rceil, \end{cases} \tag{3}$$

where $\mathbb{N} = \{1, 2, 3, ...\}$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and the notation $\lceil . \rceil$ denotes the ceiling function.

2.2 An account on the S2KCPs

Let $U_j^*(x)$ be the S2KCPs defined in the interval $[0, 1]$ by $U_j^*(x) = U_j(2\,x-1)$. These polynomials can be defined as [42]

$$U_j^*(x) = \sum_{r=0}^{j} \frac{2^{2\,r}\,(-1)^{j+r}\,(j+r+1)!}{(2\,r+1)!\,(j-r)!}\, x^r, \quad j \geq 0, \tag{4}$$

and satisfying the following orthogonality relation with respect to the weight function $\hat{w}(x) = \sqrt{x - x^2}$ [42]:

$$\int_0^1 \hat{w}(x)\, U_m^*(x)\, U_n^*(x)\, dx = \begin{cases} \frac{\pi}{8}, & \text{if } \ m = n = 0, \\ 0, & \text{if } m \neq n. \end{cases} \tag{5}$$

The recurrence relation of $U_m^*(x)$ is

$$U_m^*(x) = 2\ (2\,x-1)\ U_{m-1}^*(x) - U_{m-2}^*(x), \tag{6}$$

where $U_0^*(x) = 1, \quad U_1^*(x) = 2\,x - 1.$
Moreover, the inversion formula is [42]

$$x^r = \sum_{p=0}^{r} B_{p,r}\, U_p^*(x), \ j \geq 0, \tag{7}$$

where

$$B_{p,r} = \frac{4\,\Gamma\left(r + \frac{3}{2}\right)\, ((p+1)!\, r!)}{\sqrt{\pi}\, p!\,(r-p)!\,(p+r+2)!}. \tag{8}$$

Lemma 2.2. *([42]). Let i and m be any two nonnegative integers. The moments' formula for the S2KCPs is given by*

$$x^i\, U_m^*(x) = \sum_{k=-i-m}^{i+m} F_{k,i,m}\, U_k^*(x), \tag{9}$$

where

$$F_{k,i,m} = \frac{1}{2^{2\,i}} \binom{2\,i}{i-k+m}. \tag{10}$$

Lemma 2.3. *([43]). For all nonnegative integers m and n, the following linearization formula holds for the S2KCPs*

$$U_m^*(x)\, U_n^*(x) = \sum_{k=0}^{n} U_{2\,k+m-n}^*(x). \tag{11}$$

Lemma 2.4. *([44]). The following integral formula is valid:*

$$\int_0^x U_i^*(z)\, dz = \frac{T_{i+1}^*(x) + (-1)^i}{2i+2}, \qquad \forall i \geq 0.$$

Lemma 2.5. *([44]). The following connection formula is valid:*

$$T_{i+1}^*(x) = \frac{1}{2}\left(U_{i+1}^* - U_{i-1}^*\right), \qquad \forall i \geq 1.$$

3 Collocation approach for the time-fractional KdV-Burgers' equation

In this section, we consider the following time-fractional KdV-Burgers' equation [45]

$$D_t^\alpha \chi(x,t) + k\,\chi(x,t)\,\chi_x(x,t) - \upsilon\,\chi_{xx}(x,t) + \mu\,\chi_{xxx}(x,t) = S(x,t), \quad 0 < \alpha \leq 1, \tag{12}$$

subject to the following initial condition

$$\chi(x,0) = g(x), \quad 0 < x \leq 1, \tag{13}$$

and boundary conditions

$$\chi(0,t) = \chi(1,t) = \chi_x(1,t) = 0, \quad 0 < t \leq 1, \tag{14}$$

where $k \neq 0$, υ, μ are positive parameters and $S(x,t)$ is the source term.

3.1 Trial functions

Consider the following basis functions

$$\begin{aligned} \psi_i^*(x) &= x\,(1-x)^2\,U_i^*(x), \\ \phi_j^*(t) &= t^\alpha\,U_j^*(t). \end{aligned} \tag{15}$$

Theorem 3.1. *The following fractional derivative of $\phi_j^*(t)$ holds*

$$D_t^\alpha \phi_j^*(t) = \sum_{r=0}^{j} \lambda_{r,j}\,U_r^*(t), \tag{16}$$

where

$$\lambda_{r,j} = \frac{2(r+1)(-1)^{j+r}\Gamma(j+r+2)\Gamma(r+\alpha+1)}{\Gamma(j-r+1)}\,{}_3\tilde{F}_2\left(\begin{matrix} r+\alpha+1, r-j, j+r+2 \\ r+1, 2\,r+3 \end{matrix}\middle|\, 1\right). \tag{17}$$

Proof. The power form formula of $\phi_j^*(t)$ can be written after using relation (4) as

$$D_t^\alpha \phi_j^*(t) = \sum_{r=0}^{j} \frac{2^{2\,r}\,(-1)^{j+r}\,(j+r+1)!\,(\alpha+r)!}{(2\,r+1)!\,(j-r)!\,r!}\,t^r, \tag{18}$$

which can be rewritten with the aid of the inversion formula (7) as

$$D_t^\alpha \phi_j^*(t) = \sum_{r=0}^{j}\sum_{p=0}^{r} \frac{2\,(p+1)\,(-1)^{j+r}\,\Gamma(j+r+2)\,(\alpha+r)!}{\Gamma(j-r+1)\,\Gamma(-p+r+1)\,\Gamma(p+r+3)\,r!}\,U_p^*(t). \tag{19}$$

And hence, after rearranging and expanding the terms of the last equation, one has

$$D_t^\alpha \phi_j^*(t) = \sum_{r=0}^{j}\sum_{p=r}^{j} \frac{2\,(r+1)\,(-1)^{j+p}\,\Gamma(j+p+2)\,(\alpha+p)!}{p!\,\Gamma(j-p+1)\,\Gamma(p-r+1)\,\Gamma(p+r+3)}\,U_r^*(t). \tag{20}$$

Now, with the aid of Maple program, the following relation can be summed to give the following reduced form

$$\begin{aligned}\lambda_{r,j} &= \sum_{p=r}^{j} \frac{2\,(r+1)\,(-1)^{j+p}\,\Gamma(j+p+2)\,(\alpha+p)!}{p!\,\Gamma(j-p+1)\,\Gamma(p-r+1)\,\Gamma(p+r+3)} \\ &= \frac{2(r+1)(-1)^{j+r}\Gamma(j+r+2)\Gamma(r+\alpha+1)}{\Gamma(j-r+1)}\,{}_3\tilde{F}_2\left(\begin{array}{c} r+\alpha+1, r-j, j+r+2 \\ r+1, 2\,r+3 \end{array}\middle|\, 1\right).\end{aligned} \tag{21}$$

Therefore, we get the following relation

$$D_t^{\alpha}\phi_j^*(t) = \sum_{r=0}^{j} \lambda_{r,j}\, U_r^*(t). \tag{22}$$

This completes the proof of Theorem 3.1. ■

Remark 1. Based on the orthogonality relation (5) and the power form of $U_j^*(t)$ in (4), the following basis functions $\phi_j^*(t)$ can be written in approximation formula as

$$\phi_j^*(t) \approx \sum_{k=0}^{j}\sum_{i=0}^{M} \mathcal{H}_{k,j}\, h_{i,k}\, U_{i-j+2\,k}^*(t), \tag{23}$$

where

$$\mathcal{H}_{k,j} = \frac{2^{2\,k}\,(-1)^{j+k}\,(j+k+1)!}{(2\,k+1)!\,(j-k)!}, \tag{24}$$

and

$$h_{i,k} = \frac{8}{\pi}\int_0^1 \sqrt{t-t^2}\, t^{k+\alpha}\, U_i^*(t)\, dt. \tag{25}$$

Theorem 3.2. *The first three derivatives of $\psi_m^*(x)$ can be expressed explicitly as:*

$$\begin{aligned}\frac{d^3\,\psi_i^*(x)}{d\,x^3} &= \sum_{k=0}^{i} d_{k,i}\, U_k^*(x), \\ \frac{d^2\,\psi_i^*(x)}{d\,x^2} &= \sum_{k=0}^{i+1} \bar{c}_{k,i}\, U_k^*(x), \\ \frac{d\,\psi_i^*(x)}{d\,x} &= \sum_{k=0}^{i+2} b_{k,i}\, U_k^*(x),\end{aligned} \tag{26}$$

where

$$d_{k,i} = \begin{cases} (k+1)(3(i+2)i - k(k+2)+6), & \textit{if } (i+k) \textit{ even and } 0 \le k \le i-1, \\ (k+1)\left(-3(i+1)^2 + k^2 + 2k\right), & \textit{if } (i+k) \textit{ odd and } 0 \le k \le i-1, \\ (i+1)(i+2)(i+3), & \textit{if } k = i, \end{cases}$$

$$\bar{c}_{k,i} = \begin{cases} -\frac{d_{0,i}}{4}, & \textit{if } k=-1, \\ (-1)^{i+1}\left(\frac{3}{8}\left(2\,i\,(i+2) + (-1)^{i+1}+1\right)+1\right) - \frac{d_{1,i}}{8}, & \textit{if } k=0, \\ \frac{d_{i-1,i}}{4\,i}, & \textit{if } k=i, \\ \frac{d_{i,i}}{4\,(i+1)}, & \textit{if } k=i+1, \\ \frac{d_{k-1,i}}{4\,k} - \frac{d_{k+1,i}}{4\,(k+2)}, & \textit{otherwise}, \end{cases} \quad (27)$$

$$b_{k,i} = \begin{cases} -\frac{c_{0,i}}{4}, & \textit{if } k=-1, \\ \frac{1}{8} - \frac{c_{1,i}}{8}, & \textit{if } k=0,\ i=0, \\ \frac{(-1)^i}{4} - \frac{c_{1,i}}{8}, & \textit{if } k=0,\ i\neq 0, \\ \frac{c_{i,i}}{4\,(i+1)}, & \textit{if } k=i+1, \\ \frac{c_{i+1,i}}{4\,(i+2)}, & \textit{if } k=i+2, \\ \frac{c_{k-1,i}}{4\,k} - \frac{c_{k+1,i}}{4\,(k+2)}, & \textit{otherwise}. \end{cases}$$

Proof. The proof of the first part related to the third derivative of $\psi_i^*(x)$ can be easily obtained after replacing x by $(2\,x-1)$ in Theorem 3.2 in Ref. [33].
To prove the second part, we integrate both sides of the third-order derivative formula and then apply Lemma 2.4 followed by Lemma 2.5 and then collect like terms, rearranging the summations, and we get the desired results. The third part is typically the same as the second part but by integrating the second-order derivative formula. ■

Remark 2. Based on the recurrence relation (6) of $U_i^*(x)$, the following formula of $\psi_i^*(x)$ holds

$$\psi_i^*(x) = \frac{1}{64}\left(U_{i-3}^*(x) - 2\,U_{i-2}^*(x) - U_{i-1}^*(x) - U_{i+1}^*(x) - 2\,U_{i+2}^*(x) + U_{i+3}^*(x) + 4\,U_i^*(x)\right). \quad (28)$$

3.2 Collocation solution for the time-fractional KdV-Burgers' equation

To proceed with our proposed collocation approach, we will make use of the following transformation:

$$\chi(x,t) = u(x,t) + g(x,t), \quad (29)$$

to convert the time-fractional KdV-Burgers' equation (12) governed by the conditions (13)-(14) into the following modified equation:

$$D_t^{\alpha} u(x,t) + k\,u(x,t)\,u_x(x,t) - \upsilon\,u_{xx}(x,t) + \mu\,u_{xxx}(x,t) = f(x,t), \quad 0<\alpha\le 1, \quad (30)$$

governed by the following homogeneous conditions

$$u(x,0) = 0, \quad 0 < x \le 1, \quad (31)$$

$$u(0,t) = u(1,t) = u_x(1,t) = 0, \quad 0 < t \le 1, \quad (32)$$

where

$$
\begin{aligned}
f(x,t) = S(x,t) - D_t^{\alpha} g(x,t) - k\, g(x,t)\, g_x(x,t) + \upsilon\, g_{xx}(x,t) - \mu\, g_{xxx}(x,t) \\
- \kappa\, u\, g_x(x,t) - \kappa\, u_x\, g(x,t),
\end{aligned}
\tag{33}
$$

and $g(x,t)$ is an arbitrary function satisfying the following conditions

$$\chi(0,t) = g(0,t),\ \chi(1,t) = g(1,t),\ \chi_x(1,t) = g_x(1,t).$$

Therefore, instead of solving (12) governed by (13)-(14), we can solve the modified equation (30) governed by the homogeneous conditions (31)-(32).
Now, one may set

$$
\begin{aligned}
\zeta^M &= \text{span}\{\psi_i^*(x)\, \phi_j^*(t) : i, j = 0, 1, \ldots, M\}, \\
\Upsilon^M &= \{u \in \zeta^M : u(x,0) = u(0,t) = u(1,t) = u_x(1,t) = 0\},
\end{aligned}
\tag{34}
$$

then, any function $u^M(x,t) \in \Upsilon^M$ may be written as

$$u^M(x,t) \simeq \sum_{i=0}^{M} \sum_{j=0}^{M} c_{ij}\, \psi_i^*(x)\, \phi_j^*(t). \tag{35}$$

The residual $\mathbf{R}(x,t)$ of Eq. (30) can be written as

$$\mathbf{R}(x,t) = D_t^{\alpha} u^M(x,t) + k\, u^M(x,t)\, u_x^M(x,t) - \upsilon\, u_{xx}^M(x,t) + \mu\, u_{xxx}^M(x,t) - f(x,t). \tag{36}$$

Now, making use of Theorems 3.1 and 3.2 and Remarks 1 and 2 to get expressions for $D_t^{\alpha} u^M(x,t)$, $u^M(x,t)\, u_x^M(x,t)$, $u_{xx}^M(x,t)$ and $u_{xxx}^M(x,t)$ as following

$$
\begin{aligned}
D_t^{\alpha} u^M(x,t) &\simeq \frac{1}{64} \sum_{i=0}^{M} \sum_{j=0}^{M} \sum_{r=0}^{j} \lambda_{r,j}\, c_{ij}\, U_r^*(t) \\
&\times \left[U_{i-3}^*(x) - 2U_{i-2}^*(x) - U_{i-1}^*(x) - U_{i+1}^*(x) - 2U_{i+2}^*(x) + U_{i+3}^*(x) + 4U_i^*(x) \right],
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
u^M(x,t)\, u_x^M(x,t) &\approx \frac{1}{64} \sum_{r=0}^{M} \sum_{s=0}^{M} \sum_{i=0}^{M} \sum_{j=0}^{M} \sum_{k=0}^{i+2} \sum_{k=0}^{j} \sum_{n=0}^{M} \sum_{l=0}^{s} \sum_{p=0}^{M} \mathcal{H}_{k,j}\, \mathcal{H}_{l,s}\, h_{p,l}\, h_{n,k}\, b_{k,i}\, c_{ij}\, c_{rs} \\
&\times \left[U_{r-3}^*(x) - 2\, U_{r-2}^*(x) - U_{r-1}^*(x) - U_{r+1}^*(x) - 2U_{r+2}^*(x) + U_{r+3}^*(x) + 4U_r^*(x) \right] \\
&\times U_{p-s+2\,l}^*(t)\, U_k^*(x)\, U_{n-j+2\,k}^*(t),
\end{aligned}
\tag{38}
$$

$$u_{xx}^M(x,t) \approx \sum_{i=0}^{M} \sum_{j=0}^{M} \sum_{k=0}^{i+1} \sum_{k=0}^{j} \sum_{n=0}^{M} \mathcal{H}_{k,j}\, h_{n,k}\, \bar{c}_{k,i}\, c_{ij}\; U_k^*(x)\, U_{n-j+2\,k}^*(t), \tag{39}$$

$$u_{xxx}^M(x,t) \approx \sum_{i=0}^{M} \sum_{j=0}^{M} \sum_{k=0}^{i} \sum_{k=0}^{j} \sum_{n=0}^{M} \mathcal{H}_{k,j}\, h_{n,k}\, d_{k,i}\, c_{ij}\; U_k^*(x)\, U_{n-j+2\,k}^*(t). \tag{40}$$

Therefore, relations (37)-(40) enable us to get the residual $\mathbf{R}(x,t)$ (36) in simple form.
To get the expansion coefficients c_{ij}, we apply the spectral collocation method, by forcing the residual $\mathbf{R}(x,t)$ to be zero at some collocation points (x_i, t_j), that is, we get

$$\mathbf{R}(x_i, t_j) = 0, \tag{41}$$

where
$\{(x_i, t_j) : i, j = 2, 3, ..., M+2\}$ are the first distinct roots of $\psi^*_{M+1}(x)$ and $\phi^*_{M+1}(t)$,
or
$\{(x_i, t_j) : i, j = 1, 2, ..., M+1\}$ are the first distinct roots of $U^*_{M+1}(x)$ and $U^*_{M+1}(t)$.
And hence, we get $(M+1)^2$ nonlinear system of equations that can be solved through a suitable numerical solver such as Newton's iterative method.

Remark 3. In order to demonstrate the steps required to obtain the desired numerical solution for Eq. (30) governed by the homogeneous conditions (31)-(32), Algorithm 1 is presented to show the required steps till obtaining the numerical solution.

Algorithm 1 Coding algorithm for the proposed scheme of FDWE

Input k, υ, μ, α, M and $f(x,t)$.
Step 1. Use transformation (29) to convert Eqs. (12)-(14) into Eqs. (30)-(32).
Step 2. Assume an approximate solution $u^M(x,t)$ in (35).
Step 3. Inserting relations (37)-(40).
Step 4. Compute the residual $\mathbf{R}(x,t)$.
Step 5. Apply collocation method to obtain the system in (41).
Step 6. Use *FindRoot* command with initial guess $\{c_{ij} = 10^{-i-j}, i, j : 0, 1, ..., M\}$, to solve the system in (41) to get c_{ij}.
Output $u^M(x,t)$

4 Error bound

Let $u^M(x,t) \in \Upsilon^M$ be the best approximation of $u(x,t)$; then, the definition of the best approximation enables us to write the following inequality

$$||u(x,t) - u^M(x,t)||_\infty \leq ||u(x,t) - \hat{u}^M(x,t)||_\infty, \quad \forall \hat{u}^M(x,t) \in \Upsilon^M. \tag{42}$$

Moreover, the previous inequality is also true if $\hat{u}^M$ denotes the interpolating polynomial for $u(x,t)$ at points (x_i, t_j), where x_i are the roots of $\psi^*_i(x)$, while t_j are the roots of $\phi^*_j(t)$.
Now, if we take similar steps as in [46, 47], we get

$$\begin{aligned} u(x,t) - \hat{u}^M(x,t) = & \frac{\partial^{M+1} u(\eta,t)}{\partial x^{M+1}\,(M+1)!} \prod_{i=0}^{M}(x-x_i) + \frac{\partial^{M+1} u(x,\mu)}{\partial t^{M+1}\,(M+1)!} \prod_{j=0}^{M}(t-t_j) \\ & - \frac{\partial^{2M+2} u(\hat{\eta},\hat{\mu})}{\partial x^{M+1}\,\partial t^{M+1}\,((M+1)!)^2} \prod_{i=0}^{M}(x-x_i) \prod_{j=0}^{M}(t-t_j), \end{aligned} \tag{43}$$

where $\eta, \hat{\eta}, \mu, \hat{\mu} \in [0,1]$.
Now,

$$\begin{aligned} ||u(x,t) - \hat{u}^M(x,t)||_\infty \leq & \max_{(x,t)\in\Omega} \left|\frac{\partial^{M+1} u(\eta,t)}{\partial x^{M+1}}\right| \frac{||\prod_{i=0}^{M}(x-x_i)||_\infty}{(M+1)!} \\ & + \max_{(x,t)\in\Omega} \left|\frac{\partial^{M+1} u(x,\mu)}{\partial t^{M+1}}\right| \frac{||\prod_{j=0}^{M}(t-t_j)||_\infty}{(M+1)!} \\ & - \max_{(x,t)\in\Omega} \left|\frac{\partial^{2M+2} u(\hat{\eta},\hat{\mu})}{\partial x^{M+1}\,\partial t^{M+1}}\right| \frac{||\prod_{i=0}^{M}(x-x_i)||_\infty\,||\prod_{j=0}^{M}(t-t_j)||_\infty}{((M+1)!)^2}. \end{aligned} \tag{44}$$

Since u is a smooth function on $\Omega = [0,1]^2$, then there exist three constants σ_1, σ_2 and σ_3, such that

$$\max_{(x,t)\in\Omega}\left|\frac{\partial^{M+1} u(x,t)}{\partial x^{M+1}}\right| \le \sigma_1, \quad \max_{(x,t)\in\Omega}\left|\frac{\partial^{M+1} u(x,\mu)}{\partial t^{M+1}}\right| \le \sigma_2, \quad \max_{(x,t)\in\Omega}\left|\frac{\partial^{2M+2} u(\hat{\eta},\hat{\mu})}{\partial x^{M+1}\,\partial t^{M+1}}\right| \le \sigma_3. \tag{45}$$

To minimize the factor $||\prod_{i=0}^{M}(x - x_i)||_\infty$, let us use the one-to-one mapping $x = \frac{1}{2}(z+1)$ between the intervals $[-1,1]$ and $[0,1]$ to deduce that

$$\begin{aligned}\min_{x_i\in[0,1]}\max_{x\in[0,1]}\left|\prod_{i=0}^{M}(x-x_i)\right| &= \min_{z_i\in[-1,1]}\max_{z\in[-1,1]}\left|\prod_{i=0}^{M}\frac{1}{2}(z-z_i)\right| \\ &= \left(\frac{1}{2}\right)^{M+1}\min_{z_i\in[-1,1]}\max_{z\in[-1,1]}\left|\prod_{i=0}^{M}(z-z_i)\right| \\ &= \left(\frac{1}{2}\right)^{M+1}\min_{z_i\in[-1,1]}\max_{z\in[-1,1]}\left|\frac{\psi_{M-2}(z)}{\bar{\psi}_M}\right|,\end{aligned} \tag{46}$$

where $\bar{\psi}_M = 2^{M-5}$ is the leading coefficient of $\psi_{M-2}(z) = \left(\frac{1+z}{2}\right)\left(\frac{1-z}{2}\right)^2 U_{M-2}(z)$ and z_i are the roots of $\psi_{M+1}(z)$.

Similarly, the factor $||\prod_{j=0}^{M}(t - t_j)||_\infty$, can be minimized by using the one-to-one mapping $t = \frac{1}{2}(\bar{t}+1)$ between the intervals $[-1,1]$ and $[0,1]$ to deduce that

$$\min_{t_j\in[0,1]}\max_{t\in[0,1]}\left|\prod_{j=0}^{M}(t-t_j)\right| = \left(\frac{1}{2}\right)^{M+1}\min_{\bar{t}_j\in[-1,1]}\max_{\bar{t}\in[-1,1]}\left|\frac{\phi_M(\bar{t})}{\hat{\phi}_M}\right|, \tag{47}$$

where $\hat{\phi}_M = \alpha\, 2^{M-\alpha}$ is the leading coefficient of $\phi_M(\bar{t}) = \left(\frac{1+\bar{t}}{2}\right)^{\alpha} U_M(\bar{t})$ and $\bar{t}_j$ are the roots of $\phi_{M+1}(\bar{t})$.

It is known that

$$\max_{z\in[-1,1]}|\psi_{M-2}(z)| = \max_{z\in[-1,1]}\left|\left(\frac{1+z}{2}\right)\left(\frac{1-z}{2}\right)^2 U_{M-2}(z)\right| = \frac{4\,(M-1)}{27}, \tag{48}$$

and

$$\max_{\bar{t}\in[-1,1]}|\phi_M(\bar{t})| = \max_{\bar{t}\in[-1,1]}\left|\left(\frac{1-\bar{t}}{2}\right)^{\alpha} U_M(\bar{t})\right| = M+1. \tag{49}$$

And hence, inequality (45) along with Equations (46), (47), (48) and (49) help us to obtain the following desired result

$$\begin{aligned}||u(x,t) - u^M(x,t)||_\infty &\le ||u(x,t) - \hat{u}^M(x,t)||_\infty \\ &< \sigma_1\,\frac{4^{3-M}\,(M-1)}{27\,\Gamma(M+2)} + \sigma_2\,\frac{\alpha\,2^{\alpha-2\,M-1}}{\Gamma(M+1)} \\ &+ \sigma_3\,\frac{\alpha\,\left(M^2-1\right)2^{\alpha-4\,M+5}}{27\,\Gamma(M+2)^2},\end{aligned} \tag{50}$$

which represent an upper bound of the absolute error.

5 Illustrative examples

Test Problem 1. ([45]). Consider the time-fractional KdV-Burgers' equation of the form

$$D_t^{\alpha} u(x,t) + u(x,t)\, u_x(x,t) - u_{xx}(x,t) + u_{xxx}(x,t) = f(x,t), \quad 0 < \alpha \leq 1, \tag{51}$$

subject to the following initial and boundary conditions

$$\begin{aligned} & u(x,0) = 0, \quad 0 < x \leq 1, \\ & u(0,t) = u(1,t) = u_x(1,t) = 0, \quad 0 < t \leq 1, \end{aligned} \tag{52}$$

where $u(x,t) = t^{\alpha}\,(x-1)^4 \sin(\pi\, x)$ is the exact solution of this problem and $f(x,t)$ is determined by Equation (51) consistent with the chosen solution.

In Tables 1 and 2, we give a comparison of L_2 error between our method and the method in [45] at $\alpha = 0.4$ and $\alpha = 0.8$. Figure 1 shows the AE at different values of M when $\alpha = 0.9$. Table 3 presents the AE at different values of t and the computational time (CPU time) when $\alpha = 0.7$ and $M = 16$.

Table 1: Comparison of L_2 error of Example 1.

Method in [45] at $M = \lceil N^{\min\{r\,\alpha,2\}} \rceil$				Our method at $M = 16$	
r	N	Error at $\alpha = 0.4$	Error at $\alpha = 0.8$	Error at $\alpha = 0.4$	Error at $\alpha = 0.8$
	8	4.7587×10^{-2}	1.0273×10^{-2}		
3	16	2.2552×10^{-2}	2.5810×10^{-3}	3.52912×10^{-14}	1.22125×10^{-15}
	32	9.8238×10^{-3}	6.4395×10^{-4}		

Table 2: Comparison of L_2 error of Example 1.

Method in [45] at $N = 100, r = \frac{2}{\alpha}$			Our method at $M = 16$	
M	Error at $\alpha = 0.4$	Error at $\alpha = 0.8$	Error at $\alpha = 0.4$	Error at $\alpha = 0.8$
4	1.1873×10^{-2}	1.1888×10^{-2}		
8	2.0365×10^{-3}	2.0646×10^{-3}	3.52912×10^{-14}	1.22125×10^{-15}
16	4.7068×10^{-4}	4.5241×10^{-4}		

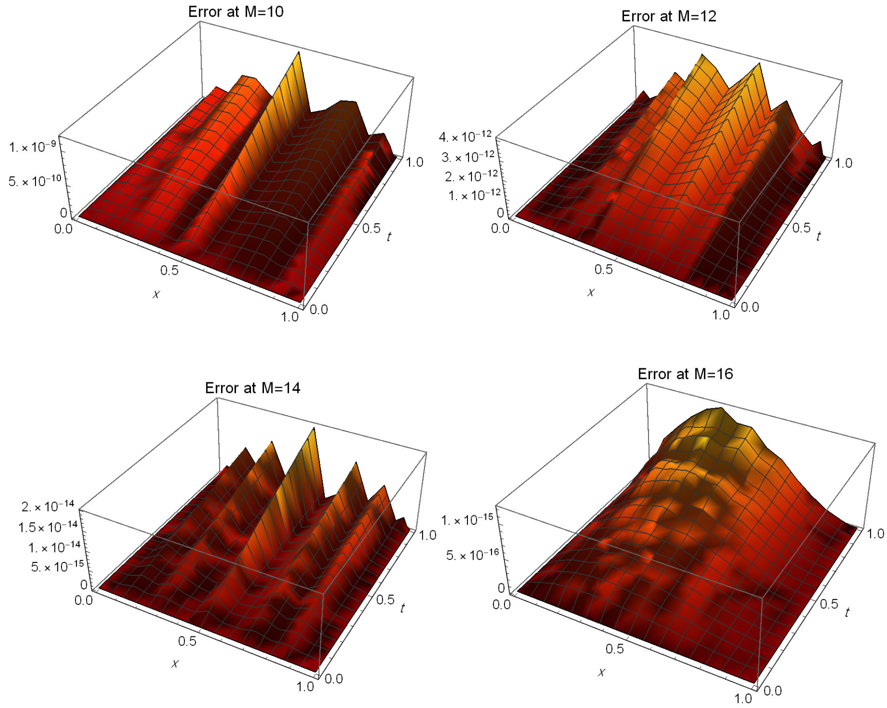

Figure 1: The AE for Example 1 at different values of M for $\alpha = 0.9$.

Table 3: The AE of Example 1 at $\alpha = 0.7, M = 16$.

x	$t = \frac{2}{10}$	$t = \frac{4}{10}$	$t = \frac{6}{10}$	$t = \frac{8}{10}$	CPU time
0.1	7.52454×10^{-14}	7.52731×10^{-14}	7.49678×10^{-14}	7.47458×10^{-14}	
0.2	1.10871×10^{-13}	1.10911×10^{-13}	1.10467×10^{-13}	1.10328×10^{-13}	
0.3	1.15921×10^{-13}	1.16061×10^{-13}	1.15657×10^{-13}	1.15352×10^{-13}	
0.4	1.01787×10^{-13}	1.01974×10^{-13}	1.01669×10^{-13}	1.01294×10^{-13}	
0.5	7.85587×10^{-14}	7.86593×10^{-14}	7.84442×10^{-14}	7.82291×10^{-14}	147.139
0.6	5.36862×10^{-14}	5.37764×10^{-14}	5.36446×10^{-14}	5.34919×10^{-14}	
0.7	3.15512×10^{-14}	3.15976×10^{-14}	3.15147×10^{-14}	3.13768×10^{-14}	
0.8	1.45151×10^{-14}	0.45489×10^{-14}	1.44871×10^{-14}	1.44422×10^{-14}	
0.9	3.75442×10^{-15}	3.76249×10^{-15}	3.75766×10^{-15}	3.74353×10^{-15}	

Test Problem 2. Consider the time-fractional KdV-Burgers' equation of the form

$$D_t^{\alpha} u(x,t) + u(x,t)\, u_x(x,t) - u_{xx}(x,t) + u_{xxx}(x,t) = f(x,t), \quad 0 < \alpha \le 1, \tag{53}$$

subject to the following initial and boundary conditions

$$\begin{aligned} &u(x,0) = 0, \quad 0 < x \le 1, \\ &u(0,t) = u(1,t) = u_x(1,t) = 0, \quad 0 < t \le 1, \end{aligned} \tag{54}$$

where $u(x,t) = x\,(x-1)^5\, t^{\alpha}$ is the exact solution of this problem and $f(x,t)$ is determined by Equation (53) consistent with the chosen solution.

Figure 2 shows the AE and approximate solution at $\alpha = 0.2$ when $M = 12$. Table 4 presents the AE at different values of t and the CPU time when $\alpha = 0.5$ and $M = 12$. Figure 3 shows the AE at different values of M when $\alpha = 0.8$.

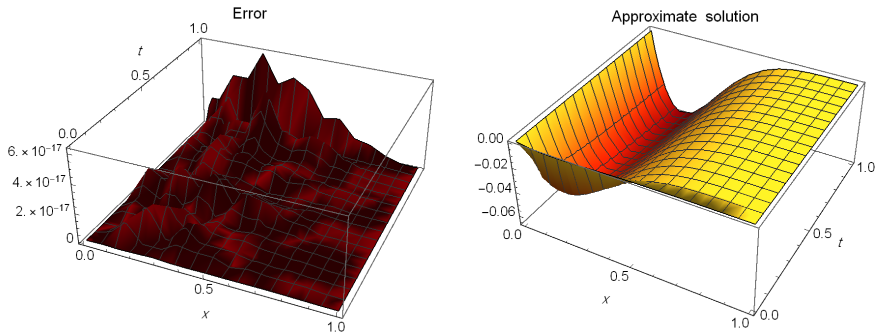

Figure 2: The AE (left) and approximate solution (right) for Example 2 at $\alpha = 0.2, M = 12$.

Table 4: The AE of Example 2 at $\alpha = 0.5, M = 12$.

x	$t = \frac{3}{10}$	$t = \frac{5}{10}$	$t = \frac{9}{10}$	CPU time
0.1	6.93889×10^{-18}	0	6.93889×10^{-18}	
0.2	6.93889×10^{-18}	6.93889×10^{-18}	1.38778×10^{-17}	
0.3	1.04083×10^{-17}	0	6.93889×10^{-18}	
0.4	6.93889×10^{-18}	3.46945×10^{-18}	1.04083×10^{-17}	
0.5	6.93889×10^{-18}	1.73472×10^{-18}	3.46945×10^{-18}	62.984
0.6	5.63785×10^{-18}	6.93889×10^{-18}	8.67362×10^{-19}	
0.7	3.03577×10^{-18}	4.33681×10^{-19}	2.60209×10^{-18}	
0.8	9.48677×10^{-19}	2.11419×10^{-18}	2.65631×10^{-18}	
0.9	1.66018×10^{-19}	2.28699×10^{-19}	7.69106×10^{-19}	

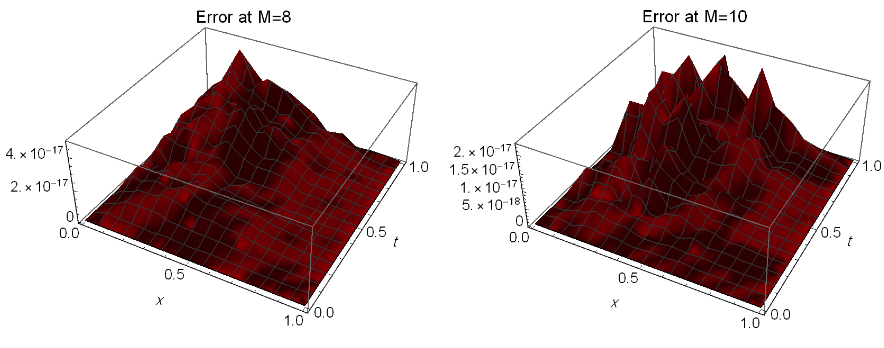

Figure 3: The AE for Example 2 at different values of M for $\alpha = 0.8$.

Test Problem 3. Consider the time-fractional KdV-Burgers' equation of the form

$$D_t^{\alpha} u(x,t) + u(x,t)\, u_x(x,t) - u_{xx}(x,t) + u_{xxx}(x,t) = f(x,t), \quad 0 < \alpha \leq 1, \tag{55}$$

subject to the following initial and boundary conditions

$$\begin{aligned} &u(x,0)=0, \quad 0<x\le 1, \\ &u(0,t)=u(1,t)=u_x(1,t)=0, \quad 0<t\le 1, \end{aligned} \tag{56}$$

where $u(x,t)=(1-x)^2\,x^{5/2}\,t^{\alpha+1}$ is the exact solution of this problem and $f(x,t)$ is determined by Equation (55) consistent with the chosen solution.

Table 5 presents the AE at different values of t and the CPU time when $\alpha=0.2$ and $M=16$. Figure 4 shows the AE and approximate solution at $\alpha=0.5$ when $M=16$. Table 6 presents the AE at different values of t and the CPU time when $\alpha=0.9$ and $M=16$.

Table 5: The AE of Example 3 at $\alpha=0.2, M=16$.

x	$t=\frac{3}{10}$	$t=\frac{5}{10}$	$t=\frac{9}{10}$	CPU time
0.1	1.22775×10^{-8}	2.26669×10^{-8}	4.58946×10^{-8}	
0.2	6.12315×10^{-8}	1.13021×10^{-7}	2.28801×10^{-7}	
0.3	1.8888×10^{-8}	3.48522×10^{-8}	7.05344×10^{-8}	
0.4	4.44177×10^{-8}	8.20105×10^{-8}	1.66068×10^{-7}	
0.5	4.19961×10^{-8}	7.7504×10^{-8}	1.56872×10^{-7}	140.594
0.6	4.15244×10^{-8}	7.66681×10^{-8}	1.55253×10^{-7}	
0.7	1.0532×10^{-8}	1.94297×10^{-8}	3.93095×10^{-8}	
0.8	5.29158×10^{-10}	9.70573×10^{-10}	1.95065×10^{-9}	
0.9	1.10494×10^{-8}	2.03984×10^{-8}	4.13017×10^{-8}	

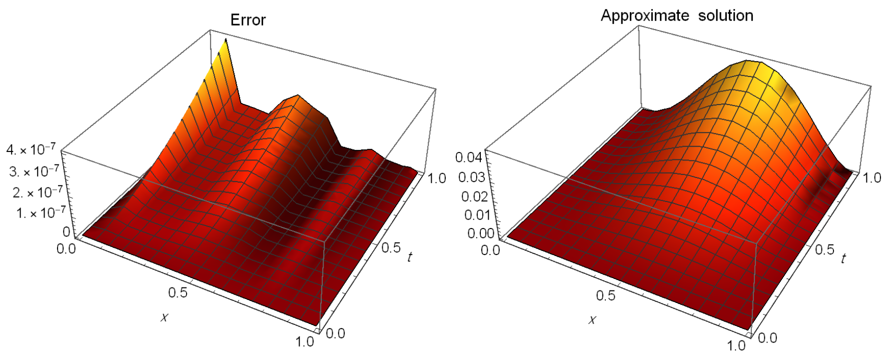

Figure 4: The AE (left) and approximate solution (right) for Example 3 at $\alpha=0.5, M=16$.

6 Concluding remarks

In this study, the third-order time-fractional KdV equation was numerically handled. A twofold expansion in terms of a Chebyshev polynomial of the second kind was used to express an approximation of the solution. The tau approach is used. Some theoretical findings aided the development of our desired approximation solutions. In addition, a formula for the third-order derivatives of a particular basis function, represented as a specific combination of the shifted

Table 6: The AE of Example 3 at $\alpha = 0.9, M = 16$.

x	$t = \frac{3}{10}$	$t = \frac{5}{10}$	$t = \frac{9}{10}$	CPU time
0.1	5.25679×10^{-9}	1.39106×10^{-8}	4.25758×10^{-8}	
0.2	2.64334×10^{-8}	6.96779×10^{-8}	2.1267×10^{-7}	
0.3	2.64334×10^{-9}	2.16063×10^{-8}	6.57156×10^{-8}	
0.4	1.89934×10^{-8}	5.02974×10^{-8}	1.54022×10^{-7}	
0.5	1.82054×10^{-8}	4.78903×10^{-8}	1.4595×10^{-7}	143.093
0.6	1.77687×10^{-8}	4.70394×10^{-8}	1.44016×10^{-7}	
0.7	4.60913×10^{-9}	1.20682×10^{-8}	3.66522×10^{-8}	
0.8	2.66452×10^{-10}	6.53017×10^{-10}	1.88272×10^{-9}	
0.9	4.74549×10^{-9}	1.25403×10^{-8}	3.83444×10^{-8}	

second-kind Chebyshev polynomials, was provided in terms of the shifted second-kind Chebyshev polynomials themselves. The algorithm was tested by giving a few instances. By making some modifications, our numerical technique can treat more complicated FDEs with non-linear power terms analogous to ours. All codes were written and debugged by Mathematica 11 on HP Z420 Workstation, Processor: Intel (R) Xeon(R) CPU E5-1620 - 3.6 GHz, 16GB Ram DDR3, and 512 GB storage. We are planning to extend the presented algorithm to handle higher-order PDEs in applied mathematics with more nonlinear terms.

Conflicts of interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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