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Research Paper

Lower Bounds on the Entire Sombor Index

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Abstract

Let $G = (V, E)$ be a graph. The entire Sombor index of graph G, $SO^{\varepsilon}(G)$ is defined as the sum of the terms $\sqrt{d_G^2(a) + d_G^2(b)}$, where a is either adjacent to or incident with b and $a, b \in V \cup E$. It is known that if T is a tree of order n, then $SO^{\varepsilon}(T) \geq 6\sqrt{5} + 8(n-3)\sqrt{2}$. We improve this result and establish best lower bounds on the entire Sombor index with given vertices number and maximum degree. Also, we determine the extremal trees achieve these bounds.

1 Introduction

Consider a graph $G = (V, E)$. For $a \in V$, the open neighborhood of a in G, $N_G(a)$, is the set $N_G(a) = \{b \in V \mid ab \in E\}$. The degree of a in G is $d_G(a) = |N_G(a)|$. The maximum degree of G is denoted by $\Delta(G) = \Delta$. Two edges e_1, e_2 of G are called adjacent if they are distinct and have a common end-vertex. The *degree* of an edge e in G is the number of edges adjacent to e and is denoted by $d_G(e)$. The distance between the vertices $a, b \in V$, $d_T(a, b)$, is the length of a shortest a, b-path in G.

The Zagreb indices [1, 2] are the oldest members of vertex–degree–based indices and they are defined as

$$M_1(G) = \sum_{a \in V} d_G^2(a), \quad \text{and} \quad M_2(G) = \sum_{ab \in E} d_G(a) d_G(b).$$

For more information on these indices we refer to [3–5].

However in the last decade, some novel variants of vertex-degree-based indices were proposed such as sum connectivity index [6], irregularity [7, 8], Lanzhou index [9, 10], and entire Zagreb indices [11, 12]. One of such variants is the Sombor index which was introduced by Gutman [13] as:

$$SO(G) = \sum_{ab \in E} \sqrt{d_G^2(a) + d_G^2(b)}.$$

For more information about the Sombor index see [14–25] and the references therein.

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In 2023, Movahedi and Akhbari [26] extended the concept of Sombor index to the vertex and edge degrees, conceiving the so-called *entire Sombor index*. This index is defined as:

$$SO^{\varepsilon}(G) = \sum_{\substack{a \text{ is either adjacentto} \\ \text{or incident with b}}} \sqrt{d_G^2(a) + d_G^2(b)}.$$

Our primary motivation for the present paper is the following result:

Theorem 1.1. *([26]). If T is a tree of order n, then*

$$SO^{\varepsilon}(T) \geq 6\sqrt{5} + 8(n-3)\sqrt{2}\,.$$

The equality is hold if and only if $T = P_n$.

In this paper we extend the bound of Theorem 1.1 by establishing the sharp lower bounds for the entire Sombor index of trees of given order and maximum degree. We also determine the extremal trees achieving these bounds.

2 Lower bound

A *rooted tree* is a tree together with a special vertex chosen as the *root* of the tree. A *leaf* is a vertex of degree one. A tree with exactly one vertex of degree greater than two is called a *spider*. The high degree vertex of a spider T is the *center* of T. A *leg* of a spider is a path from its center to a leaf. A star is a spider such that all legs have length one. Also a path is a spider with one or two leg.

In this section, T denotes a rooted tree with root a, where $d_T(a) = \Delta$ and $N_T(a) = \{a_1, a_2, \ldots, a_\Delta\}$. For positive integers n and Δ, let $\mathcal{T}_{n,\Delta}$ be the set of all trees with n vertices and maximum degree Δ.

Lemma 2.1. *Let $T \in \mathcal{T}_{n,\Delta}$ has a vertex b of degree more than two in maximum distance from a. Then, there is a tree $T' \in \mathcal{T}_{n,\Delta}$ such that $SO^{\varepsilon}(T') < SO^{\varepsilon}(T)$.*

Proof. Let $b \neq a$ be a vertex of T with $d_T(b) = \beta \geq 3$ and $N_T(b) = \{b_1, b_2, \ldots, b_\beta\}$, where b_β lies on the a,b-path in T. By our assumption, we have $d_T(b_i) \in \{1, 2\}$ for $1 \leq i \leq \beta - 1$. We distinguish the following cases:

Case 1. b is adjacent to at least two leaves b_1 and b_2. Let $T' = (T - \{bb_1\}) \cup \{b_1 b_2\}$. Then,

$$\begin{aligned}
\alpha_1 = &\sum_{i=3}^{\beta} \left(\sqrt{d_T^2(b_i) + d_T^2(b)} + \sqrt{d_T^2(b_i) + d_T^2(bb_i)} + \sqrt{d_T^2(bb_i) + d_T^2(b)} \right) \\
&- \sum_{i=3}^{\beta} \left(\sqrt{d_{T'}^2(b_i) + d_{T'}^2(b)} + \sqrt{d_{T'}^2(b_i) + d_{T'}^2(bb_i)} + \sqrt{d_{T'}^2(bb_i) + d_{T'}^2(b)} \right) \\
= &\sum_{i=3}^{\beta} \left(\sqrt{d_T^2(b_i) + d_T^2(b)} + \sqrt{d_T^2(b_i) + d_T^2(bb_i)} + \sqrt{d_T^2(bb_i) + d_T^2(b)} \right) \\
&- \sum_{i=3}^{\beta} \left(\sqrt{d_T^2(b_i) + (d_T(b) - 1)^2} + \sqrt{d_T^2(b_i) + (d_T(bb_i) - 1)^2} \right. \\
&\left. + \sqrt{(d_T(bb_i) - 1)^2 + (d_T(b) - 1)^2} \right) > 0,
\end{aligned}$$

and

$$\begin{aligned}
\alpha_2 = & \sum_{3\le i<j\le\beta} \sqrt{d_T^2(bb_i)+d_T^2(bb_j)} - \sum_{3\le i<j\le\beta} \sqrt{d_{T'}^2(bb_i)+d_{T'}^2(bb_j)} \\
& + \sum_{3\le i\le\beta} \sqrt{d_T^2(bb_i)+d_T^2(bb_2)} - \sum_{3\le i\le\beta} \sqrt{d_{T'}^2(bb_i)+d_{T'}^2(bb_2)} \\
& + \sum_{i=3}^{\beta} \sum_{x\in N_T(b_i)-\{b\}} \sqrt{d_T^2(bb_i)+d_T^2(xb_i)} - \sum_{i=3}^{\beta} \sum_{x\in N_{T'}(b_i)-\{b\}} \sqrt{d_{T'}^2(bb_i)+d_{T'}^2(xb_i)} \\
= & \sum_{3\le i<j\le\beta} \sqrt{d_T^2(bb_i)+d_T^2(bb_j)} - \sum_{3\le i<j\le\beta} \sqrt{(d_T(bb_i)-1)^2+(d_T(bb_j)-1)^2} \\
& + \sum_{3\le i\le\beta} \sqrt{d_T^2(bb_i)+d_T^2(bb_2)} - \sum_{3\le i\le\beta} \sqrt{(d_T(bb_i)-1)^2+d_T^2(bb_2)} \\
& + \sum_{i=3}^{\beta} \sum_{x\in N_T(b_i)-\{b\}} \sqrt{d_T^2(bb_i)+d_T^2(xb_i)} \\
& - \sum_{i=3}^{\beta} \sum_{x\in N_T(b_i)-\{b\}} \sqrt{(d_T(bb_i)-1)^2+d_T^2(xb_i)} > 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
SO^{\varepsilon}(T)-SO^{\varepsilon}(T') \ge & \alpha_1+\alpha_2+\sqrt{d_T^2(b_1)+d_T^2(b)}+\sqrt{d_T^2(b_1)+d_T^2(bb_1)} \\
& + \sqrt{d_T^2(b)+d_T^2(bb_1)}+\sqrt{d_T^2(bb_1)+d_T^2(bb_2)} \\
& + \sqrt{d_T(b_2)^2+d_T^2(bb_2)}+\sqrt{d_T^2(b)+d_T^2(bb_2)} \\
& + \sqrt{d_T(b_2)^2+d_T^2(b)}-\sqrt{d_{T'}^2(b_1)+d_{T'}^2(b_2)} \\
& - \sqrt{d_{T'}^2(b_1)+d_{T'}^2(b_1b_2)}-\sqrt{d_{T'}(b_1b_2)^2+d_{T'}^2(b_2)} \\
& - \sqrt{d_{T'}^2(b_1b_2)+d_{T'}^2(bb_2)}-\sqrt{d_{T'}^2(b_2)+d_{T'}^2(bb_2)} \\
& - \sqrt{d_{T'}^2(b_2)+d_{T'}^2(b)}-\sqrt{d_{T'}^2(b)+d_{T'}^2(bb_2)} \\
> & \sqrt{\beta^2+1}+\sqrt{(\beta-1)^2+1}+\sqrt{\beta^2+(\beta-1)^2}+\sqrt{2}(\beta-1) \\
& + \sqrt{\beta^2+1}+\sqrt{\beta^2+(\beta-1)^2}+\sqrt{(\beta-1)^2+1} \\
& - \sqrt{5}-\sqrt{2}-\sqrt{5}-\sqrt{(\beta-1)^2+1}-\sqrt{(\beta-1)^2+4} \\
& - \sqrt{(\beta-1)^2+4}-\sqrt{2}(\beta-1) > 0.
\end{aligned}$$

Case 2. b is adjacent to exactly one leaf, say b_1. Assume that $bc_1c_2\dots c_l$ is a path in T and

$b_2 = c_1$, where $l \geq 2$. Let $T' = (T - \{bb_1\}) \cup \{b_1c_l\}$. Then,

$$\begin{aligned}\alpha_1 =& \sum_{i=3}^{\beta}\Big(\sqrt{d_T^2(b_i)+d_T^2(b)}+\sqrt{d_T^2(b_i)+d_T^2(bb_i)}+\sqrt{d_T^2(bb_i)+d_T^2(b)}\Big)\\ &-\sum_{i=3}^{\beta}\Big(\sqrt{d_{T'}^2(b_i)+d_{T'}^2(b)}+\sqrt{d_{T'}^2(b_i)+d_{T'}^2(bb_i)}+\sqrt{d_{T'}^2(bb_i)+d_{T'}^2(b)}\Big)\\ =&\sum_{i=3}^{\beta}\Big(\sqrt{d_T^2(b_i)+d_T^2(b)}+\sqrt{d_T^2(b_i)+d_T^2(bb_i)}+\sqrt{d_T^2(bb_i)+d_T^2(b)}\Big)\\ &-\sum_{i=3}^{\beta}\Big(\sqrt{d_T^2(b_i)+(d_T(b)-1)^2}+\sqrt{d_T^2(b_i)+(d_T(bb_i)-1)^2}\\ &+\sqrt{(d_T(bb_i)-1)^2+(d_T(b)-1)^2}\Big)>0,\end{aligned}$$

and

$$\begin{aligned}\alpha_2 =& \sum_{3\leq i<j\leq\beta}\sqrt{d_T^2(bb_i)+d_T^2(bb_j)}-\sum_{3\leq i<j\leq\beta}\sqrt{d_{T'}^2(bb_i)+d_{T'}^2(bb_j)}\\ &+\sum_{i=3}^{\beta}\sum_{x\in N_T(b_i)-\{b\}}\sqrt{d_T^2(bb_i)+d_T^2(xb_i)}-\sum_{i=3}^{\beta}\sum_{x\in N_{T'}(b_i)-\{b\}}\sqrt{d_{T'}^2(bb_i)+d_{T'}^2(xb_i)}\\ =&\sum_{3\leq i<j\leq\beta}\sqrt{d_T^2(bb_i)+d_T^2(bb_j)}-\sum_{3\leq i<j\leq\beta}\sqrt{(d_T(bb_i)-1)^2+(d_T(bb_j)-1)^2}\\ &+\sum_{i=3}^{\beta}\sum_{x\in N_T(b_i)-\{b\}}\sqrt{d_T^2(bb_i)+d_T^2(xb_i)}\\ &-\sum_{i=3}^{\beta}\sum_{x\in N_T(b_i)-\{b\}}\sqrt{(d_T(bb_i)-1)^2+d_T^2(xb_i)}>0.\end{aligned}$$

Therefore,

$$\begin{aligned}SO^{\varepsilon}(T)-SO^{\varepsilon}(T')\geq&\alpha_1+\alpha_2+\sqrt{d_T^2(b_1)+d_T^2(b)}+\sqrt{d_T^2(b_1)+d_T^2(bb_1)}\\ &+\sqrt{d_T^2(bb_1)+d_T^2(b)}+\sqrt{d_T^2(c_l)+d_T^2(c_{l-1})}\\ &+\sqrt{d_T^2(c_l)+d_T^2(c_lc_{l-1})}+\sqrt{d_T^2(c_{l-1})+d_T^2(c_lc_{l-1})}\\ &+\sqrt{d_T^2(c_{l-1})+d_T^2(c_{l-1}c_{l-2})}+\sqrt{d_T^2(c_{l-1}c_{l-2})+d_T^2(c_lc_{l-1})}\\ &+\sqrt{d_T^2(bb_1)+d_T^2(bb_2)}-\sqrt{d_{T'}^2(b_1)+d_{T'}^2(c_l)}\\ &-\sqrt{d_{T'}^2(b_1)+d_{T'}^2(b_1c_l)}-\sqrt{d_{T'}^2(c_l)+d_{T'}^2(b_1c_l)}\\ &-\sqrt{d_{T'}^2(b_1c_l)+d_{T'}^2(c_lc_{l-1})}-\sqrt{d_{T'}^2(c_l)+d_{T'}^2(c_{l-1})}\\ &-\sqrt{d_{T'}^2(c_l)+d_{T'}^2(c_lc_{l-1})}-\sqrt{d_{T'}^2(c_{l-1})+d_{T'}^2(c_lc_{l-1})}\\ &-\sqrt{d_{T'}^2(c_{l-1})+d_{T'}^2(c_{l-1}c_{l-2})}-\sqrt{d_{T'}^2(c_{l-1}c_{l-2})+d_{T'}^2(c_lc_{l-1})}.\end{aligned}$$

If $l = 2$, then $c_{l-2} = b$ and

$$\begin{aligned} SO^{\varepsilon}(T) - SO^{\varepsilon}(T') > & \sqrt{\beta^2+1} + \sqrt{(\beta-1)^2+1} + \sqrt{\beta^2+(\beta-1)^2} \\ & + \sqrt{5} + \sqrt{2} + \sqrt{5} + \sqrt{\beta^2+4} + 2\sqrt{\beta^2+1} \\ & - \sqrt{5} - \sqrt{2} - 2\sqrt{5} - 3\sqrt{8} \\ & - \sqrt{(\beta-1)^2+4} - \sqrt{(\beta-1)^2+4} > 0. \end{aligned}$$

If $l \geq 3$, then $c_{l-2} \neq b$ and

$$\begin{aligned} SO^{\varepsilon}(T) - SO^{\varepsilon}(T') > & \sqrt{\beta^2+1} + \sqrt{(\beta-1)^2+1} + \sqrt{\beta^2+(\beta-1)^2} \\ & + \sqrt{5} + \sqrt{2} + \sqrt{5} + \sqrt{8} + \sqrt{5} + \sqrt{\beta^2+1} \\ & - \sqrt{5} - \sqrt{2} - 2\sqrt{5} - 5\sqrt{8} > 0. \end{aligned}$$

Case 3. All vertices adjacent to b are of degree at least two. Let $bc_1c_2\ldots c_t$ and $bd_1d_2\ldots d_l$ be two paths in T such that $l, t \geq 2$, $b_1 = c_1$ and $b_2 = d_1$. Let T' be the tree deduced from $T - \{c_1, c_2, \ldots, c_t\}$ by attaching the path $d_lc_1c_2\ldots c_t$. Then,

$$\begin{aligned} \alpha_1 = & \sum_{i=3}^{\beta} \Big(\sqrt{d_T^2(b_i) + d_T^2(b)} + \sqrt{d_T^2(b_i) + d_T^2(bb_i)} + \sqrt{d_T^2(bb_i) + d_T^2(b)} \Big) \\ & - \sum_{i=3}^{\beta} \Big(\sqrt{d_{T'}^2(b_i) + d_{T'}^2(b)} + \sqrt{d_{T'}^2(b_i) + d_{T'}^2(bb_i)} + \sqrt{d_{T'}^2(bb_i) + d_{T'}^2(b)} \Big) \\ = & \sum_{i=3}^{\beta} \Big(\sqrt{d_T^2(b_i) + d_T^2(b)} + \sqrt{d_T^2(b_i) + d_T^2(bb_i)} + \sqrt{d_T^2(bb_i) + d_T^2(b)} \Big) \\ & - \sum_{i=3}^{\beta} \Big(\sqrt{d_T^2(b_i) + (d_T(b)-1)^2} + \sqrt{d_T^2(b_i) + (d_T(bb_i)-1)^2} \\ & + \sqrt{(d_T(bb_i)-1)^2 + (d_T(b)-1)^2} \Big) > 0, \end{aligned}$$

and

$$\begin{aligned} \alpha_2 = & \sum_{3 \leq i < j \leq \beta} \sqrt{d_T^2(bb_i) + d_T^2(bb_j)} - \sum_{3 \leq i < j \leq \beta} \sqrt{d_{T'}^2(bb_i) + d_{T'}^2(bb_j)} \\ & + \sum_{i=3}^{\beta} \sum_{x \in N_T(b_i) - \{b\}} \sqrt{d_T^2(bb_i) + d_T^2(xb_i)} - \sum_{i=3}^{\beta} \sum_{x \in N_{T'}(b_i) - \{b\}} \sqrt{d_{T'}^2(bb_i) + d_{T'}^2(xb_i)} \\ = & \sum_{3 \leq i < j \leq \beta} \sqrt{d_T^2(bb_i) + d_T^2(bb_j)} - \sum_{3 \leq i < j \leq \beta} \sqrt{(d_T(bb_i)-1)^2 + (d_T(bb_j)-1)^2} \\ & + \sum_{i=3}^{\beta} \sum_{x \in N_T(b_i) - \{b\}} \sqrt{d_T^2(bb_i) + d_T^2(xb_i)} \\ & - \sum_{i=3}^{\beta} \sum_{x \in N_T(b_i) - \{b\}} \sqrt{(d_T(bb_i)-1)^2 + d_T^2(xb_i)} > 0. \end{aligned}$$

Therefore,

$$\begin{aligned}SO^{\varepsilon}(T) - SO^{\varepsilon}(T') \geq & \alpha_1 + \alpha_2 + \sqrt{d_T^2(b_1) + d_T^2(b)} + \sqrt{d_T^2(b_1) + d_T^2(bb_1)} \\ & + \sqrt{d_T^2(bb_1) + d_T^2(b)} + \sqrt{d_T^2(bb_1) + d_T^2(b_1c_2)} \\ & + \sqrt{d_T^2(d_l) + d_T^2(d_{l-1})} + \sqrt{d_T^2(d_l) + d_T^2(d_ld_{l-1})} \\ & + \sqrt{d_T^2(d_{l-1}) + d_T^2(d_ld_{l-1})} + \sqrt{d_T^2(d_{l-1}) + d_T^2(d_{l-1}d_{l-2})} \\ & + \sqrt{d_T^2(d_{l-1}d_{l-2}) + d_T^2(d_ld_{l-1})} + \sqrt{d_T^2(bb_1) + d_T^2(bb_2)} \\ & - \sqrt{d_{T'}^2(b_1) + d_{T'}^2(d_l)} - \sqrt{d_{T'}^2(b_1) + d_{T'}^2(b_1d_l)} \\ & - \sqrt{d_{T'}^2(d_l) + d_{T'}^2(b_1d_l)} - \sqrt{d_{T'}^2(b_1d_l) + d_{T'}^2(d_ld_{l-1})} \\ & - \sqrt{d_{T'}^2(d_l) + d_{T'}^2(d_{l-1})} - \sqrt{d_{T'}^2(d_l) + d_{T'}^2(d_ld_{l-1})} \\ & - \sqrt{d_{T'}^2(d_{l-1}) + d_{T'}^2(d_ld_{l-1})} - \sqrt{d_{T'}^2(d_{l-1}) + d_{T'}^2(d_{l-1}d_{l-2})} \\ & - \sqrt{d_{T'}^2(d_{l-1}d_{l-2}) + d_{T'}^2(d_ld_{l-1})} - \sqrt{d_{T'}^2(b_1d_l) + d_{T'}^2(b_1c_2)}.\end{aligned}$$

If $l = 2$, then $d_{l-2} = b$ and

$$\begin{aligned}SO^{\varepsilon}(T) - SO^{\varepsilon}(T') > & 2\sqrt{\beta^2+4} + \sqrt{2}\beta + \sqrt{\beta^2 + d_{T'}^2(b_1c_2)} \\ & + \sqrt{5} + \sqrt{2} + \sqrt{5} + \sqrt{\beta^2+4} + \sqrt{\beta^2+1} + \sqrt{2}\beta \\ & - 7\sqrt{8} - 2\sqrt{(\beta-1)^2+4} - \sqrt{4 + d_{T'}^2(b_1c_2)} > 0.\end{aligned}$$

If $l \geq 3$, then $d_{l-2} \neq b$ and

$$\begin{aligned}SO^{\varepsilon}(T) - SO^{\varepsilon}(T') > & 2\sqrt{\beta^2+4} + \sqrt{2}\beta + \sqrt{\beta^2 + d_{T'}^2(b_1c_2)} \\ & + \sqrt{5} + \sqrt{2} + \sqrt{5} + \sqrt{8} + \sqrt{5} + \sqrt{2}\beta \\ & - 9\sqrt{8} - \sqrt{4 + d_{T'}^2(b_1c_2)} > 0.\end{aligned}$$

This completes the proof. ■

Lemma 2.2. *Let $T \in \mathcal{T}_{n,\Delta}$ be a spider with $\Delta \geq 3$ and T has two legs of length more than one. Then there exists a spider $T' \in \mathcal{T}_{n,\Delta}$ such that $SO^{\varepsilon}(T') < SO^{\varepsilon}(T)$.*

Proof. Let $d_T(a) = \Delta$ and $ab_1b_2\ldots b_t$ and $ac_1c_2\ldots c_l$ be two legs of length more than one. Assume that T' be the tree deduced from $T - \{b_2, \ldots, b_t\}$ with attaching the path $c_lb_2\ldots b_t$.

Firstly, let $l = t = 2$. Then,

$$
\begin{aligned}
SO^{\varepsilon}(T) - SO^{\varepsilon}(T') \geq & \sqrt{d_T^2(b_1) + d_T^2(b_2)} + \sqrt{d_T^2(b_2) + d_T^2(b_1b_2)} \\
& + \sqrt{d_T^2(b_1) + d_T^2(b_1b_2)} + \sqrt{d_T^2(b_1b_2) + d_T^2(b_1a)} \\
& + \sqrt{d_T^2(b_1) + d_T^2(b_1a)} + \sqrt{d_T^2(b_1) + d_T^2(a)} \\
& + \sqrt{d_T^2(a) + d_T^2(b_1a)} + \sqrt{d_T^2(b_1a) + d_T^2(ac_1)} \\
& + \sqrt{d_T^2(c_2) + d_T^2(c_1c_2)} + \sqrt{d_T^2(c_1) + d_T^2(c_1c_2)} \\
& + \sqrt{d_T^2(c_1) + d_T^2(c_2)} + \sqrt{d_T^2(c_1a) + d_T^2(c_1c_2)} \\
& - \sqrt{d_{T'}^2(b_1a) + d_{T'}^2(c_1a)} - \sqrt{d_{T'}^2(b_1) + d_{T'}^2(b_1a)} \\
& - \sqrt{d_{T'}^2(b_1) + d_{T'}^2(a)} - \sqrt{d_{T'}^2(a) + d_{T'}^2(b_1a)} \\
& - \sqrt{d_{T'}^2(b_2) + d_{T'}^2(b_2c_2)} - \sqrt{d_{T'}^2(b_2) + d_{T'}^2(c_2)} \\
& - \sqrt{d_{T'}^2(b_2c_2) + d_{T'}^2(c_2)} - \sqrt{d_{T'}^2(c_1) + d_{T'}^2(c_2)} \\
& - \sqrt{d_{T'}^2(c_1c_2) + d_{T'}^2(b_2c_2)} - \sqrt{d_{T'}^2(c_1) + d_{T'}^2(c_1c_2)} \\
& - \sqrt{d_{T'}^2(c_1c_2) + d_{T'}^2(c_2)} - \sqrt{d_{T'}^2(c_1c_2) + d_{T'}^2(ac_1)} \\
= & 2\sqrt{2}\Delta + 2\sqrt{\Delta^2+4} + 2\sqrt{\Delta^2+1} + 3\sqrt{5} + 2\sqrt{2} \\
& - 2\sqrt{\Delta^2 + (\Delta-1)^2} - \sqrt{\Delta^2+4} - \sqrt{\Delta^2+1} \\
& - \sqrt{(\Delta-1)^2+1} - 3\sqrt{8} - 3\sqrt{5} - \sqrt{2} \\
\geq & \sqrt{\Delta^2+4} + \sqrt{5} - 4\sqrt{2} \geq \sqrt{13} + \sqrt{5} - 4\sqrt{2} \approx 0.1847 > 0.
\end{aligned}
$$

If $t = 2$ and $l \geq 3$, then

$$
\begin{aligned}
SO^{\varepsilon}(T) - SO^{\varepsilon}(T') \geq & \sqrt{d_T^2(b_1) + d_T^2(b_2)} + \sqrt{d_T^2(b_2) + d_T^2(b_1b_2)} \\
& + \sqrt{d_T^2(b_1) + d_T^2(b_1b_2)} + \sqrt{d_T^2(b_1b_2) + d_T^2(b_1a)} \\
& + \sqrt{d_T^2(b_1) + d_T^2(b_1a)} + \sqrt{d_T^2(b_1) + d_T^2(a)} \\
& + \sqrt{d_T^2(c_{l-1}) + d_T^2(c_{l-1}c_l)} + \sqrt{d_T^2(c_l) + d_T^2(c_{l-1}c_l)} \\
& + \sqrt{d_T^2(c_{l-1}) + d_T^2(c_l)} + \sqrt{d_T^2(c_lc_{l-1}) + d_T^2(c_{l-1}c_{l-2})} \\
& + \sqrt{d_T^2(b_1a) + d_T^2(ac_1)} + \sqrt{d_T^2(a) + d_T^2(b_1a)}
\end{aligned}
$$

$$
\begin{aligned}
&- \sqrt{d^2_{T'}(b_1a) + d^2_{T'}(c_1a)} - \sqrt{d^2_{T'}(b_1) + d^2_{T'}(b_1a)} - \sqrt{d^2_{T'}(b_1) + d^2_{T'}(a)}\\
&- \sqrt{d^2_{T'}(a) + d^2_{T'}(b_1a)} - \sqrt{d^2_{T'}(b_2) + d^2_{T'}(b_2c_l)} - \sqrt{d^2_{T'}(b_2) + d^2_{T'}(c_l)}\\
&- \sqrt{d^2_{T'}(b_2c_l) + d^2_{T'}(c_l)} - \sqrt{d^2_{T'}(c_{l-1}) + d^2_{T'}(c_l)} - \sqrt{d^2_{T'}(c_{l-1}c_l) + d^2_{T'}(b_2c_l)}\\
&- \sqrt{d^2_{T'}(c_{l-1}) + d^2_{T'}(c_{l-1}c_l)} - \sqrt{d^2_{T'}(c_{l-1}c_l) + d^2_{T'}(c_l)}\\
&- \sqrt{d^2_{T'}(c_{l-1}c_l) + d^2_{T'}(c_{l-1}c_{l-2})}\\
=&2\sqrt{2}\Delta + 2\sqrt{\Delta^2+4} + \sqrt{\Delta^2+1} + 5\sqrt{5} + 2\sqrt{2}\\
&- 2\sqrt{\Delta^2+(\Delta-1)^2} - \sqrt{\Delta^2+1} - \sqrt{(\Delta-1)^2+1} - 4\sqrt{8} - 3\sqrt{5} - \sqrt{2}\\
\geq&\sqrt{5} - \sqrt{2} \approx 0.8218 > 0.
\end{aligned}
$$

Finally, let $l, t \geq 3$. Then,

$$
\begin{aligned}
SO^{\varepsilon}(T) - SO^{\varepsilon}(T') \geq&\sqrt{d^2_T(b_1) + d^2_T(b_2)} + \sqrt{d^2_T(b_2) + d^2_T(b_1b_2)}\\
&+ \sqrt{d^2_T(b_1) + d^2_T(b_1b_2)} + \sqrt{d^2_T(b_1b_2) + d^2_T(b_1a)}\\
&+ \sqrt{d^2_T(b_1) + d^2_T(b_1a)} + \sqrt{d^2_T(b_1) + d^2_T(a)}\\
&+ \sqrt{d^2_T(c_{l-1}) + d^2_T(c_{l-1}c_l)} + \sqrt{d^2_T(c_l) + d^2_T(c_{l-1}c_l)}\\
&+ \sqrt{d^2_T(c_{l-1}) + d^2_T(c_l)} + \sqrt{d^2_T(c_lc_{l-1}) + d^2_T(c_{l-1}c_{l-2})}\\
&+ \sqrt{d^2_T(b_1a) + d^2_T(ac_1)} + \sqrt{d^2_T(a) + d^2_T(b_1a)}\\
&- \sqrt{d^2_{T'}(b_1a) + d^2_{T'}(c_1a)} - \sqrt{d^2_{T'}(b_1) + d^2_{T'}(b_1a)}\\
&- \sqrt{d^2_{T'}(b_1) + d^2_{T'}(a)} - \sqrt{d^2_{T'}(a) + d^2_{T'}(b_1a)}\\
&- \sqrt{d^2_{T'}(b_2) + d^2_{T'}(b_2c_l)} - \sqrt{d^2_{T'}(b_2) + d^2_{T'}(c_l)}\\
&- \sqrt{d^2_{T'}(b_2c_l) + d^2_{T'}(c_l)} - \sqrt{d^2_{T'}(c_{l-1}) + d^2_{T'}(c_l)}\\
&- \sqrt{d^2_{T'}(c_{l-1}c_l) + d^2_{T'}(b_2c_l)} - \sqrt{d^2_{T'}(c_{l-1}) + d^2_{T'}(c_{l-1}c_l)}\\
&- \sqrt{d^2_{T'}(c_{l-1}c_l) + d^2_{T'}(c_l)} - \sqrt{d^2_{T'}(c_{l-1}c_l) + d^2_{T'}(c_{l-1}c_{l-2})}\\
=&2\sqrt{2}\Delta + 3\sqrt{\Delta^2+4} + 3\sqrt{5} + \sqrt{2}\\
&- 2\sqrt{\Delta^2+(\Delta-1)^2} - \sqrt{\Delta^2+1} - \sqrt{(\Delta-1)^2+1} - 5\sqrt{8}\\
\geq&\sqrt{13} + 2\sqrt{5} - \sqrt{10} - 3\sqrt{2} \approx 0.6727 > 0.
\end{aligned}
$$

This completes the proof. ■

Theorem 2.3. *Let $T \in \mathcal{T}_{n,\Delta}$. If $n \geq 4$, then*

$$
\begin{aligned}
SO^{\varepsilon}(T) \geq&(\Delta-1)[2\sqrt{\Delta^2+(\Delta-1)^2} + \sqrt{\Delta^2+1} + \sqrt{(\Delta-1)^2+1}]\\
&+ 3\sqrt{\Delta^2+4} + \frac{1}{\sqrt{2}}(\Delta-1)^2(\Delta-2) + (8n-8\Delta-17)\sqrt{2} + 3\sqrt{5},
\end{aligned}
$$

when $n > \Delta - 2$,

$$\begin{aligned} SO^{\varepsilon}(T) \geq &(\Delta-1)[2\sqrt{\Delta^2+(\Delta-1)^2}+\sqrt{\Delta^2+1}+\sqrt{(\Delta-1)^2+1}] \\ &+2\sqrt{\Delta^2+4}+\sqrt{\Delta^2+1}+\frac{1}{\sqrt{2}}(\Delta-1)^2(\Delta-2)+2\sqrt{5}+\sqrt{2}, \end{aligned}$$

when $\Delta = n-2$, *and*

$$SO^{\varepsilon}(T) = \Delta[\sqrt{\Delta^2+(\Delta-1)^2}+\sqrt{\Delta^2+1}+\sqrt{(\Delta-1)^2+1}]+\frac{1}{\sqrt{2}}\Delta(\Delta-1)^2,$$

when $\Delta = n-1$. *The equality is hold if and only if* T *is a spider with at most one leg of length more than one.*

Proof. Assume that $T^* \in \mathcal{T}_{n,\Delta}$ with $SO^{\varepsilon}(T^*) \leq SO^{\varepsilon}(T)$ for all $T \in \mathcal{T}_{n,\Delta}$. Let a be a vertex with maximum degree Δ and root T^* at a. If $\Delta = 2$, then T is a path and by Theorem 1.1, $SO^{\varepsilon}(T) = 6\sqrt{5}+8(n-3)\sqrt{2}$. If $\Delta \geq 3$, then by Lemma 2.1, T^* is a spider with center a and by Lemma 2.2, T^* has at most one leg of length more than one. If all legs of T^* have length one, then T^* is a star and

$$SO^{\varepsilon}(T^*) = \Delta[\sqrt{\Delta^2+(\Delta-1)^2}+\sqrt{\Delta^2+1}+\sqrt{(\Delta-1)^2+1}]+\frac{1}{\sqrt{2}}\Delta(\Delta-1)^2.$$

Let T^* is not a star and T^* have only one leg of length more than one. If $\Delta = n-2$, then

$$\begin{aligned} SO^{\varepsilon}(T^*) = &(\Delta-1)[2\sqrt{\Delta^2+(\Delta-1)^2}+\sqrt{\Delta^2+1}+\sqrt{(\Delta-1)^2+1}] \\ &+2\sqrt{\Delta^2+4}+\sqrt{\Delta^2+1}+\frac{1}{\sqrt{2}}(\Delta-1)^2(\Delta-2)+2\sqrt{5}+\sqrt{2}, \end{aligned}$$

and if $n > \Delta - 2$, then

$$\begin{aligned} SO^{\varepsilon}(T^*) = &(\Delta-1)[2\sqrt{\Delta^2+(\Delta-1)^2}+\sqrt{\Delta^2+1}+\sqrt{(\Delta-1)^2+1}] \\ &+3\sqrt{\Delta^2+4}+\frac{1}{\sqrt{2}}(\Delta-1)^2(\Delta-2)+(8n-8\Delta-17)\sqrt{2}+3\sqrt{5}. \end{aligned}$$

Now, the proof is complete. ■

By definition of entire Sombor index, we have the following observation:

Observation 2.4. *Let* G *be a graph. Then, for every edge* $e \notin E(G)$,

$$SO^{\varepsilon}(G+e) > SO^{\varepsilon}(G).$$

By Observation 2.4, we obtain the following Theorem:

Theorem 2.5. *Let* G *be a graph of order* $n \geq 4$ *with maximum degree* Δ. *Then,*

$$\begin{aligned} SO^{\varepsilon}(G) \geq &(\Delta-1)[2\sqrt{\Delta^2+(\Delta-1)^2}+\sqrt{\Delta^2+1}+\sqrt{(\Delta-1)^2+1}] \\ &+3\sqrt{\Delta^2+4}+\frac{1}{\sqrt{2}}(\Delta-1)^2(\Delta-2)+(8n-8\Delta-17)\sqrt{2}+3\sqrt{5}, \end{aligned}$$

when $n > \Delta - 2$,

$$\begin{aligned} SO^{\varepsilon}(G) \geq & (\Delta-1)[2\sqrt{\Delta^2+(\Delta-1)^2}+\sqrt{\Delta^2+1}+\sqrt{(\Delta-1)^2+1}] \\ & +2\sqrt{\Delta^2+4}+\sqrt{\Delta^2+1}+\frac{1}{\sqrt{2}}(\Delta-1)^2(\Delta-2)+2\sqrt{5}+\sqrt{2}, \end{aligned}$$

when $\Delta = n - 2$, *and*

$$SO^{\varepsilon}(G) \geq \Delta[\sqrt{\Delta^2+(\Delta-1)^2}+\sqrt{\Delta^2+1}+\sqrt{(\Delta-1)^2+1}]+\frac{1}{\sqrt{2}}\Delta(\Delta-1)^2,$$

when $\Delta = n - 1$. *The equality is hold if and only if* G *is a spider with at most one leg of length more than one.*

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