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Research Paper

General Randić Index of Uniform Hypergraphs

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Abstract

The general Randić index of a graph $G = (V, E)$ is defined as $R_\alpha = \sum_{u,v \in V} (d_u d_v)^\alpha$, where d_u is the degree of vertex u and α is an arbitrary real number. In this paper, we define the Randić index of a uniform hypergraph and obtain lower and upper bounds for R_α depending on different values of α.

1 Introduction

Graphs have long been used as suitable and a practical tool in modeling. For instance to study the properties of alkanes, the graph corresponding to their molecule is examined and some numerical schemes are defined, such as different types of energies of graphs and Randić index.

The Randić index was first defined by Randić in 1975 [1]. He considered some weights for each edge uv of the associated hydrogen-suppressed graph based on the degree of vertices belonging to the edge as $(d_u d_v)^{-1}$ or $(d_u d_v)^{-\frac{1}{2}}$, where d_u is the degree of vertex u. Then, he defined the Randić index of a graph $G = (V, E)$ as the sum of all these latter weights over the edges,

$$R(G) = \sum_{u,v \in V} (d_u d_v)^{-\frac{1}{2}}. \tag{1}$$

The Randić index, often called the connectivity index, describes some important molecular characteristics and is related to many chemical properties of alkanes such as boiling point, enthalpy of formation, surface area, and solubility in water.

After that, finding some upper and lower bounds of the Randić index and the graphs having the maximum and minimum Randić index attracted the attention of many researchers [2–7]. Also, some researchers tried to find the relationship between the Randić index and other topological indices of a graph [8, 9].

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In 1998, Bollobaś and Erdös introduced the general Randić index for graph $G = (V, E)$ as:

$$R_\alpha(G) = \sum_{u,v \in V} (d_u d_v)^\alpha, \tag{2}$$

where α is an arbitrary real number [3]. Note that when $\alpha = 1$, it is the second Zagreb index which is another important chemical index [10]. Similarly, several researches have been done on proposing some upper and lower bounds for the general Randić index [11, 12].

Since in recent years, the use of hypergraphs in different sciences has attracted much attention from researchers, in this paper we define the general Randić index for uniform hypergraphs and determine some of its upper and lower bounds for different values of α.

2 Preliminaries

Here, we present some required concepts of uniform hypergraphs, see [13] for comprehensive references.

An undirected hypergraph $\mathcal{H} = (V, E)$ with vertex set V, which is labeled as $[n] = \{1, 2, \cdots, n\}$, and edge set $E = \{e_1, e_2, \ldots, e_m\}$, in which $e_s \subset V$ for $s \in [m]$, is called k-uniform hypergraph if $|e_s| = k$ for $s \in [m]$. The order of the hypergraph is n and its size is m. The degree of the ith vertex, which is denoted by d_i, is defined by $|\{e \in E \mid i \in e\}|$ and $\bar{d}$ is the average of degree of vertices. Vertex $i \in V$ is an isolated vertex if $d_i = 0$. Vertices i, j are called adjacent and denoted by $i \sim j$ if there exists an edge that contains both of them. A hypergraph is d-regular if all it's vertices have the same degree d. Two different vertices i and j are connected to each other if there exists a sequence of edges $(e_{l_1}, \ldots, e_{l_p})$ such that $i \in e_{l_1}$, $j \in e_{l_p}$ and $e_{l_s} \cap e_{l_{s+1}} \neq \phi$ for all $s \in \{1, \ldots, p-1\}$. A hypergraph is called connected if every pair of distinct vertices in $\mathcal{H}$ is connected. Let $\lfloor \mathbf{a} \rfloor$ denote the maximum integer smaller than or equal to $\mathbf{a}$ for a real number $\mathbf{a}$.

Definition 2.1. Let $\mathcal{K}_n = (V, E)$ be a k-uniform hypergraph and n be a positive integer number. We call it complete hypergraph of order n if E consists of all possible edges; in other words, every k distinct vertices form an edge. It is clear that the degree of each vertex in $\mathcal{K}_n$ is $\binom{n-1}{k-1}$.

Definition 2.2. Let $\mathcal{S}_{(m)} = (V, E)$ be a k-uniform hypergraph and m be a positive integer number. We call it hyperstar of size m if there exists a disjoint partition of the vertex set V as $V = V_0 \cup V_1 \cup \cdots \cup V_m$ such that $|V_0| = 1$ and $|V_1| = \cdots = |V_m| = k-1$ and $E = \{V_0 \cup V_i \mid i \in [m]\}$.

It is clear that in hyperstar $\mathcal{S}_{(m)}$, there is only one vertex of degree m and the degree of the other vertices is one. The vertex of degree m is called the heart which is connected with all other vertices.

3 General Randić index of a hypergraph

As mentioned earlier, in 1975, Randić first defined the Randić index for a graph. He defined a weight for each edge and then called the sum of all these weights the Randić index. Let $\mathcal{H} = (V, E)$ be a k-uniform hypergraph with n vertices. There are two ways to generalize this concept to $\mathcal{H}$ and define the general Randić index for it.

1) **Graph approach**: Cosider the weight $w_\alpha(e)$ for each $e \in E$ as $w_\alpha(e) = \frac{1}{k-1}\sum_{i,j\in e}(d_i d_j)^\alpha$ and then define the general Randić index as follows:

$$R_\alpha(\mathcal{H}) = \sum_{e\in E} w_\alpha(e) = \sum_{i\sim j} a_{ij}(d_i d_j)^\alpha,$$

where $A = [a_{ij}]_{n\times n}$ is the adjacency matrix of $\mathcal{H}$ and is defined as follows [14]:

$$a_{ij} = \begin{cases} \frac{1}{k-1}|\{e \in E \ : \ i, j \in e\}|, & i \sim j, \\ 0, & i \nsim j. \end{cases}$$

2) **Hypergraph approach**: Consider the weight $w_\alpha(e)$ for each $e \in E$ as

$$w_\alpha(e) = (d_{i_1} d_{i_2} \cdots d_{i_k})^\alpha, \qquad \textit{where} \quad e = \{i_1, i_2, \cdots, i_k\}, \tag{3}$$

and then define the general Randić index as

$$R_\alpha(\mathcal{H}) = \sum_{e\in E} w_\alpha(e). \tag{4}$$

Although in recent years, the use of the graph approach in the study of hypergraphs, especially in studying the spectral theory of hypergraphs, has received much attention from researchers, we use the second method to define $R_\alpha(\mathcal{H})$. Because it seems that the second definition is more suitable and well-defined.

In the next section, we will find some upper and lower bounds for the general Randić index and corresponding extremal hypergraphs, considering different values of α.

4 Upper and lower bounds for the general Randić index

We consider a few cases for different values of α and find upper and lower bounds for the general Randić index and corresponding extremal hypergraphs in each cases.

4.1 Case: $\alpha = -1$.

Theorem 4.1. *Let $\mathcal{H} = (V, E)$ be a k-uniform hypergraph and let e_0 be an edge with minimum weight w_{-1} in $\mathcal{H}$ such that it cotains at least two vertices of degree greater than 1. Then we have:*

$$R_{-1}(\mathcal{H} - e_0) > R_{-1}(\mathcal{H}).$$

Proof. Without loss of generality, we can assume that $e_0 = \{1, 2, \cdots, k\}$ and $e_* = \{1, 2, \cdots, l\}$ be vertices of e_0 of degree greater than 1, where $l \geq 2$. Now we define the following

$$E^{(i)} = \{e \in E \ : \ |e \cap e_*| = i\}, \qquad \textit{for i=1, } \cdots \textit{, l-1,}$$
$$E^{(l)} = \{e \in E \ , \ e \neq e_0 \ : \ |e \cap e_*| = l\}.$$

Now consider the following notations:

$$\beta_j^{(1)} = \left|\left\{e \in E^{(1)} \ : \ \{j\} = e \cap e_*\right\}\right|, \qquad \forall\, j \in e_*, \tag{5}$$
$$\beta_{j_1\cdots j_t}^{(t)} = \left|\left\{e \in E^{(t)} \ : \ \{j_1, \cdots, j_t\} = e \cap e_*\right\}\right|, \qquad \forall\, j_1, \cdots, j_t \in e_* \ \&\ 1 \leq t \leq l,$$
$$\beta_{j_1\cdots j_t}^{(r)} = \left|\left\{e \in E^{(r)} \ : \ \{j_1, \cdots, j_t\} \subset e \cap e_*\right\}\right|, \qquad \forall\, j_1, \cdots, j_t \in e_* \ \&\ 1 \leq t < r \leq l,$$

It's clear that

$$d_j - 1 = \sum_{t=2}^{l}\Big(\sum_{j_2,\cdots,j_t\in e_*}\beta^{(t)}_{jj_2\cdots j_t}\Big) + \beta^{(1)}_j \tag{6}$$

$$= \sum_{t=1}^{l}\beta^{(t)}_j, \qquad \forall j\in e_*. \tag{7}$$

Equations (6) and (7) result from not considering e_0 in the βs notations. In other words, in the above summations, all edges with vertex j are considered except for e_0.

Now suppose that $e\in E^{(1)}$. Then there exists a vertex i suth that $e_*\cap e=\{i\}$. Let $w^*_{-1}(e)$ be the weight of edge e in $\mathcal{H}-e_0$. Then we have $w^*_{-1}(e)=\frac{d_i}{d_i-1}w_{-1}(e)$, and therefore:

$$w^*_{-1}(e)-w_{-1}(e)=\frac{1}{d_i-1}w_{-1}(e)\geq\frac{1}{d_i-1}w_{-1}(e_0).$$

Thus,

$$\sum_{e\in E^{(1)}}\Big(w^*_{-1}(e)-w_{-1}(e)\Big)\geq\Big(\sum_{i=1}^{l}\frac{\beta^{(1)}_i}{d_i-1}\Big)w_{-1}(e_0).$$

Similarly suppose that $e\in E^{(2)}$,

$$\exists\, i,j \ s.t.\ e\cap e_*=\{i,j\},\quad w^*_{-1}(e)=\frac{d_id_j}{(d_i-1)(d_j-1)}w_{-1}(e).$$

Therefore,

$$w^*_{-1}(e)-w_{-1}(e)=(\frac{1}{d_i-1}+\frac{1}{d_j-1}+\frac{1}{(d_i-1)(d_j-1)})w_{-1}(e)$$
$$\geq(\frac{1}{d_i-1}+\frac{1}{d_j-1})w_{-1}(e_0).$$

So,

$$\sum_{e\in E^{(2)}}\Big(w^*_{-1}(e)-w_{-1}(e)\Big)\geq\Big(\sum_{i,j\in e^*}\beta^{(2)}_{ij}(\frac{1}{d_i-1}+\frac{1}{d_j-1})\Big)w_{-1}(e_0).$$

Then similarly for $e\in E^{(t)}$ and $3\leq t\leq l$ we have:

$$\exists\, i_1,\cdots,i_t \ s.t.\ e\cap e_*=\{i_1,\cdots,i_t\},$$
$$w^*_{-1}(e)-w_{-1}(e)\geq\Big(\sum_{j=1}^{t}\frac{1}{d_{i_j}-1}\Big)w_{-1}(e_0).$$

Thus,

$$\sum_{e\in E^{(t)}}\Big(w^*_{-1}(e)-w_{-1}(e)\Big)\geq\Bigg(\sum_{i_1,\cdots,i_t\in e^*}\beta^{(t)}_{i_1\cdots i_t}\Big(\sum_{j=1}^{t}\frac{1}{d_{i_j}-1}\Big)\Bigg)w_{-1}(e_0).$$

Now we have:

$$
\begin{aligned}
& R_{-1}(\mathcal{H}-e_0)-R_{-1}(\mathcal{H}) \\
=& \sum_{e\in E\setminus e_0}\Big(w^*_{-1}(e)-w_{-1}(e)\Big)-w_{-1}(e_0)\\
=& \sum_{e\in E^{(1)}}\Big(w^*_{-1}(e)-w_{-1}(e)\Big)+\cdots+\sum_{e\in E^{(l)}}\Big(w^*_{-1}(e)-w_{-1}(e)\Big)-w_{-1}(e_0)\\
\geq& \Bigg(\sum_{i=1}^{l}\frac{\beta_i^{(1)}}{d_i-1}+\cdots+\sum_{i_1,\cdots,i_l\in e^*}\beta^{(l)}_{i_1\cdots i_l}\Big(\sum_{j=1}^{l}\frac{1}{d_{i_j}-1}\Big)\Bigg)w_{-1}(e_0)-w_{-1}(e_0)\\
=& \Bigg[\frac{1}{d_1-1}\Big(\beta_1^{(1)}+\sum_{j=1}^{l}\beta^{(2)}_{1j}+\cdots+\sum_{i_1,\cdots,i_{l-1}\in e^*}\beta^{(l)}_{1i_1\cdots i_{l-1}}\Big)\\
&+\cdots\\
&+\frac{1}{d_l-1}\Big(\beta_1^{(1)}+\sum_{j=1}^{l}\beta^{(2)}_{lj}+\cdots+\sum_{i_1,\cdots,i_{l-1}\in e^*}\beta^{(l)}_{li_1\cdots i_{l-1}}\Big)\Bigg]w_{-1}(e_0)-w_{-1}(e_0)\\
=& \Bigg(\frac{1}{d_1-1}\Big(\sum_{t=1}^{l}\beta_1^{(t)}\Big)+\cdots+\frac{1}{d_l-1}\Big(\sum_{t=1}^{l}\beta_l^{(t)}\Big)\Bigg)w_{-1}(e_0)-w_{-1}(e_0)\\
=& \Big(\frac{d_1-1}{d_1-1}+\cdots+\frac{d_l-1}{d_l-1}\Big)w_{-1}(e_0)-w_{-1}(e_0)\\
=& l w_{-1}(e_0)-w_{-1}(e_0)>0.
\end{aligned}
$$

■

Lemma 4.2. *Let $\mathcal{S}_{(m)}$ be a hyperstar of size m. Then $R_{-1}(\mathcal{S}_{(m)})=1$.*

Proof. It is clear that the weight of each edge is $\frac{1}{m}$. The result is easily obtained by considering that the number of edges is m. ■

In the following, we will specify precisely the hypergraphs with maximum and minimum R_{-1}.

Lemma 4.3. *Let $\mathcal{H}=(V,E)$ be a k-uniform hypergraph of order n. If $\mathcal{H}$ has the maximum value of R_{-1}, then it composed of hyperstars.*

Proof. It suffices to prove that each edge of $\mathcal{H}$ contains at most one vertex of degree greater than 1. By cotradiction, suppose that there exists an edge with at least two vertices with degree greater than 1 and with minimum weight (w_{-1}). Then by Theorem 4.1, we obtain a hypergraph of order n with larger value of $R_{-1}(\mathcal{H})$ by deleting this edge, which contradicts our assumption. ■

Theorem 4.4. *Let $\mathcal{H}=(V,E)$ be a k-uniform hypergraph of order n. Then we have*

$$R_{-1}(\mathcal{H})\leq\lfloor\frac{n}{k}\rfloor,$$

with equality if and only if $\mathcal{H}$ is composed of $\frac{n}{k}$ disjoint edges for $n=qk$ or is composed of a k-uniform hyperstar of size 2 and $\lfloor\frac{n}{k}\rfloor-1$ disjoint edges for $n=qk+(k-1)$.

Proof. By Lemma 4.3, a hypergraph of order n with maximum value of R_{-1} must be composed of hyperstars. Since the value of R_{-1} for each hyperstar is equal to 1 (see Lemma 4.2), the hypergraph with maximum value of R_{-1} must be composed of most hyperstar components. There are two cases:

1) If $n = qk$, then the hypergraph must be composed of q disjoint edges. In this case, the value of R_{-1} is equal to $q = \lfloor \frac{n}{k} \rfloor$.

2) If $n = qk + r$ and $0 < r \leq k-1$, then the hypergraph must be composed of several disjoint edeges and a hyperstar $\mathcal{S}_{(m)}$ of size m. We want to determine the value of m. Suppose that $\mathcal{S}_{(m)}$ has $pk + r$ vertices. On the other hand, a k-uniform hyperstar of size m must have $k + (m-1)(k-1)$ vertices. Therefore, we have:

$$\begin{aligned}
&(p-1)k + r = (m-1)(k-1), \text{ so} \\
&p(k-1) + p + r - k = (m-1)(k-1), \text{ thus} \\
&p = m - 1 \ \ \& \ \ p + r - k = 0, \text{ therefore,} \\
&m = k - r + 1.
\end{aligned}$$

Then the hypergraph must be composed of $\frac{(n-r)-(k-r)k}{k}$ disjoint edges and a hyperstar of size $k - r + 1$. In this case, the value of R_{-1} is equal to $q - k + r + 1 \leq q = \lfloor \frac{n}{k} \rfloor$. The last inequality holds because $r \leq k - 1$, and equality holds if and only if $r = k - 1$.

Therefore, the result is valid in both cases. ■

Theorem 4.5. *Let $\mathcal{H} = (V, E)$ be a k-uniform hypergraph of order n containing no isolated vertices. Then we have the following:*

$$R_{-1}(\mathcal{H}) \geq \frac{n}{k\binom{n-1}{k-1}^{k-1}},$$

with equality if and only if $\mathcal{H}$ is a complete hypergraph.

Proof. Suppose that v is an arbitrary vertex in V and let W_v denote the sum of weights of all edges containing the vertex v. Then

$$W_v = \sum_{\substack{e \in E \\ v \in e}} \prod_{u \in e} \frac{1}{d_u} \geq \frac{d_v}{\binom{n-1}{k-1}^{k-1} d_v} = \frac{1}{\binom{n-1}{k-1}^{k-1}}.$$

$$R_{-1}(\mathcal{H}) = \sum_{i_1, \cdots, i_k \in E} \frac{1}{d_{i_1} \cdots d_{i_k}} = \frac{1}{k} \sum_{v \in V} W_v \geq \frac{n}{k} \frac{1}{\binom{n-1}{k-1}^{k-1}},$$

equality holds if and only if the degree of all vertices are $\binom{n-1}{k-1}$ or, equivalently, $\mathcal{H}$ is a complete hypergraph. ■

4.2 Case: $\alpha \geq 0$

According to the definition of the general Randić Index and relations (3) and (4), it is clear that in this case, adding edges increases the value of R_α while removing edges decreases it. Among all n-vertex k-uniform hypergraphs without isolated vertices, the minimum R_α corresponds to the hypergraph with the fewest number of edges and with all its vertices having degree one. The complete hypergraph $\mathcal{K}_n$ has maximum R_α. Therefore, we have the following theorem.

Theorem 4.6. *Let $\mathcal{H} = (V, E)$ be a k-uniform hypergraph of order n containing no isolated vertices and $\alpha > 0$. Then we have*

$$\frac{n}{k} \leq R_\alpha(\mathcal{H}) \leq \binom{n}{k}^{k\alpha+1} (\frac{k}{n})^{k\alpha},$$

for $n = qk$, and

$$(q-k+r) + (k-r+1)^{\alpha+1} \leq R_\alpha(\mathcal{H}) \leq \binom{n}{k}^{k\alpha+1} (\frac{k}{n})^{k\alpha},$$

for $n = qk + r$ and $0 < r \leq k-1$. The right equality holds if and only if $\mathcal{H}$ is a complete hypergraph, while the left equality holds if and only if $\mathcal{H}$ is composed of $\frac{n}{k}$ disjoint edges for $n = qk$ or is composed of $q-k+r$ disjoint edges and a k-uniform hyperstar of size $k-r+1$ for $n = qk+r$ where $0 < r \leq k-1$.

4.3 Case: -$\frac{1}{k} < \alpha < 0$.

In this case, we only propose an upper bound for R_α.

Lemma 4.7. *[15] Let $k \geq 2$, be an integer number and let $x_1, \cdots, x_k$ be nonnegative real numbers. Then we have*

$$x_1^k + \cdots + x_k^k \pm kx_1 \cdots x_k \geq 0.$$

Theorem 4.8. *Let $\mathcal{H} = (V, E)$ be a k-uniform hypergraph of order n containing no isolated vertices and let $\frac{-1}{k} < \alpha < 0$. Then we have*

$$R_\alpha(\mathcal{H}) \leq \frac{n}{k}\binom{n-1}{k-1}^{k\alpha+1}.$$

Equality holds if and only if $\mathcal{H}$ is a complete hypergraph.

Proof. By Lemma 4.7, for $d_{i_1}^\alpha, \cdots, d_{i_k}^\alpha$, we have $k(d_{i_1} \cdots d_{i_k})^\alpha \leq d_{i_1}^{k\alpha} + \cdots + d_{i_k}^{k\alpha}$. Thus

$$\begin{aligned} R_\alpha(\mathcal{H}) &= \sum_{\{i_1,\cdots,i_k\}\in E} (d_{i_1} \cdots d_{i_k})^\alpha \\ &\leq \frac{1}{k} \sum_{\{i_1,\cdots,i_k\}\in E} (d_{i_1}^{k\alpha} + \cdots + d_{i_k}^{k\alpha}) \\ &= \frac{1}{k}\sum_{v\in V} d_v d_v^{k\alpha} = \frac{1}{k}\sum_{v\in V} d_v^{k\alpha+1} \\ &\leq \frac{n}{k}\binom{n-1}{k-1}^{k\alpha+1}. \end{aligned}$$

It is clear that equality holds if and only if the degree of all vertices is equal to $\binom{n-1}{k-1}$: that is, $\mathcal{H}$ is a complete hypergraph. ■

Theorem 4.9. *Let $\mathcal{H} = (V, E)$ be a k-uniform hypergraph of order n and size m containing no isolated vertices. When $\frac{-1}{k} \leq \alpha < 0$ and $\bar{d}$ denotes the average degree of $\mathcal{H}$, we have*

$$R_\alpha(\mathcal{H}) \leq m\bar{d}^{k\alpha},$$

with equality if and only if $\mathcal{H}$ is composed of regular components.

Proof. By Lemma 4.7 and Jencen's inequality for the Concave function $\phi(x) = x^{k\alpha+1}$, we have the following:

$$\begin{aligned} R_\alpha(\mathcal{H}) &= \sum_{\{i_1,\cdots,i_k\}\in E} (d_{i_1}\cdots d_{i_k})^\alpha \\ &\leq \frac{1}{k} \sum_{\{i_1,\cdots,i_k\}\in E} (d_{i_1}^{k\alpha} + \cdots + d_{i_k}^{k\alpha}) \\ &= \frac{1}{k}\sum_{v\in V} d_v^{k\alpha+1} \leq \frac{n}{k}\bar{d}^{k\alpha+1} \\ &= m\bar{d}^{k\alpha}. \end{aligned}$$

The equality holds if and only if $d_{i_1} = \cdots = d_{i_k}$, for every $e = \{i_1, \cdots, i_k\} \in E$ i.e. all components of $\mathcal{H}$ are regular. ∎

4.4 Case: $\alpha = -\frac{1}{k}$.

This case is very similar to the previous one. By Lemma 4.7, we have following theorem.

Theorem 4.10. *Let $\mathcal{H} = (V, E)$ be a k-uniform hypergraph of order n containing no isolated vertices. Then we have the following:*

$$R_{-\frac{1}{k}}(\mathcal{H}) \leq \frac{n}{k},$$

equality holds if and only if $\mathcal{H}$ is a complete hypergraph or is composed of $\frac{n}{k}$ disjoint edges for $n = qk$.

Proof.

$$\begin{aligned} R_{-\frac{1}{k}}(\mathcal{H}) &= \sum_{\{i_1,\cdots,i_k\}\in E} (d_{i_1}\cdots d_{i_k})^{\frac{-1}{k}} \\ &\leq \frac{1}{k} \sum_{\{i_1,\cdots,i_k\}\in E} (d_{i_1}^{\frac{-k}{k}} + \cdots + d_{i_k}^{\frac{-k}{k}}) \\ &= \frac{1}{k}\sum_{v\in V} d_v d_v^- 1 = \frac{n}{k}. \end{aligned}$$

Equality holds if and only if $d_{i_1} = \cdots = d_{i_k}$, for every $e = \{i_1, \cdots, i_k\} \in E$. The only cases where this occurs are when, $\mathcal{H}$ is a complete hypergraph or is composed of $\frac{n}{k}$ disjoint edges for $n = qk$. ∎

4.5 Case: $-1 < \alpha < -\frac{1}{k}$.

In this case, we propose three upper bounds for R_α.

Theorem 4.11. *Let $\mathcal{H} = (V, E)$ be a k-uniform hypergraph of order n containing no isolated vertices and let $-1 < \alpha < -\frac{1}{k}$. Then we have*

$$R_\alpha(\mathcal{H}) \leq \frac{n}{k},$$

with equality if and only if $\mathcal{H}$ is composed of $\frac{n}{k}$ disjoint edges for $n = qk$.

Proof. It is clear that if $\alpha \leq \beta$ then $R_\alpha(\mathcal{H}) \leq R_\beta(\mathcal{H})$. Therefore, $R_\alpha(\mathcal{H}) \leq R_{-\frac{1}{k}}(\mathcal{H}) \leq \frac{n}{k}$, thus, $R_\alpha(\mathcal{H}) \leq \frac{n}{k}$. It is easy to see that if $n = qk$ and $\mathcal{H}$ is composed of $\frac{n}{k}$ disjoint edges, then $R_\alpha(\mathcal{H}) = \frac{n}{k}$. Conversely, suppose that $\mathcal{H}$ is a hypergraph with $R_\alpha(\mathcal{H}) = \frac{n}{k}$. Then we have

$$R_\alpha(\mathcal{H}) \leq R_{-\frac{1}{k}}(\mathcal{H}), \qquad \text{thus} \qquad R_{-\frac{1}{k}}(\mathcal{H}) = \frac{n}{k}.$$

Therefore, by Theorem 4.10, $\mathcal{H}$ is either a complete hypergraph or is composed of $\frac{n}{k}$ disjoint edges. On the other hand, if $\mathcal{H}$ is a complete hypergraph, then

$$R_\alpha(\mathcal{H}) = \frac{n}{k}\binom{n-1}{k-1}^{k\alpha+1} < \frac{n}{k}.$$

The above inequality holds because $k\alpha + 1 < 0$. Thus, $\mathcal{H}$ is composed of $\frac{n}{k}$ disjoint edges. ■

Theorem 4.12. *Let $\mathcal{H} = (V, E)$ be a k-uniform hypergraph of order n containing no isolated vertices and let $\alpha \leq -\frac{1}{k}$. If δ denotes the minimum degree of $\mathcal{H}$, then we have*

$$R_\alpha(\mathcal{H}) \leq \frac{n}{k}\delta^{k\alpha+1}.$$

Equality holds if and only if $\mathcal{H}$ is regular.

Proof. By Lemma 4.7, we have the following:

$$\begin{aligned} R_\alpha(\mathcal{H}) &= \sum_{\{i_1,\cdots,i_k\}\in E} (d_{i_1}\cdots d_{i_k})^\alpha \\ &\leq \frac{1}{k}\sum_{\{i_1,\cdots,i_k\}\in E} (d_{i_1}^{k\alpha} + \cdots + d_{i_k}^{k\alpha}) \\ &= \frac{1}{k}\sum_{v\in V} d_v^{k\alpha+1} \leq \frac{n}{k}\delta^{k\alpha+1}, \end{aligned}$$

with equality if and only if $\mathcal{H}$ is regular. ■

Theorem 4.13. *Let $\mathcal{H} = (V, E)$ be a k-uniform hypergraph of order n containing no isolated vertices and let $-1 \leq \alpha \leq 0$. If δ and Δ denote the minimum and maximum degrees of $\mathcal{H}$, respectively, then we have*

$$R_\alpha(\mathcal{H}) \geq \frac{n}{k}\Delta^{(k-1)\alpha}\delta^{\alpha+1}.$$

Equality holds if and only if $\mathcal{H}$ is regular.

Proof. By Lemma 4.7, we have

$$\begin{aligned} R_\alpha(\mathcal{H}) &= \sum_{\{i_1,\cdots,i_k\}\in E} (d_{i_1}\cdots d_{i_k})^\alpha \\ &\geq \sum_{\{i_1,\cdots,i_k\}\in E} \Delta^{(k-1)\alpha} d_{i_1}^\alpha = \frac{\Delta^{(k-1)\alpha}}{k}\sum_{v\in V} d_v^{\alpha+1} \\ &\geq \frac{\Delta^{(k-1)\alpha}}{k} n\delta^{\alpha+1}, \end{aligned}$$

with equality if and only if $\mathcal{H}$ is regular. ■

4.6 Case: $\alpha < -1$

This case is similar to case $\alpha = -1$, with a slight difference.

Lemma 4.14. *Let $\alpha < 0$ and let $x_1, \cdots, x_n \geq 2$ be real numbers. Then*

$$\frac{(x_1-1)^\alpha \cdots (x_n-1)^\alpha}{x_1^\alpha \cdots x_n^\alpha} - 1 > \frac{(x_1-1)^\alpha}{x_1^\alpha} - 1 + \cdots + \frac{(x_n-1)^\alpha}{x_n^\alpha} - 1.$$

Proof. The proof is by induction on n. For $n = 2$, we have

$$\begin{aligned}\frac{(x_1-1)^\alpha(x_2-1)^\alpha}{x_1^\alpha x_2^\alpha} - 1 &= \frac{(x_1-1)^\alpha(x_2-1)^\alpha - x_1^\alpha(x_2-1)^\alpha + x_1^\alpha(x_2-1)^\alpha}{x_1^\alpha x_2^\alpha} - 1 \\ &= \frac{x_1^\alpha((x_2-1)^\alpha - x_2^\alpha)}{x_1^\alpha x_2^\alpha} + (x_2-1)^\alpha \frac{((x_1-1)^\alpha - x_1^\alpha)}{x_1^\alpha x_2^\alpha} \\ &> \frac{(x_2-1)^\alpha}{x_2^\alpha} - 1 + \frac{(x_1-1)^\alpha}{x_1^\alpha} - 1.\end{aligned}$$

The last inequality is valid due to $(x_2-1)^\alpha > x_2^\alpha$ for $\alpha < 0$. Now suppose that the result is true for $n-1$, we show that the result holds for n.

$$\begin{aligned}&\frac{(x_1-1)^\alpha \cdots (x_n-1)^\alpha}{x_1^\alpha \cdots x_n^\alpha} - 1 \\ =&\frac{(x_1-1)^\alpha \cdots (x_n-1)^\alpha - x_1^\alpha(x_2-1)^\alpha \cdots (x_n-1)^\alpha + x_1^\alpha(x_2-1)^\alpha \cdots (x_n-1)^\alpha}{x_1^\alpha \cdots x_n^\alpha} - 1 \\ =&\frac{(x_2-1)^\alpha \cdots (x_n-1)^\alpha\Big((x_1-1)^\alpha - x_1^\alpha\Big)}{x_1^\alpha \cdots x_n^\alpha} + \frac{x_1^\alpha\Big((x_2-1)^\alpha \cdots (x_n-1)^\alpha - x_2^\alpha \cdots x_n^\alpha\Big)}{x_1^\alpha \cdots x_n^\alpha} \\ >&\frac{x_2^\alpha \cdots x_n^\alpha\Big((x_1-1)^\alpha - x_1^\alpha\Big)}{x_1^\alpha \cdots x_n^\alpha} + \frac{(x_2-1)^\alpha}{x_2^\alpha} - 1 + \cdots + \frac{(x_n-1)^\alpha}{x_n^\alpha} - 1 \\ >&\frac{(x_1-1)^\alpha}{x_1^\alpha} - 1 + \frac{(x_2-1)^\alpha}{x_2^\alpha} - 1 + \cdots + \frac{(x_n-1)^\alpha}{x_n^\alpha} - 1.\end{aligned}$$

■

Theorem 4.15. *Let $\mathcal{H} = (V, E)$ be a k-uniform hypergraph of order n and let e_0 be an edge with minimum weight w_α in $\mathcal{H}$ such that it cotains at least two vertices of degree greater than 1 and $\alpha < -1$. Then we have*

$$R_\alpha(\mathcal{H} - e_0) > R_\alpha(\mathcal{H}).$$

Proof. By notations in Theorem 4.1 and by Lemma 4.14 we have:

$$\sum_{e \in E^{(t)}} \Big(w_\alpha^*(e) - w_\alpha(e)\Big) \geq \Bigg(\sum_{i_1,\cdots,i_t \in e^*} \beta_{i_1\cdots i_t}^{(t)} \Big(\sum_{j=1}^{t} \frac{1}{d_{i_j} - 1}\Big)\Bigg) w_\alpha(e_0).$$

Therefore

$$
\begin{aligned}
&R_\alpha(\mathcal{H}-e_0)-R_\alpha(\mathcal{H})\\
=&\sum_{e\in E\setminus e_0}\Big(w^*_\alpha(e)-w_\alpha(e)\Big)-w_\alpha(e_0)\\
=&\sum_{e\in E^{(1)}}\Big(w^*_\alpha(e)-w_\alpha(e)\Big)+\cdots+\sum_{e\in E^{(l)}}\Big(w^*_\alpha(e)-w_\alpha(e)\Big)-w_\alpha(e_0)\\
>&\Bigg(\sum_{i=1}^{l}\beta_i^{(1)}\Big(\frac{(d_i-1)^\alpha}{d_i^\alpha}-1\Big)+\cdots+\sum_{i_1,\cdots,i_l\in e^*}\beta^{(l)}_{i_1\cdots i_l}\sum_{j=1}^{l}\Big(\frac{(d_{i_j}-1)^\alpha}{d_{i_j}^\alpha}-1\Big)\Bigg)w_\alpha(e_0)-w_\alpha(e_0)\\
=&\Bigg[\Big(\frac{(d_1-1)^\alpha}{d_1^\alpha}-1\Big)\Big(\beta_1^{(1)}+\sum_{j=1}^{l}\beta^{(2)}_{1j}+\cdots+\sum_{i_1,\cdots,i_{l-1}\in e^*}\beta^{(l)}_{1i_1\cdots i_{l-1}}\Big)\\
&+\cdots+\\
&\Big(\frac{(d_l-1)^\alpha}{d_l^\alpha}-1\Big)\Big(\beta_1^{(1)}+\sum_{j=1}^{l}\beta^{(2)}_{lj}+\cdots+\sum_{i_1,\cdots,i_{l-1}\in e^*}\beta^{(l)}_{li_1\cdots i_{l-1}}\Big)\Bigg]w_\alpha(e_0)-w_\alpha(e_0)\\
=&\Big((d_1-1)\Big(\frac{(d_1-1)^\alpha}{d_1^\alpha}-1\Big)+\cdots+(d_l-1)\Big(\frac{(d_l-1)^\alpha}{d_l^\alpha}-1\Big)\Big)w_\alpha(e_0)-w_\alpha(e_0)\\
=&\Bigg((d_1-1)\Big((1+\frac{1}{d_1-1})^{-\alpha}-1\Big)+\cdots+(d_l-1)\Big((1+\frac{1}{d_l-1})^{-\alpha}-1\Big)\Bigg)w_\alpha(e_0)-w_\alpha(e_0)\\
>&\Big((d_1-1)(1-\alpha\frac{1}{d_1-1}-1)+\cdots+(d_l-1)(1-\alpha\frac{1}{d_l-1}-1)\Big)w_\alpha(e_0)-w_\alpha(e_0)\\
=&(-l\alpha-1)w_\alpha(e_0)>0.
\end{aligned}
$$

■

Theorem 4.16. *Let $\mathcal{H}=(V,E)$ be a k-uniform hypergraph of order n and $\alpha<-1$. Then we have*

$$R_\alpha(\mathcal{H})\leq\frac{n}{k},$$

for $n=qk$ and

$$R_\alpha(\mathcal{H})\leq\lfloor\frac{n}{k}\rfloor-(k-r)+(k-r+1)^{\alpha+1},$$

for $n=qk+r$ and $0<r<k$. With equality if and only if $\mathcal{H}$ is composed of $\frac{n}{k}$ disjoint edges for $n=qk$ or is composed of $\lfloor\frac{n}{k}\rfloor-(k-r)$ disjoint edges and a k-uniform hyperstar of size $k-r+1$ for $n=qk+r$ and $0<r<k$.

Proof. The proof is similar to the case $\alpha=-1$ in Theorem 4.4. ■

Theorem 4.17. *Let $\mathcal{H} = (V, E)$ be a k-uniform hypergraph of order n containing no isolated vertices and let $\alpha < -1$. Then we have*

$$R_\alpha(\mathcal{H}) \geq \frac{n}{k}\binom{n-1}{k-1}^{k\alpha+1},$$

with equality if and only if $\mathcal{H}$ is a complete hypergraph.

Proof. Let $e = \{i_1, i_2, \cdots, i_k\}$ be an arbitrary edge in $\mathcal{H}$. Then

$$\begin{aligned} d_{i_1}^\alpha + \cdots + d_{i_k}^\alpha &= \frac{1}{d_{i_1}^{-\alpha}} + \cdots + \frac{1}{d_{i_k}^{-\alpha}} \\ &= \frac{d_{i_2}^{-\alpha} \cdots d_{i_k}^{-\alpha}}{d_{i_1}^{-\alpha} \cdots d_{i_k}^{-\alpha}} + \cdots + \frac{d_{i_1}^{-\alpha} \cdots d_{i_{k-1}}^{-\alpha}}{d_{i_1}^{-\alpha} \cdots d_{i_k}^{-\alpha}} \\ &\leq \frac{k\binom{n-1}{k-1}^{(k-1)(-\alpha)}}{d_{i_1}^{-\alpha} \cdots d_{i_k}^{-\alpha}}, \end{aligned}$$

therefore

$$\frac{1}{k}\binom{n-1}{k-1}^{(k-1)\alpha}(d_{i_1}^\alpha + \cdots + d_{i_k}^\alpha) \leq (d_{i_1} \cdots d_{i_k})^\alpha. \tag{8}$$

Thus, by (8) we have

$$\begin{aligned} R_\alpha(\mathcal{H}) &= \sum_{\{i_1, \cdots, i_k\} \in E} (d_{i_1} \cdots d_{i_k})^\alpha \\ &\geq \sum_{v \in V} \frac{1}{k} d_v d_v^\alpha \binom{n-1}{k-1}^{(k-1)\alpha} \\ &\geq \frac{1}{k}\binom{n-1}{k-1}^{(k-1)\alpha} \sum_{v \in V} \binom{n-1}{k-1}^{\alpha+1} = \frac{n}{k}\binom{n-1}{k-1}^{k\alpha+1} \\ &= \binom{n}{k}\binom{n-1}{k-1}^{k\alpha+} = R_\alpha(\mathcal{K}_n). \end{aligned}$$

It is clear that equality holds if and only if $\mathcal{H}$ is a complete hypergraph. ■

5 Conclusion

The general Randić index of a graph $G = (V, E)$ is defined as $R_\alpha = \sum_{u,v \in V} (d_u d_v)^\alpha$, where d_u is the degree of vertex u and α is an arbitrary real number. In 1975, the Randić index of a graph was first defined as $R(G) = \sum_{u,v \in V} (d_u d_v)^{-\frac{1}{2}}$, which has been widely used in studying the chemical properties of alkanes. In recent years, the use of hypergraphs in various sciences has attracted much attention from researchers. In this paper, we define the Randić index for a uniform hypergraph and obtain lower and upper bounds for R_α depending for different values of α.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

References

[1] M. Randić, Characterization of molecular branching, *J. Am. Chem. Soc.* **97** (23) (1975) 6609–6615.

[2] M. Atapour, A. Jahanbani and R. Khoeilar, New bounds for the Randić index of graphs, *J. Math.* **2021** (2021) 1–8, https://doi.org/10.1155/2021/9938406.

[3] B. Bollobás and P. Erdös, Graphs of Extremal Weights, *Ars comb.* **50** (1998) p. 225.

[4] G. Caporossi, I. Gutman, P. Hansen and L. Pavlović, Graphs with maximum connectivity index, *Comput. Biol. Chem.* **27** (1) (2003) 85–90, https://doi.org/10.1016/S0097-8485(02)00016-5.

[5] L. H. Clark and J. W. Moon, On the general Randić index for certain families of trees, *Ars Comb.* **54** (2000) 223–235.

[6] C. Delorme, O. Favaron and D. Rautenbach, On the Randić index, *Discrete Math.* **257** (1) (2002) 29–38.

[7] P. Yu, An upper bound for the Randić indices of tree, *J. Math. Studies* **31** (1998) 225–230 (chinese).

[8] G. Arizmendi and O. Arizmendi, Energy of a graph and Randić index, *Linear Algebra Appl.* **609** (2021) 332–338, https://doi.org/10.1016/j.laa.2020.09.025.

[9] Z. Du, A. Jahanbani and S. M. Sheikholeslami, Relationships between Randić index and other topological indices, *Commun. comb. optim.* **6** (1) (2021) 137–154, https://doi.org/10.22049/CCO.2020.26751.1138.

[10] N. Trinajstic, *Chemical Graph Theory*, CRC Press, Boca Raton, 1992.

[11] A. Ali, M. Javaid, M. Matejić, I. Milovanović and E. Milovanović, Some new bounds on the general sum-connectivity index, *Commun. Comb. Optim.* **5** (2) (2020) 97–109, https://doi.org/10.22049/CCO.2019.26618.1125.

[12] X. Li and Y. Yang, Sharp bounds for the general Randić index, *MATCH Commun. Math. Comput. Chem.* **51** (2004) 155–166.

[13] C. Berge, *Hypergraphs, Combinatorics of Finite Sets*, North-Holland, Amsterdam, 1989.

[14] A. Banerjee, On the spectrum of hypergraphs, *Linear Algebra Appl.* **614** (2021) 82–110, https://doi.org/10.1016/j.laa.2020.01.012.

[15] J. Y. Shao , H. Y. Shan and B. F. Wu, Some spectral properties and characterizations of connected odd-bipartite uniform hypergraphs, *Linear Multilinear Algebra* **63** (12) (2015) 2359–2372, https://doi.org/10.1080/03081087.2015.1009061.