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Research Paper

Edge Metric Dimension of Fullerenes

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Abstract

A $(k, 6)$-fullerene graph is a planar 3-connected cubic graph whose faces are k-gons and hexagons. The aim of this paper is to study the edge metric dimension of $(3, 6)$- and $(4, 6)$-fullerene graphs.

1 Introduction

In a *molecular graph* G, *atoms* are represented as vertices and their relationships (*bonds*) as edges with the property that the number of edges of G incident with each vertex is at most four. The number of bonds incident with a given atom u is called the *degree* of u and denoted by $\deg(u)$. The molecular graph G is *connected* if for any two atoms u and v there exists a path between u and v. The *distance* between two atoms u and v is the number of edges in a shortest sequence of vertices from u to v. As usual, we use notation $d(u, v)$ for the distance between two atoms u and v. This graph is 3-connected, if G has at least three atoms and remains connected whenever fewer than three atoms are removed from G. A molecular graph is said *cubic*, if each atom's degree is equal to 3. A molecular graph is called *planar*, if it can be drawn in the plane in such a way that bonds meet only at atoms corresponding to their common ends. A planar 3-connected cubic graph whose faces are only r-gons and hexagons is called an $(r, 6)$-fullerene

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graph. These molecular graphs are topological models of fullerene molecules [1]. In [2], it was proved that 3, 4 and 5 are the only values of k for which a $(k, 6)$-fullerene exists. By Euler's formula, we also know that a $(3, 6)$-fullerene graph has exactly four faces of size 3 and $\frac{n}{2} - 2$ hexagons.

If $S = \{v_1, \ldots, v_k\}$ is an ordered subset of $V(G)$, then the *S-code* of an edge $e \in E(G)$ is the vector $r_G(e|S) = (d_G(v_1, e), \ldots, d_G(v_k, e))$. The set S *distinguishes* edges e and e' if $r_G(e|S) \neq r_G(e'|S)$ and S is an *edge metric generator* for G if each pair of edges of G is distinguished by S. A metric generator of the smallest cardinality is called an *edge metric basis* for G and its cardinality is said to the *edge metric dimension* of G and denoted by $\mathrm{edim}(G)$.

The source for the edge metric dimension is the paper [3]. The complexity of computing the edge metric dimension was investigated in [3]. One can also see [4] for application of edge metric generators in the intelligent transportation system (ITS). We recommend papers [5–9] for more information about mathematical properties of this invariant. In the present work, we are motivated by [10] to compute the edge metric dimension of $(3, 6)$- and $(4, 6)$-fullerene graphs.

2 Main results

Let $F_1[n]$ be $(3, 6)$-fullerene depicted in Figure 1 of order $8n + 4$. In the following, we proceed with labeling shown in this figure.

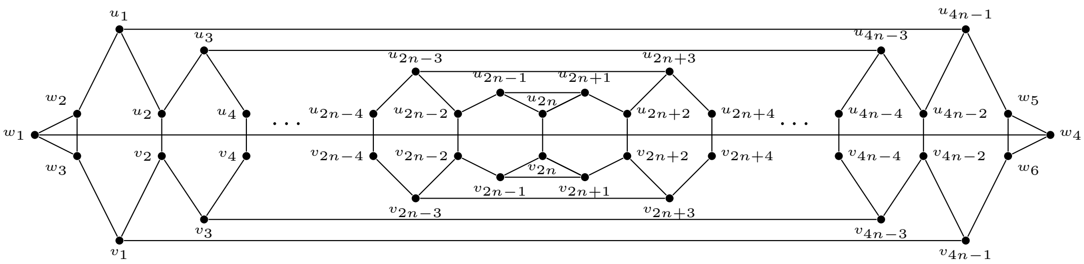

Figure 1: The graph $F_1[n]$.

Theorem 2.1. *The edge metric dimension of fullerene graph $F_1[n]$ is equal to 3.*

Proof. First of all, we prove that $\mathrm{edim}(F_1[n]) \leq 3$. To achieve this aim, let

$$S = \{u_{2n-1}, v_{2n-1}, w_1\} \subset V(F_1[n]).$$

We claim that S is an edge metric generator of $F_1[n]$. Hence, we investigate the S-code of edges of $E(F_1[n])$. Let $\{w_1, w_2, w_3\}$ and $\{w_4, w_5, w_6\}$ be the vertex sets of outer triangles of $F_1[n]$. The S-code of edges of $F_1[n]$ are as follows:

$$r(w_1w_2|S) = (2n-1, 2n, 0), \qquad r(w_2w_3|S) = (2n-1, 2n-1, 1),$$
$$r(w_1w_3|S) = (2n, 2n-1, 0), \qquad r(w_1w_4|S) = (2n, 2n, 0),$$
$$r(w_3v_1|S) = (2n, 2n-2, 1), \qquad r(v_{4n-1}w_6|S) = (2n-2, 2n, 1),$$
$$r(w_4w_5|S) = (2n, 2n+1, 1), \qquad r(w_5w_6|S) = (2n, 2n, 2),$$
$$r(w_4w_6|S) = (2n+1, 2n, 1), \qquad r(w_2u_1|S) = (2n-2, 2n-1, 1),$$
$$r(u_{4n-1}w_5|S) = (2n-1, 2n+1, 2).$$

Also, for $j = i + 1$, we have

$$r(u_iu_j|S) = \begin{cases} (2n-2-i, 2n-i-1, i+1); & \text{if } 1 \le i < 2n-1, \\ (0, 2, 2n); & \text{if } i = 2n-1, \\ (1, 2, 2n+1); & \text{if } i = 2n, \\ (1, 3, 2n+1); & \text{if } i = 2n+1, \\ (i-2n, i-2n+1, 4n-i+1); & \text{if } 2n+1 < i \leqslant 4n-1. \end{cases}$$

In addition, in the case that $j = 4n - i$, we have $r(u_iu_j|S) = (2n - i - 1, 2n - i, i + 1)$, where $i = 2k - 1$ and k is a natural number. Moreover, the S-codes of the lower half of the fullerene graph $F_1[n]$ for $j = i + 1$ are as follows:

$$r(v_iv_j|S) = \begin{cases} (2n-1-i, 2n-i-2, i+1); & \text{if } 1 \le i < 2n-1, \\ (2, 0, 2n); & \text{if } i = 2n-1, \\ (2, 1, 2n+1); & \text{if } i = 2n, \\ (3, 1, 2n+1); & \text{if } i = 2n+1, \\ (i-2n+1, i-2n, 4n-i+1); & \text{if } 2n+1 < i \leqslant 4n-1. \end{cases}$$

Furthermore, when $j = 4n - i$, we have $r(v_iv_j|S) = (2n - i, 2n - i - 1, i + 1)$, where $i = 2k - 1$ and k is a natural number. Besides,

$$r(u_iv_i|S) = \begin{cases} (2n-1-i, 2n-1-i, i+1); & \text{if } 1 \le i \le 2n-1, \\ (1, 1, i+1); & \text{if } i = 2n, \\ (i-2n, i-2n, 4n-i+2); & \text{if } 2n+1 \le i < 4n-1, \end{cases}$$

where $i = 2k$.

Clearly, the above information about the S-codes of $E(F_1[n])$ shows that all edges of $F_1[n]$ have different S-codes and consequently, S is an edge metric generator of $F_1[n]$. Thus $\text{edim}(F_1[n]) \le |S| = 3$.

It remains to prove that $\text{edim}(F_1[n]) \ge 3$. To achieve this aim, let $A = \{w_1, w_2, w_3, w_4, w_5, w_6\}$ be the set of vertices of outer triangles of $F_1[n]$. Assume, to the contrary, that $\text{edim}(F_1[n]) = 2$ and S' is an edge metric generator with $|S'| = 2$. We have the following cases:

Case 1. If both vertices of S' are in the upper half of $F_1[n]$, then the S-codes of pair of edges w_4w_5, w_5w_6 and w_1w_2, w_2w_3 are the same. Thus S' is not an edge metric generator of $F_1[n]$.

Case 2. If both vertices of S' belong to A, then

subcase 2.1. If $S' = \{w_1w_2\}$, then the S'-codes of pair of edges w_4w_5, w_4w_6 are the same.

subcase 2.2. If $S' = \{w_1, w_4\}$, then $r(w_4w_5|S') = r(w_4w_6|S')$ and $r(w_1w_2|S') = r(w_1w_3|S')$.

subcase 2.3. If $S' = \{w_3, w_2\}$, then the S'-codes of pair of edges w_2u_1, w_1w_2 are the same.

subcase 2.4. If $S' = \{w_6, w_2\}$, then $r(w_6w_5|S') = r(v_{4n-1}w_6|S')$.

subcase 2.5. If $S' = \{w_2, w_5\}$, then the S'-code of pair of edges $u_{4n-1}w_5, w_4w_5$ and w_1w_2, w_2w_1 are the same.

Case 3. If one vertex of S' belongs to the upper half of $F_1[n]$ and the other vertex belongs to A, i. e., ($S' = \{u_i, w_j\}$), then we have the following subcases:

subcase 3.1. For $i \leq 2n-1$ and $j = 2$, we have $r(w_1w_2|S') = r(w_2w_3|S')$.

subcase 3.2. If $i \leq 2n-1$ and $j = 5$, then the S'-codes of pair of edges w_4w_5, w_5w_6 are the same.

subcase 3.3. If $i \leq 2n-1$ and $j = 3$, then the S'-codes of pair of edges w_1w_3 and v_1w_3 are the same.

subcase 3.4. For $i \leq 2n-1$ and $j = 6$, we have $r(w_1w_4|S') = r(w_4w_5|S')$.

subcase 3.5. For $i \leq 2n-1$ and $j = 1$, we have $r(w_1w_3|S') = r(w_1w_4|S')$.

subcase 3.6. If $i \leq 2n-1$ and $j = 4$, then the S'-codes of pair of edges w_1w_3 and w_5w_6 are the same.

Case 4. If one vertex of S' is from $\{u_1, u_2, \ldots, x_{4n-1}\}$ and other vertex is from $\{v_1, v_2, \ldots, v_{4n-1}\}$, then we have the following subcases:

subcase 4.1. For $i < j \leqslant 2n-1$, we have $r(v_{2n}v_{2n-1}|S') = r(v_{2n-1}v_{2n+1}|S')$.

subcase 4.2. If $j < i \leqslant 2n-1$, then the representation of pair of edges $v_{4n-2}y_{4n-1}, v_{4n-1}z_6$ are the same.

subcase 4.3. If $j = i \leqslant 2n-1$, then the S'-codes of pair of edges $w_1w_2, u_{4n-1}w_5$ are the same.

subcase 4.4. For $i = j = 1$, we have $r(u_{4n-2}u_{4n-1}|S') = r(w_5u_{4n-1}|S')$.

Therefore, in each case, we reach a contradiction and consequently, there does not exit an edge metric generator S' of size 2 for $F_1[n]$. Thus $\mathrm{edim}(F_1[n]) \geq 3$ which completes the proof. ■

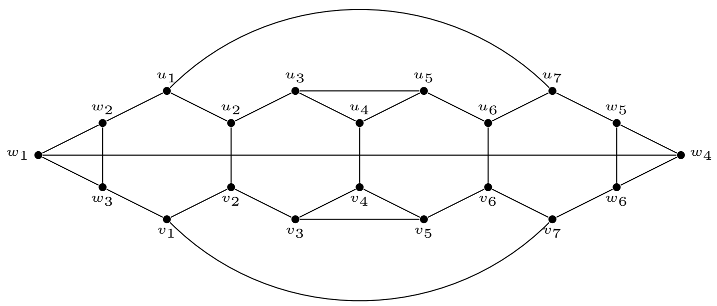

Figure 2: The graph $F_1[2]$.

For more illustration, we implement the proof of Theorem 2.1 on $F_1[2]$. Consider $F_1[2]$ shown in Figure 2. Let $\{z_1, z_2, z_3\}$ and $\{z_4, z_5, z_6\}$ be the vertex sets of outer triangles of $F_1[2]$, and $S = \{x_3, y_3, z_1\} \subset V(F_1[2])$. Now, we prove that S is an edge metric generator of $F_1[n]$. To achieve this, we give the S-codes of $E(F_1[2])$ as follows:

$r(w_1w_2|S) = (1,2,0)$, $r(w_2w_3|S) = (1,1,1)$, $r(w_1w_3|S) = (2,1,0)$, $r(w_1w_4|S) = (2,2,0)$,
$r(w_4w_5|S) = (2,3,1)$, $r(w_5w_6|S) = (2,2,2)$, $r(w_6w_4|S) = (3,2,1)$, $r(w_2u_1|S) = (0,2,1)$,
$r(w_3v_1|S) = (2,0,1)$, $r(u_1u_2|S) = (0,2,2)$, $r(u_2u_3|S) = (1,2,2)$, $r(u_1u_3|S) = (0,3,2)$,
$r(u_2v_2|S) = (1,1,3)$, $r(u_3w_5|S) = (1,3,2)$, $r(v_1v_2|S) = (2,0,2)$, $r(v_2v_3|S) = (2,1,3)$,
$r(v_1v_3|S) = (3,0,2)$, $r(v_3w_6|S) = (3,1,2)$, $r(u_2u_3|S) = (1,2,2)$, $r(u_1u_3|S) = (0,3,2)$.

Thus, all the edges of this graph have different S-codes which implies that S is an edge metric generator of $F_1[2]$.

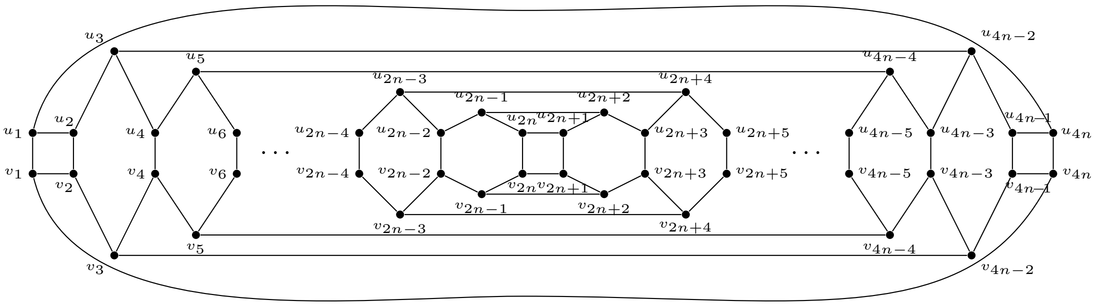

Figure 3: The graph $G_1[n]$.

Theorem 2.2. *The edge metric dimension of* $G_1[n]$*, shown in Figure 3, is equal to* 3 *for* $n \geq 2$.

Proof. For $S = \{x_1, y_1, x_{4n}\} \subset V(G_1[n])$, we need to show that S is an edge metric generator of $G_1[n]$. We first prove that $\dim(G_1[n]) \leq 3$. For this aim, we give the representation of the edges $G_1[n]$ with respect to S.
The representation of edges of the fullerene graph $G_1[n]$ is given below:

$$r(x_1x_{i+1}|S) = \begin{cases} (i-1, i, i); & \text{if } 1 \leq i \leq 2n, \\ (4n-1, 4n-i+1, 4n-i+1); & \text{if } 2n+1 \leq i \leq 4n-1. \end{cases}$$

The S-code of edges of lower half of $G_1[n]$ is given below:

$$r(y_iy_{i+1}|S) = \begin{cases} (i, i-1, i+1); & \text{if } 1 \leq i \leq 2n, \\ (4n-i+1, 4n-i, 4n-i); & \text{if } 2n+1 \leq i \leq 4n-1. \end{cases}$$

Moreover, the S-codes of edges x_iy_i $(i = 1, 2, 4, 6, \ldots, 4n-2, 4n-1, 4n)$ are as follows:

$$r(x_iy_i|S) = \begin{cases} (i-1, i-1, i); & \text{if } 1 \leq i \leq 2n, \\ (4n-i+1, 4n-i+1, 4n-i); & \text{if } 2n+1 \leq i \leq 4n, \end{cases}$$

$$r(x_1x_{4n-i+1}|S) = (i-1, i, i-1).$$

Thus, all edges of $E(G_1[n])$ can be resolved with respect to S and consequently, S is an edge metric generator of $G_1[n]$.

Now we show that $\text{edim}(G_1[n]) \neq 2$. To achieve this aim, we consider the following cases:

Case 1. If both vertices are in the upper half of $G_1[n]$ and the edge metric generator is $S' = \{u_s, u_t\}$, for $1 \leq s \leq t \leq 4n$, then the S'-code of pair of edges u_iu_{i-1} and $v_{i-2}v_{i-1}$, for $2n+1 \leq i \leq 4n-1$ are the same. Thus S' is not an edge metric generator of $G_1[n]$. Therefore, the edge metric generator is not a subset of $\{u_1, u_2, \ldots, u_{4n}\}$.

Case 2. If both vertices are in the lower half of $G_1[n]$ and the edge metric generator is $S' = \{v_s, v_t\}$, for $1 \leq s \leq t \leq 4n$, then the S'-codes of pair of edges v_iv_{i-1} and $u_{i-2}u_{i-1}$, for $2n+1 \leq i \leq 4n-1$ are the same. Thus, S' is not an edge metric generator of $G_1[n]$. Therefore, the edge metric generator is not a subset of $\{v_1, v_2, \ldots, v_{4n}\}$.

Case 3. Assume that one vertex belongs to the set of vertices $\{u_1, u_2, \ldots, u_{4n}\}$ and the other one is in the set of vertices $\{v_1, v_2, \ldots, v_{4n}\}$. Without loss of generality, we may assume that the edge metric generator is $S' = \{u_s, v_t\}$, where $1 \leq s \leq 4n$ and $1 \leq t \leq 4n$. We have the following subcases:

subcase 3.1. If $s = t$, then the S'-codes of pair of edges $u_su_{s+1}, u_{s-1}u_s$ and v_tv_{t-1}, v_tv_{t+1} are the same.

subcase 3.2. If $s < t$, then the S'-codes of pair of $u_{2n}u_{2n+1}$ and $u_{2n+1}u_{2n+2}$ are the same.

subcase 3.3. If $s > t$, then the S'-codes of pair of $v_{2n}v_{2n+1}, v_{2n+1}v_{2n+2}$ are the same.

Thus in every subcases, we get a contradiction.

Based on the above cases, we conclude that there is no edge metric generator S' with $|S'| = 2$, and so $\mathrm{edim}(G_1[n]) \geq 3$. ■

For more illustration, we implement the proof of Theorem 2.1 on $G_1[3]$, shown in Figure 4. Now, let $S = \{u_1, v_1, u_{12}\}$. We show that S is an edge metric generator of $G_1[3]$. To do this, we determine the S-codes of all edges as follows:

$$
\begin{array}{rrr}
r(u_1u_2|S) = (0,1,1) & r(y_1y_2|S) = (1,0,2) & r(u_2u_3|S) = (1,2,2) \\
r(v_2v_3|S) = (2,1,3) & r(u_3u_4|S) = (2,3,3) & r(v_3v_4|S) = (3,2,4) \\
r(u_4u_5|S) = (3,4,4) & r(v_4v_5|S) = (4,3,5) & r(u_5u_6|S) = (4,5,5) \\
r(v_5v_6|S) = (5,4,6) & r(u_6u_7|S) = (5,6,5) & r(v_6v_7|S) = (6,5,6) \\
r(u_7u_8|S) = (5,6,4) & r(v_7v_8|S) = (6,5,5) & r(u_8u_9|S) = (4,5,3) \\
r(v_8v_9|S) = (5,4,4) & r(u_9u_{10}|S) = (3,4,2) & r(v_9v_{10}|S) = (4,3,3) \\
r(u_{10}u_{11}|S) = (2,3,1) & r(v_{10}v_{11}|S) = (3,2,2) & r(u_{11}u_{12}|S) = (1,2,0) \\
r(v_{11}v_{12}|S) = (2,1,1) & r(u_1u_{12}|S) = (0,1,0) & r(v_1v_{12}|S) = (1,0,1) \\
r(u_3u_{10}|S) = (2,3,2) & r(v_3v_{10}|S) = (3,2,2) & r(u_5u_8|S) = (4,5,4) \\
r(v_5v_8|S) = (5,4,5) & r(u_1v_1|S) = (0,0,1) & r(u_2v_2|S) = (1,1,2) \\
r(u_4v_4|S) = (3,3,4) & r(u_6v_6|S) = (5,5,6) & r(u_7v_7|S) = (6,6,5) \\
r(u_9v_9|S) = (4,4,3) & r(u_{11}v_{11}|S) = (2,2,1) & r(u_{12}v_{12}|S) = (1,1,0)
\end{array}
$$

All the edges of this graph have different S-codes. This implies that S is an edge metric generator of $G_1[3]$. Thus $\mathrm{edim}(G_1[3]) \leq 3$. In Theorem 2.2, we showed that $\mathrm{edim}(G_1[n]) \geq 3$. Thus, the edge metric dimension of $G_1[3]$ is equal to 3.

$r(w_1w_2\|S) = (3,4,0)$	$r(w_2w_3\|S) = (3,3,1)$	$r(w_1w_3\|S) = (4,3,0)$
$r(w_1w_4)\|S) = (4,4,0)$	$r(w_4w_5\|S) = (4,5,1)$	$r(w_5w_6)\|S) = (4,4,2)$
$r(w_6w_4)\|S) = (5,4,1)$	$r(w_2u_1\|S) = (2,3,1)$	$r(w_3v_1)\|S) = (3,2,1)$
$r(w_1u_2\|S) = (1,2,2)$	$r(u_2u_3\|S) = (0,2,3)$	$r(u_3u_4\|S) = (0,2,4)$
$r(w_4u_5\|S) = (1,2,5)$	$r(u_5u_6\|S)) = (1,3,5)$	$r(u_6u_7\|S) = (2,3,3)$
$r(w_3u_5\|S) = (0,3,4)$	$r(u_1u_7\|S) = (2,3,2)$	$r(u_2v_2\|S) = (1,1,3)$
$r(w_4v_4\|S) = (1,1,5)$	$r(u_6v_6\|S) = (2,2,4)$	$r(u_7w_5\|S) = (3,4,3)$
$r(v_1v_2\|S) = (2,1,2)$	$r(v_2v_3\|S) = (2,0,3)$	$r(v_3v_4\|S) = (2,0,4)$
$r(v_4v_5\|S) = (2,1,5)$	$r(v_5v_6\|S) = (3,1,4)$	$r(v_6v_7\|S) = (3,2,3)$
$r(v_7w_6\|S) = (4,3,3)$	$r(v_3v_5\|S) = (3,0,4)$	$r(v_1v_7\|S) = (3,2,2)$

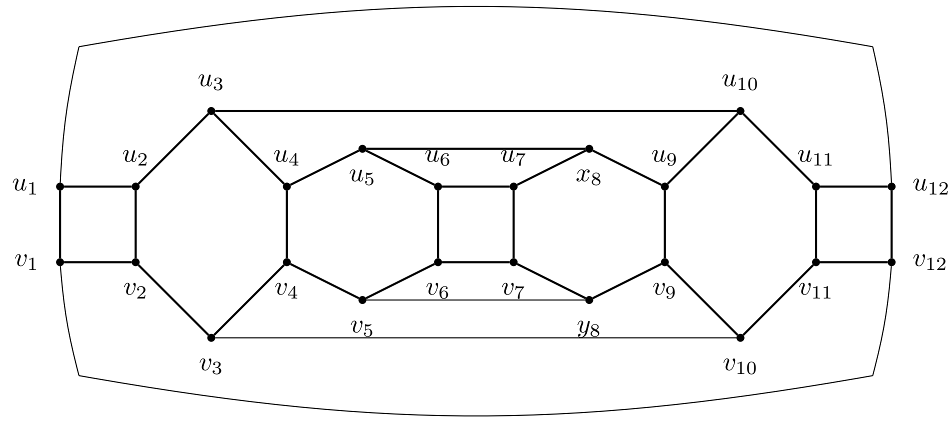

Figure 4: $G_1[3]$

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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