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# *Exponential Growth of Graph Resolvent*

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**ABSTRACT**

The resolvent matrix is a matrix with this property that all of its eigenvalues are outside the spectra of G. In this paper, we study the exponential growth of the resolvent matrix of a graph G. The exponential growth of resolvent energy of graph G was established.



## 1. INTRODUCTION

The resolvent matrix of a given matrix $A$ of finite (or infinite) order is defined as $R = (\zeta\, I - A)^{-1}$ where $\zeta$ is a complex (or real) number and $I$ is the unit matrix. Its well-known that the relationship between the resolvent matrix and the power of $A$ can be represented by Taylor series such as $(\zeta\, I - A)^{-1} = \sum_{k=0}^{\infty} \frac{A^k}{\zeta^{k+1}}$.

In what follows, by $G$ we mean a graph of order $n$, $V(G) = \{v_1, v_2, \ldots, v_n\}$ is the set of vertices and $E(G) = \{ u_1, u, \ldots, u_n\}$ is the set of edges. By $A(G)$, we mean the $(0,1)$ adjacency matrix of $G$. The set of eigenvalues of $A(G)$ is said to be the *spectrum of* $G$. Let

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$\lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G)$, the largest eigenvalue $\lambda_1$ is called *the spectral radius*. Let $M_{m,n}$ be a space of $m \times n$ complex matrices. A *matrix norm* is a positive function $||.||$ defined on $M_{m,n}$ such that: (a) $||A|| = 0$ if and only if $A = 0$; (b) $||cA|| = |c|\,||A||$ for every complex number $c$ and (c) $||A + B|| \leq ||A|| + ||B||$ for every $A, B \in M_{m,n}$. A max-norm is an example of matrix norm defined as $||A||_{max} := \max_{i,j}|a_{ij}|$ for any matrix $A := [a_{i,j}]$.

In this paper, Section 1 devotes to study the resolvent of $A(G)$ and obtain its exponential growth with finite order and normal type in term of matrix norm. In Section 2, we will give a condition under which the norm of the resolvent of the adjacency matrix $A(G)$ grows exponentially. In Section 3, we introduce the exponential growth of the resolvent energy of a graph $G$ and establish its lower bound. In Section 4, computational studies to the resolvent energy of exponential growth were provided.

## 2. EXPONENTIAL GROWTH OF THE RESOLVENT OF $A(G)$

**Theorem 2.1.** Let $G$ be a graph of order $n$ and $A(G)$ be its adjacency matrix. The resolvent of $G$ holds exponential growth with finite order $\gamma$ and normal type $\mu$ such as

$$||(\zeta I - A(G))^{-1}||_{max} \leq Ce^{\mu(|\lambda|-1)^{-\gamma}}, \text{ for } \left|\frac{\lambda}{n}\right| < 1$$

where $C$ is a constant, only if the norm of the power of its adjacency matrix $||A^n(G)||, n \in N$ holds exponential growth with finite order $0 < \beta < 1$ and normal type $\omega > 0$ such as $||A^n(G)||_{max} \leq C\, e^{\omega n^{\beta}}$, where $\gamma = \frac{\beta}{1-\beta}$ *and* $\mu = \frac{(\beta\omega)^{\beta}}{\gamma}$.

To prove theorem, we need to provide brief introduction to the theory of entire function with the following peculiarities:

1. Entire function with variable $\frac{1}{1-x}$;
2. For $x \to 1$, the quantity $1 - x$ is equivalent to $-\ln x = \ln\frac{1}{x} =: t$.

Let $f(x) > 0$ be a function defined for $0 < x < 1$. The phenomena of order with $x \to 1$, can be introduced with scale $e^{\rho(1-x)^{-\alpha}}$. We say that the function $f(x)$ is of finite order (precisely, finite exponential order), if there exists a constant $\alpha$ such that

$$f(x) \leq e^{(1-x)^{-\alpha}} \qquad x_0 < x < 1.$$

In particular, the lower bound of those $\alpha$'s, is called the order of the function $f$ and denote by $\gamma(f)$. The order of $f$ can be obtained by

$$\gamma(f) = -\overline{\lim_{x\to 1}}\frac{\ln\ln f(x)}{\ln(1-x)}.$$

Moreover, let $f$ be entire function of finite order $\gamma(f) > 0$. If there exists a constant $\eta > 0$ such that

$$f(x) \leq e^{\eta(1-x)^{-\alpha}} \qquad x_0 < x < 1.$$

The lower bound of those $\eta$ 's, is called the type of function $f$ and denote by $\rho(f)$. Its well-known that if $\rho(f) > 0$, then the function $f$ holds normal type. The type of function $f$ of finite order $\gamma(f)$ can be obtained by

$$\rho(f) = -\overline{\lim_{x\to 1}} \frac{\ln(f)}{(1-x)^{-\gamma}}.$$

Similarly, one can definite the order and type of any sequence of numbers such as $\phi_n = e^{\omega(n)^\beta}$, where $\beta, \omega > 0$ are the order and the type of $\phi_n$, respectively. For more details about entire function, we refer to [5]. One more object needed to prove Theorem 2.1, is the Legendre transformation. The Legendre transformation is given by (see [6]):

$$f^* = (\sup_x xs - f(x));\ g^* = (\sup_s xs - g(s)).$$

It is well-known that evaluation of the Legendre transformation (in short, LT.) is given by $(f^*)^*(x) = f(x)$.

**Lemma 2.1.** The Legendre transformation of the function:

$$f_\gamma(t) = \begin{cases} \frac{1}{\gamma} t^{-\gamma} & t > 0 \\ +\infty & t \le 0 \end{cases}$$

is

$$g_\beta(s) = \begin{cases} \frac{-1}{\beta}(-s)^\beta & t > 0 \\ +\infty & t > 0 \end{cases} \tag{1}$$

**Proof.** By definition of LT., we have $f^*(s) = \sup_t\ [ts - f(t)] = \sup_{t>0}\left[ts - \frac{1}{\gamma}\ t^{-\gamma}\right]$. Obviously, $f_\gamma^*(s) = +\infty$ for $s > 0$. On the other hand, for $s < 0$, we have

$$t = (-s)^{-1/(\gamma+1)}$$

and

$$f_\gamma^*(s) = (-s)^{-1/(\gamma+1)} s - \frac{1}{\gamma}\left[-s^{-\frac{1}{\gamma+1}}\right]^{-\gamma} = -\frac{1}{\beta}(-s)^\beta,$$

where $\beta = \frac{\gamma}{1+\gamma}$. ■

**Corollary 2.1.** Let $f_\gamma(t)$ be a function defined in Equation (1). Then for $C > 0$, the Legendre transformation of function $f(t) = C\ f_\gamma(t)$ is

$$f^*(s) = C^{1-\beta}\ f_\gamma^*(s) = \begin{cases} -\frac{\frac{C1}{\gamma+1}}{\beta} & s \le 0 \\ +\infty & s > 0 \end{cases}.$$

**Proof.** The proof follows from Lemma 2.1 and definition of the Legendre transformation of a function $f$ multiplied by $C > 0$ such as $[Cf]^*(s) = C\ f^*\left(\frac{s}{C}\right)$. ■

**Theorem 2.2.** Let $\varphi(z) = \sum_0^\infty \varphi_n z^n$ be an analytical function. The function $\varphi(z)$ holds finite order $\gamma > 0$ and normal type $\rho > 0$, only if the sequence of coefficients $\varphi_n$ has finite order $0 < \beta < 1$ and normal type $\omega > 0$, where $\gamma = \frac{\beta}{1-\beta}$ and $\rho = \frac{(\beta\omega)^\beta}{\gamma}$.

**Proof**. Without lose of generality, we assume that for $\gamma, \rho > 0$, the function $\varphi(r)$ satisfies the following inequality

$$M_\varphi(r) \leq e^{\rho\left(\frac{1}{r}\right)^{-\gamma}}.$$

It is well known that the coefficients hold the Cauchy inequality such as $|\varphi_n| \leq \frac{M_{\varphi(r)}}{r^n}$. From the above inequality, we have $\ln|\varphi_n| \leq g_n(t) \coloneqq \rho t^{-\gamma} + nt$. Thus

$$\ln|\varphi_n| \leq \min_t[\rho t^{-\gamma} + nt\,] = \max_t\left[-nt - \frac{\rho\gamma}{\gamma}t^{-\gamma}\right].$$

The quantity $\max_t\left[-nt - \frac{\rho\gamma}{\gamma}t^{-\gamma}\right]$ is the Legendre transformation of function $\frac{\rho\gamma}{\gamma}t^{-\gamma}$ given in Lemma 2.1 at point $s = -n$. Therefore, Corollary 2.1 and Lemma 2.1 imply that

$$\ln|\varphi_n| \leq -\left[\rho\gamma f_\gamma\right]^*(-n) = \frac{\rho\gamma^{\frac{1}{\gamma+1}}}{\beta n^\beta}.$$

This implies that the order of $\varphi_n$ is

$$\lim_n \ln\ln\frac{|\varphi_n|}{\ln n} \leq \beta = \frac{\gamma}{\gamma+1}$$

and the type is

$$\lim_n \frac{\ln|\varphi_n|}{n^\beta} \leq \omega = \frac{(\rho\gamma)^{\frac{1}{\gamma+1}}}{\beta}.$$

Now, the proof is a consequence of Lemma 2.1 for $\left|\frac{\lambda}{n}\right| < 1$. ∎

## 3. EXPONENTIAL GROWTH OF RESOLVENT ENERGY

The *energy of graph* is the sum of absolute values of the eigenvalues of $A(G)$, i.e. $E(G) = \sum_{i=1}^{n}|\lambda_i|$. This graph invariant has important applications in chemical graph theory and had been extensively studied. For more details, we refer to [1,2,3] and [10]. Let's remind that the $k$th spectral moment of graph $G$ is $M_k(G) = \sum_{i=1}^{n}(\lambda_i)^k$, with $M_0 = n, M_1 = 0, M_2 = 2m$ and $M_k = 0$ for all odd values of $k$ if and only if $G$ is bipartite, see [4] for details.

Estrada and Higham in [7] was proposed an invariant of graphs based on Taylor series expansion of spectral moments terms such as $EE(G,c) = \sum_{k=0}^{\infty} c_k M_k(G)$. The series above have been investigated with the following $c_k$. For $c_k = \frac{1}{k!}$, the $EE(G,c)$ is called Estrada index;

I. For $c_k = \frac{1}{(n-1)^k}$, the $EE_r(G,c)$ is called Estrada resolvent index;

II. For $c_k = \frac{1}{n}$, the $ER(G,c)$ is called the resolvent energy.

The index numbered by I is a graph-spectrum-based invariant found by Estrada in [8], which provided 3D geometric characteristics of biologically, while the index numbered by II was established by Estrada and Higham in [7] with noteworthy applications, both in biochemistry and in complex networks.

The eigenvalues of the resolvent matrix of $A(G)$ are $\frac{1}{\zeta-\lambda_i}$, $i = 1,2,\dots,n$. Note that the eigenvalues of resolvent matrix are lies outside the spectrum of $G$. In [9], the resolvent energy of $G$ was defined as $ER(G) = \sum_{i=1}^{n} |(n-\lambda_i)^{-1}|$, Here, we will consider the exponential growth of the resolvent energy of graph $G$. Let $\gamma,\mu > 0$, we say that the resolvent of graph $G$ has exponential growth of finite order $\gamma$ and normal type $\mu$, if it is satisfied the following

$$ER_{\gamma\mu}(G) = \sum_{i=1}^{n} e^{|\mu(n-\lambda_i)^{-\gamma}|}. \qquad (2)$$

**Theorem 3.1.** Let $G$ be an $(n,m)$-graph. Then the exponential resolvent energy holds the following lower bound:

$$ER_{\gamma\mu}(G) \geq \exp[\mu\,(\text{n} \ln n)^{-\gamma}] - \exp[\mu(n \ln 2m)^{-\gamma}],$$

where $\gamma\mu > 0$.

**Proof.** According to Equation (2), we have

$$\begin{aligned} ER_{\gamma\mu}(G) &= \sum_{i=1}^{n} e^{|\mu(n-\lambda_i)^{-\gamma}|}. \\ &= \sum_{k=0}^{\infty}\sum_{i=1}^{n} \left|\frac{(\mu(n-\lambda_i)^{-\gamma})^k}{k!}\right| \\ &= \sum_{k=0}^{\infty}\sum_{i=1}^{n} \left|\frac{[\mu^k n^{-k\gamma}((1-\lambda_i)^{-\gamma})^k]}{k!}\right| \\ &= \sum_{k=0}^{\infty}\sum_{i=1}^{n} \left|\frac{[\mu^k n^{-k\gamma}\left(\left(\ln\frac{n}{\lambda_i}\right)^{-\gamma}\right)^k]}{k!}\right| \\ &= \sum_{k=0}^{\infty}\sum_{i=1}^{n} \left|\frac{[\mu^k n^{-k\gamma}(\ln n - \ln \lambda_i)^{-\gamma k}]}{k!}\right|. \end{aligned}$$

For any positive integer $\eta$ the following inequality is satisfied

$$f(x,y) = (\ln x - \ln y)^{\eta} > (\ln x^2)^{\eta} - (\ln y^2)^{\eta}.$$

Consequently, $ER_{\gamma\mu}(G) \geq \exp[\mu\,(\text{n} \ln n)^{-\gamma}] - \exp[\mu(n \ln 2m)^{-\gamma}]$. ∎

Simply one can check the following:

**Corollary 3.1** Let $G$ be a graph of order $n$. For $\gamma,\mu > 0$, the following statements are hold:

I. $ER(G) < ER_{\gamma\mu}(G)$;

II. If $\lambda_1 > 1$ is the greatest eigenvalue of $A(G)$ and $M_2(G) = 2m$, then we have

$$\gamma > \frac{\ln\ln\lambda_1}{\ln 2m - \ln n} \quad \text{and} \quad \mu > \frac{\ln\ln\lambda_1}{\gamma(\ln 2m - \ln n)}.$$

## 4. COMPUTATIONAL STUDIES ON EXPONENTIAL RESOLVENT ENERGY

For better understanding to the properties of the exponential growth of resolvent energy of graphs, we have undertaken extensive computer-aided studies. The $ER_{\gamma\mu}$-values of all trees and connected unicyclic, and bicyclic graphs up to 15 vertices were computed, and the structure of the extremal members of these classes was established. For $\gamma\mu > 0$, studies are the following observations. Note that the studies here are related to the computational studies in [9] and [11] and the below graphs were plotted in the mentioned references.

1. Among trees of order $n$, the path $P_n$ has smallest and the tree $P_n^*$ second-smallest exponential resolvent energy $ER_{\gamma\mu}$. Among trees of order $n$, the star $S_n$ has greatest and the tree $S_n^*$ second-greatest exponential resolvent energy $ER_{\gamma\mu}$. These graphs are depicted in Figures 1,2.

**Figure 1.** Trees with extremal exponential resolvent energy of type $P_n, P_n^*$.

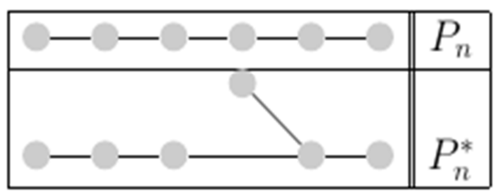
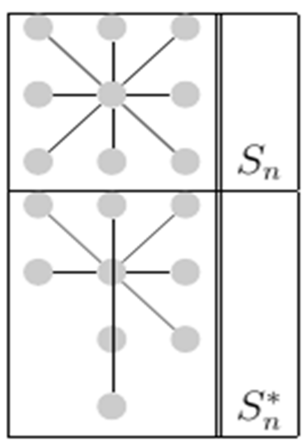



**Figure 2.** Trees with extremal exponential resolvent energy of type $S_n, S_n^*$.




2. Among connected unicyclic graphs of order $n$, $(n \leq 4)$, the cycle $C_n$ has smallest and the graph $C_n^*$ second-smallest exponential resolvent energy $ER_{\gamma\mu}$. Among these graphs

of order $n$, $(n \leq 5)$, the graphs $n$ $X_n$ and $X_n^*$ have, respectively, greatest and second-greatest exponential resolvent energy $ER_{\gamma\mu}$. These graphs are depicted in Figures 3, 4.

**Figure 3.** Unicyclic graphs with extremal exponential resolvent energy of type $C_n, C_n^*$.

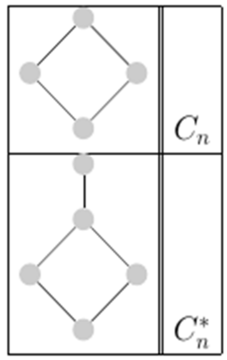
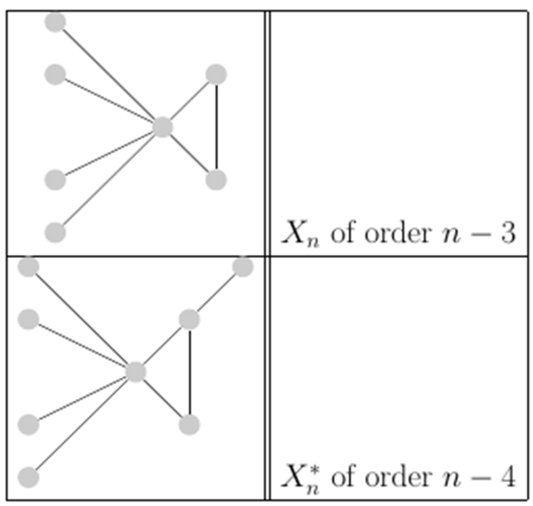
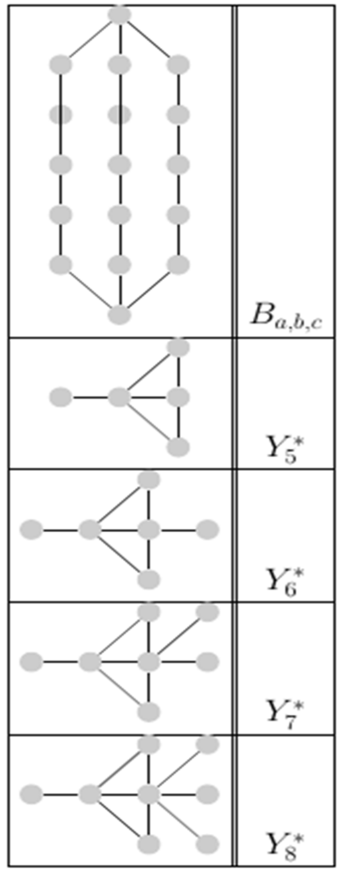



**Figure 4.** Unicyclic graphs with extremal exponential resolvent energy of type $X_n, X_n^*$.




3. Among connected bicyclic graphs of order $n$, those with the smallest exponential resolvent energy $ER_{\gamma\mu}$ are: $B_{p-1,p-1,p}$ $if$ $n = 3p; p \geq 2$, $B_{p-1,p,p}$ $if$ $n = 3p + 1; p \geq 2$ $and$ $B_{p,p,p}$ $if$ $n = 3p + 2; p \geq 1$. The graphs with second-smallest exponential resolvent energy $ER_{\gamma\mu}$ are $B_{p-2,p,p}$ $if$ $n = 3p; p \geq 2$, $B_{p-1,p-1,p+1}$ $if$ $n = 3p + 1; p \geq 2$ $and$ $B_{p-1,p,p+1}$ $if$ $n = 3p + 2; p \geq 1$. Among these graphs of order $n$; $n \geq 5$, the graph $Y_n$ has greatest exponential resolvent energy $ER_{\gamma\mu}$. For $n \geq 9$, the graph $Y_n^*$ has second-greatest exponential resolvent energy $ER_{\gamma\mu}$, where $Y_5^*, Y_6^*, Y_7^*$ and $Y_8^*$ are exceptions. Those graphs are depicted in Figure 5.

**Figure 5.** Bicyclic graphs with extremal exponential resolvent energy.




Comparing the studies above with the investigation in [9] and [11], one can verify that for any of the above considered graphs, we have $ER(G) \subset EE_{r(G)} \subset ER_{1,\mu}(G)$ for $\mu > 0$, for example, $ER(C_6^*) = 1.0464$, $EE_r(C_6^*) = 6.4387$ and $ER_{1,2}(C_6^*) = 7.1498$ while for any $\mu, \gamma > 0$, we have $ER(G) \subset ER_{\gamma,\mu}(G) \subset EE_{r(G)}$, see $ER_{300,2}(C_6^*) = 6$.

## 5. CONCLUSION

In this paper, we studied the exponential growth of the resolvent of graph $G$ in term of max-norm and the relationship between the max-norm of resolvent and power of matrix $A(G)$. Resolvent energy of graphs shows very important application in chemical graph theory and network complex, in Section 4, we applied the exponential growth of the resolvent of graph $G$ to resolvent energy. The exponential growth of resolvent energy shows rapid growth for special case, then the results obtained in [9] and [11]. In this work still there are open questions, like, studying the relationship between the exponential resolvent energy and the Estrada index and Estrada resolvent index.

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