Iranian J. Math. Chem. 12 (2) June (2021) 101 − 109

IRANIAN JOURNAL OF
Mathematical Chemistry
Journal homepage: ijmc.kashanu.ac.ir

Original Scientific Paper

Steiner Wiener Index of Complete m–Ary Trees

MESFIN MASRE LEGESE•

Department of Mathematics, Woldia University, Woldia, Ethiopia

ARTICLE INFO

Article History:

Received: 11 March 2021
Accepted: 28 June 2021
Published online: 30 June 2021
Academic Editor: Boris Furtula

Keywords:

Steiner distance
Wiener index
Steiner Wiener index
Binary trees
Complete m-ary trees

ABSTRACT

Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. For a subset S of $V(G)$, the Steiner distance $d(S)$ of S is the minimum size of a connected subgraph whose vertex set contains S. For an integer k with $2 \leq k \leq n-1$, the k-th Steiner Wiener index of a graph G is defined as

$$SW_k(G) = \sum_{\substack{S \subseteq V(G) \\ |S|=k}} d(S)$$

In this paper, we present exact values of the k-th Steiner Wiener index of complete m-ary trees by using inclusion-exclusion principle for various values of k.

1. INTRODUCTION

Molecular descriptors called topological indices are graph invariants that play a significant role in chemistry, materials science, pharmaceutical sciences and engineering, since they can be correlated with a large number of physio-chemical properties of molecules. Topological indices are used in the process of correlating the chemical structures with various characteristics such as boiling points and molar heats of formation. Graph theory is used to characterize these chemical structures. Binary and m-ary trees have extensive applications in chemistry and computer science, since these trees can represent chemical structures and various useful networks. We consider connected graphs without loops and multiple edges. Let $V(G)$ be the vertex set and let $E(G)$ be the edge set of a graph G. The

•Corresponding Author: mesfinmas@wldu.edu.et
DOI: 10.22052/IJMC.2021.242136.1552

distance $d(u,v)$ between two vertices $u,v \in V(G)$ is the number of edges in a shortest path connecting them.

The Wiener index W of a connected graph G, introduced by wiener in 1947, is defined as

$$W(G) = 1/2 \sum_{u,v\in V(G)} d(u,v),$$

where $d(u,v)$ is the distance between vertices u and v of G. The Wiener index is an important distance-based graph invariant. It was proposed by Harold Wiener [12] in 1947. He found that there exist correlations between the boiling points of paraffins and their molecular structure. The study of the Wiener index in mathematics dates back to the 1970s [2]. The Wiener index obtained wide attention and numerous results have been worked out, see the surveys [3, 4, 5, 13], the recent papers [1, 6, 7, 8] and the references cited therein.

The Steiner distance in a graph, introduced by Chartrand et al. in 1989, is a natural generalization of the concept of classical graph distance. For a connected graph G of order at least 2 and $S \subseteq V(G)$, the Steiner distance $d(S)$ of the vertices of S is the minimum size of a connected subgraph whose vertex set is S. The k-th Steiner Wiener index of a graph G is defined by

$$\mathrm{SW_k(G)} = \sum_{\substack{S\subseteq V(G)\\ |S|=k}} d(S).$$

In this definition, we have $SW_1(G) = 0$ and $SW_n(G) = n-1$. So we consider $2 \le k \le n-1$. We denote the number of $S \subseteq V(G)$ such that $|S| = k$ and $d(S) = t$ by $n_k(G,t)$. Based on this notation, we can define the k-th Steiner Wiener index of a graph G of order n alternatively by

$$\mathrm{SW_k(G)} = \sum_{t=k-1}^{n-1} t n_k(G,t).$$

For some recent investigations on Steiner Wiener index see [9, 10].

A tree T is a connected graph containing no cycles. A leaf is a vertex of T of degree 1 and all the other vertices will be called internal vertices. In a rooted tree, the level of a vertex v is its distance from the root vertex. The height of a rooted tree is the length of a longest path from the root. For $m \ge 2$, an m-ary tree is a rooted tree in which every vertex has at most m children A complete m-ary tree is an m-ary tree in which every internal vertex has exactly m children and all leaves have the same level. The complete m-ary tree of height h will be denoted by $T_{h,m}$ and its root vertex by v_0. $T_{h,2}$ is a complete binary tree of height h. Wiener index of complete m-ary tree is found in [11]. Let $N_i(v)$ be the set of vertices at distance i from v in G.

Let $A = V(T_{h,m}) - N_h(v_0)$, $x = |A| = \frac{m^h-m}{m-1}$ and $n = |V(T_{h,m})| = \frac{m^{h+1}-1}{m-1} = x + m^h + 1$. For a vertex $v \in A$, we denote $N^c(v)$ to be the set of children of v and $N^c[v] = N^c(v) \cup \{v\}$.

2. PRELIMINARY RESULTS

For positive integers a and b with $1 \le b \le a-1$, we have $\binom{a}{b} = \binom{a-1}{b} + \binom{a-1}{b-1}$. The Pascal's and Vandermonde's identity are stated as,

Lemma 2.1. For positive integers a, b and t, we have

$$\sum_{p=0}^{t} \binom{a}{p}\binom{b}{t-p} = \binom{a+b}{t}.$$

Lemma 2.2. For positive integers a, b, c and t with $t \le c$, we have

$$\sum_{p=0}^{c} \binom{a}{p}\binom{b}{t-p} = \binom{a+b}{t} - \binom{c}{t}.$$

From the Vandermonde's identity we get the following lemmas which help us to get our main results.

Lemma 2.3. If a, b and t are positive integers with $1 \le b \le a-1$, then

$$\sum_{p=0}^{t} p\binom{a}{p}\binom{b}{t-p} = a\binom{a+b-1}{t-1}.$$

Lemma 2.4. If a, b and t are positive integers with $1 \le b \le a-1$, then

$$\sum_{p=0}^{t} p^2\binom{a}{p}\binom{b}{t-p} = a\binom{a+b-2}{t-1} + a^2\binom{a+b-2}{t-2}.$$

3. MAIN RESULTS

In this section, we will present our main results.

Theorem 3.1. For $h \ge 2$ and $1 \le k \le m$ or $h = 1$ and $1 \le k < m$, we have

$$SW_{n-k}(T_{h,m}) = (n-1)\binom{n}{k} - m^h\binom{n-1}{k-1}.$$

Proof. Let $S \subseteq V(T_{h,m})$ such that $|S| = n-k$. We have two cases to be considered.

Case 1: $v_0 \in \bar{S}$. Let $|\bar{S} \cap N_h(v_0)| = r$. Then we have $0 \le r \le k-1$, $|\bar{S} \cap A| = k-r-1$ and $d(S) = n-r-1$. There are $\binom{m^h}{r}\binom{x}{k-r-1}$ such vertices. This case contributes to $SW_{n-k}(T_{h,m})$ by

$$\sum_{r=0}^{k-1}\binom{m^h}{r}\binom{x}{k-r-1}(n-r-1) = (n-1)\sum_{r=0}^{k-1}\binom{m^h}{r}\binom{x}{k-r-1} - \sum_{r=0}^{k-1} r\binom{m^h}{r}\binom{x}{k-r-1} \quad (1)$$

By Lemma 2.1, we have

$$\sum_{r=0}^{k-1}\binom{m^h}{r}\binom{x}{k-r-1} = \binom{m^h+x}{k-1} \quad (2)$$

By Lemma 2.3, we have

$$\sum_{r=0}^{k-1} r\binom{m^h}{r}\binom{x}{k-r-1} = m^h\binom{m^h+x-1}{k-2} \tag{3}$$

From Equations (1), (2) and (3), we have

$$\begin{aligned}\sum_{r=0}^{k-1}\binom{m^h}{r}\binom{x}{k-r-1}(n-r-1) &= (n-1)\sum_{r=0}^{k-1}\binom{m^h}{r}\binom{x}{k-r-1} - \sum_{r=0}^{k-1} r\binom{m^h}{r}\binom{x}{k-r-1}\\ &= (n-1)\binom{m^h+x}{k-1} - m^h\binom{m^h+x-1}{k-2}\\ &= (n-1)\binom{n-1}{k-1} - m^h\binom{n-2}{k-2}.\end{aligned}$$

Case 2: $v_0 \notin \bar{S}$. Let $|\bar{S}\cap N_h(v_{0})| = r$. Then $0 \le r \le k$, $|\bar{S}\cap A| = k-r$ and $d(S) = n-r-1$. There are $\binom{m^h}{r}\binom{x}{k-r}$ such vertices. This case contributes to $SW_{n-k}(T_{h,m})$ by $\sum_{r=0}^{k}\binom{m^h}{r}\binom{x}{k-r}(n-r-1)$. Then by Lemmas 2.1 and 2.3, we have

$$\begin{aligned}\sum_{r=0}^{k}\binom{m^h}{r}\binom{x}{k-r}(n-r-1) &= (n-1)\sum_{r=0}^{k}\binom{m^h}{r}\binom{x}{k-r} - \sum_{r=0}^{k} r\binom{m^h}{r}\binom{x}{k-r}\\ &= (n-1)\binom{m^h+x}{k} - m^h\binom{m^h+x-1}{k-1}\\ &= (n-1)\binom{n-1}{k} - m^h\binom{n-2}{k-1}.\end{aligned}$$

From Case 1 and Case 2, we have

$$\begin{aligned}SW_{n-k}(T_{h,m}) &= \sum_{r=0}^{k-1}\binom{m^h}{r}\binom{x}{k-r-1}(n-r-1) + \sum_{r=0}^{k}\binom{m^h}{r}\binom{x}{k-r}(n-r-1)\\ &= (n-1)\binom{n-1}{k-1} - m^h\binom{n-2}{k-2} + (n-1)\binom{n-1}{k} - m^h\binom{n-2}{k-1}\\ &= (n-1)\left[\binom{n-1}{k-1} + \binom{n-1}{k}\right] - m^h\left[\binom{n-2}{k-2} + \binom{n-2}{k-1}\right]\\ &= (n-1)\binom{n}{k} - m^h\binom{n-1}{k-1}.\end{aligned}$$

This completes the proof. □

Theorem 3.2. Let $1 \le a \le m$ be a fixed integer and $k = am + b$ for some b, $1 \le b \le m$. Then for $h \ge 2$ or $ab > 1$, we have

$$SW_{n-k}(T_{h,m}) = (n-1)\binom{n}{k} - m^h\binom{n-1}{k-1} - m^{h-1}\binom{n-m-1}{k-m-1}.$$

Proof. For $0 \le r \le k$, let $U_{r,k} = \{T \subseteq V(T_{h,m}) \mid |T| = k \text{ and } |T\cap N_h(v_0)| = r\}$. Then

$$|U_{r,k}| = \binom{m^h}{r}\binom{x}{k-r} \tag{4}$$

Let $S \subseteq V(T_{h,m})$ such that $|S| = n-k$. Then $\bar{S} \in U_{r,k}$ for some r. Let $N_{h-1}(v_0) = \{u_1, u_2, \ldots, u_{m^{h-1}}\}$. For $0 \le i \le m^{h-1}$, let $B_i = \{T \in U_{r,k} | N^c[u_i] \subseteq T$. Then $|B_i| = \binom{x-1}{k-r-1}\binom{m^h-m}{r-m}$ and $|\bigcap_{j=1}^{t} B_{i_j}| = \binom{m^{h-1}}{t}\binom{x-t}{k-r-t}\binom{m^h-tm}{r-tm}$ for all t, $1 \le t \le m^{h-1}$. By inclusion-exclusion principle, we have

$$\begin{aligned}|\bigcup_{1\le j\le t} B_{i_j}| &= \sum_{1\le j\le t}\left|B_{i_j}\right| - \sum_{1\le s<z\le t}\left|B_{i_s}\cap B_{i_z}\right|\\ &\quad + \sum_{1\le d<s<z\le t}\left|B_{i_d}\cap B_{i_s}\cap B_{i_z}\right| + \ldots + (-1)^{t+1}|\bigcap_{1\le j\le t} B_{i_j}|\end{aligned}$$

$$= \binom{m^{h-1}}{1}\binom{x-1}{k-r-1}\binom{m^h-m}{r-m} - \binom{m^{h-1}}{2}\binom{x-2}{k-r-2}\binom{m^h-2m}{r-2m}$$
$$+ \binom{m^{h-1}}{3}\binom{x-3}{k-r-3}\binom{m^h-3m}{r-3m} - \cdots + (-1)^{h+1}\binom{m^{h-1}}{t}\binom{x-t}{k-r-t}\binom{m^h-tm}{r-tm}$$
$$= \sum_{p=1}^{t}(-1)^{p+1}\binom{m^{h-1}}{p}\binom{x-p}{k-r-p}\binom{m^h-pm}{r-pm}.$$

Thus for any $t, 1 \le t \le m^{h-1}$, we have

$$|\cup_{1\le j\le t} B_{i_j}| = \sum_{p=1}^{t}(-1)^{p+1}\binom{m^{h-1}}{p}\binom{x-p}{k-r-p}\binom{m^h-pm}{r-pm} \quad (5)$$

We have two cases to be considered.

Case 1: $v_0 \notin \bar{S}$. Then $|\bar{S} \cap A| = k - r$. If $0 \le r \le m-1$ or $r = k$, then $d(S) = n - r - 1$ and $n_{n-k}(T_{h,m}, n-r-1) = \binom{m^h}{r}\binom{x}{k-r}$. Suppose $m \le r \le k-1$ (that is, $m \le r \le am + b - 1$). If $N^c[u_i] \nsubseteq \bar{S}$ for all $i, 1 \le i \le m^{h-1}$, then $d(S) = n - r - 1$ and $n_{n-k}(T_{h,m}, n-r-1) = |U_{r,k}| - |\cup_{1\le j\le a} B_{i_j}|$. Then using Equations (4) and (5), we have

$$n_{n-k}(T_{h,m}, n-r-1) = \binom{m^h}{r}\binom{x}{k-r} - \sum_{p=1}^{a}(-1)^{p+1}\binom{m^{h-1}}{p}\binom{x-p}{k-r-p}\binom{m^h-pm}{r-pm}.$$

For $1 \le t \le a$, if there exists $\{i_1, i_2, \ldots, i_t\} \subseteq \{1, 2, \ldots, m^{h-1}\}$ such that $\cap_{j=1}^{t} B_{i_j} \subseteq \bar{S}$ but $B_z \nsubseteq \bar{S}$ for all $z \in \{1, 2, \ldots, m^{h-1}\} - \{i_1, i_2, \ldots, i_t\}$, then $d(S) = n - r - 1$ and

$$n_{n-k}(T_{h,m}, n-r-1) = \left|\cap_{j=1}^{t} B_{i_j}\right| - \binom{t+1}{t}\left|\cap_{j=1}^{t+1} B_{i_j}\right| + \binom{t+2}{t}\left|\cap_{j=1}^{t+2} B_{i_j}\right|$$
$$- \cdots + (-1)^{a-t}\binom{a}{t}\left|\cap_{j=1}^{a} B_{i_j}\right|$$
$$= \binom{m^{h-1}}{t}\binom{x-t}{k-r-t}\binom{m^h-tm}{r-tm}$$
$$-\binom{t+1}{t}\binom{m^{h-1}}{t+1}\binom{x-(t+1)}{k-r-(t+1)}\binom{m^h-(t+1)m}{r-(t+1)m}$$
$$+\binom{t+2}{t}\binom{m^{h-1}}{t+2}\binom{x-(t+2)}{k-r-(t+2)}\binom{m^h-(t+2)m}{r-(t+2)m}$$
$$- \cdots + (-1)^{a-t}\binom{a}{t}\binom{m^{h-1}}{a}\binom{x-a}{k-r-a}\binom{m^h-am}{r-am}$$
$$= \sum_{p=0}^{a-t}(-1)^{p}\binom{t+p}{t}\binom{m^{h-1}}{t+p}\binom{x-(t+p)}{k-r-(t+p)}\binom{m^h-(t+p)m}{r-(t+p)m}.$$

This implies that, the first case contributes to $SW_{n-k}(T_{h,m})$ by

$$\sum_{\substack{S\subseteq V(T_{h,m})\\ v_0\in S, |S|=n-k}} d(S) = \sum_{r=0}^{k}\sum_{q=n-r-a-1}^{n-r-1} q\, n_{n-k}(T_{h,m}, q)$$
$$= \sum_{r=0}^{m-1}(n-r-1)n_{n-k}(T_{h,m}, n-r-1) + (n-k-1)n_{n-k}(T_{h,m}, n-k-1)$$
$$+ \sum_{\substack{r=m\\ u\in N_{h-1}(v_0)\\ N^c[u]\subseteq \bar{S}}}^{k-1} \sum_{q=n-r-a-1}^{n-r-1} q\, n_{n-k}(T_{h,m}, q)$$

$$+\sum_{\substack{r=m\\ u\in N_{h-1}(v_0)\\ N^c[u]\not\subseteq \bar{S}}}^{k-1} \sum_{q=n-r-a-1}^{n-r-1} q n_{n-k}(T_{h,m}, q)$$

$$= \sum_{r=0}^{m-1} \binom{m^h}{r}\binom{x}{k-r}(n-r-1) + \binom{m^h}{k}(n-k-1)$$

$$+ \sum_{r=m}^{k-1}\sum_{t=1}^{a}\sum_{p=0}^{a-t}(-1)^p\binom{t+p}{t}\binom{m^{h-1}}{t+p}\binom{x-(t+p)}{k-r-(t+p)}\binom{m^h-(t+p)m}{r-(t+p)m}(n-r-t-1)$$

$$+ \sum_{r=m}^{k-1}\binom{m^h}{r}\binom{x}{k-r}(n-r-1)$$

$$+\sum_{r=m}^{k-1}\sum_{z=1}^{a}(-1)^z\binom{m^{h-1}}{z}\binom{x-z}{k-r-z}\binom{m^h-zm}{r-zm}(n-r-1)$$

$$= \sum_{r=0}^{k}\binom{m^h}{r}\binom{x}{k-r}(n-r-1)$$

$$+\sum_{r=m}^{k-1}\sum_{z=1}^{a}(-1)^z\binom{m^{h-1}}{z}\binom{x-z}{k-r-z}\binom{m^h-zm}{r-zm}(n-r-1)$$

$$+ \sum_{r=m}^{k-1}\sum_{t=1}^{a}\sum_{p=0}^{a-t}(-1)^p\binom{t+p}{t}\binom{m^{h-1}}{t+p}\binom{x-(t+p)}{k-r-(t+p)}\binom{m^h-(t+p)m}{r-(t+p)m}(n-r-t-1)$$

To simplify this expression, first let us simplify sum in the third line.

$$\sum_{t=1}^{a}\sum_{p=0}^{a-t}(-1)^p\binom{t+p}{t}\binom{m^{h-1}}{t+p}\binom{x-(t+p)}{k-r-(t+p)}\binom{m^h-(t+p)m}{r-(t+p)m}(n-r-t-1)$$

$$= \binom{1}{1}\binom{m^{h-1}}{1}\binom{x-1}{k-r-1}\binom{m^h-m}{r-m}(n-r-2)$$

$$-\binom{2}{1}\binom{m^{h-1}}{2}\binom{x-2}{k-r-2}\binom{m^h-2m}{r-2m}(n-r-2)$$

$$+\binom{2}{2}\binom{m^{h-1}}{2}\binom{x-2}{k-r-2}\binom{m^h-2m}{r-2m}(n-r-3)$$

$$+\binom{3}{1}\binom{m^{h-1}}{3}\binom{x-3}{k-r-3}\binom{m^h-3m}{r-3m}(n-r-2)$$

$$-\binom{3}{2}\binom{m^{h-1}}{3}\binom{x-3}{k-r-3}\binom{m^h-3m}{r-3m}(n-r-3)$$

$$+\binom{3}{3}\binom{m^{h-1}}{3}\binom{x-3}{k-r-3}\binom{m^h-3m}{r-3m}(n-r-4)$$

$$+\cdots+(-1)^{a-1}\binom{a}{1}\binom{m^{h-1}}{a}\binom{x-a}{k-r-a}\binom{m^h-am}{r-am}(n-r-2)$$

$$+\cdots+\binom{a}{a}\binom{m^{h-1}}{a}\binom{x-a}{k-r-a}\binom{m^h-am}{r-am}(n-r-a-1).$$

For $2 \le z \le a$, the z^{th} row of this equation is

$$\sum_{i=1}^{z}(-1)^{z-i}\binom{z}{i}\binom{m^{h-1}}{z}\binom{x-z}{k-r-z}\binom{m^h-zm}{r-zm}(n-r-i-1)$$

$$= (-1)^z\binom{m^{h-1}}{z}\binom{x-z}{k-r-z}\binom{m^h-zm}{r-zm}\sum_{i=1}^{z}(-1)^i\binom{z}{i}(n-r-i-1)$$

$$= (-1)^z\binom{m^{h-1}}{z}\binom{x-z}{k-r-z}\binom{m^h-zm}{r-zm}\left[(n-r-1)\sum_{i=1}^{z}(-1)^i\binom{z}{i} - \sum_{i=1}^{z}(-1)^i i\binom{z}{i}\right]$$

$$= (-1)^z\binom{m^{h-1}}{z}\binom{x-z}{k-r-z}\binom{m^h-zm}{r-zm}(n-r-1)(-1)$$

$$= (-1)^{z+1}\binom{m^{h-1}}{z}\binom{x-z}{k-r-z}\binom{m^h-zm}{r-zm}(n-r-1).$$

Therefore,

$$\sum_{\substack{S\subseteq V(T_{h,m})\\ v_0\in S,|S|=n-k}} d(S)$$

$$= \sum_{r=0}^{k}\binom{m^h}{r}\binom{x}{k-r}(n-r-1) + \sum_{r=m}^{k-1} m^{h-1}\binom{x-1}{k-r-1}\binom{m^h-m}{r-m}(n-r-2)$$

$$+ \sum_{r=m}^{k-1}\sum_{z=2}^{a}(-1)^{z+1}\binom{m^{h-1}}{z}\binom{x-z}{k-r-z}\binom{m^h-zm}{r-zm}(n-r-1)$$

$$+ \sum_{r=m}^{k-1}\sum_{z=1}^{a}(-1)^{z}\binom{m^{h-1}}{z}\binom{x-z}{k-r-z}\binom{m^h-zm}{r-zm}(n-r-1)$$

$$= \sum_{r=0}^{k}\binom{m^h}{r}\binom{x}{k-r}(n-r-1)$$

$$+ \sum_{r=m}^{k-1} m^{h-1}\binom{x-1}{k-r-1}\binom{m^h-m}{r-m}(n-r-2-(n-r-1))$$

$$+ \sum_{r=m}^{k-1}\sum_{z=2}^{a}(-1)^{z}\binom{m^{h-1}}{z}\binom{x-z}{k-r-z}\binom{m^h-zm}{r-zm}(n-r-1-(n-r-1))$$

$$= \sum_{r=0}^{k}\binom{m^h}{r}\binom{x}{k-r}(n-r-1) - \sum_{r=m}^{k-1} m^{h-1}\binom{x-1}{k-r-1}\binom{m^h-m}{r-m}$$

$$= (n-1)\sum_{r=0}^{k}\binom{m^h}{r}\binom{x}{k-r} - \sum_{r=0}^{k} r\binom{m^h}{r}\binom{x}{k-r}$$

$$-m^{h-1}\sum_{p=0}^{k-m-1}\binom{x-1}{k-m-1-p}\binom{m^h-m}{p}$$

$$= (n-1)\binom{x+m^h}{k} - m^h\binom{x+m^h-1}{k-1} - m^{h-1}\binom{x+m^h-m-1}{k-m-1}$$

$$= (n-1)\binom{n-1}{k} - m^h\binom{n-2}{k-1} - m^{h-1}\binom{n-m-2}{k-m-1}.$$

This implies that,

$$\sum_{\substack{S\subseteq V(T_{h,m})\\ v_0\in S,|S|=n-k}} d(S) = (n-1)\binom{n-1}{k} - m^h\binom{n-2}{k-1} - m^{h-1}\binom{n-m-2}{k-m-1} \quad (6)$$

Case 2: $v_0 \in \bar{S}$. Then $|\bar{S}\cap A| = k-r-1$. Similar to the first case, the second case also contributes to $SW_{n-k}(T_{h,m})$ by

$$\sum_{\substack{S\subseteq V(T_{h,m})\\ v_0\notin S,|S|=n-k}} d(S) = (n-1)\binom{n-1}{k-1} - m^h\binom{n-2}{k-2} - m^{h-1}\binom{n-m-2}{k-m-2} \quad (7)$$

From Equations (6) and (7), we have

$$SW_{n-k}(T_{h,m}) = \sum_{\substack{S\subseteq V(T_{h,m})\\ v_0\in S,|S|=n-k}} d(S) + \sum_{\substack{S\subseteq V(T_{h,m})\\ v_0\notin S,|S|=n-k}} d(S)$$

$$= (n-1)\binom{n-1}{k} - m^h\binom{n-2}{k-1} - m^{h-1}\binom{n-m-2}{k-m-1}$$

$$+(n-1)\binom{n-1}{k-1} - m^h\binom{n-2}{k-2} - m^{h-1}\binom{n-m-2}{k-m-2}$$

$$= (n-1)\left[\binom{n-1}{k} + \binom{n-1}{k-1}\right] - m^h\left[\binom{n-2}{k-1} + \binom{n-2}{k-2}\right]$$

$$-m^{h-1}\left[\binom{n-m-2}{k-m-1} + \binom{n-m-2}{k-m-2}\right]$$

$$= (n-1)\binom{n}{k} - m^h\binom{n-1}{k-1} - m^{h-1}\binom{n-m-1}{k-m-1}.$$

This completes the proof. □

4. CONCLUDING REMARKS

In this paper, we presented exact values of the k-th Steiner Wiener index of complete m-ary trees by using inclusion-exclusion principle for various values of k.

REFERENCES

1. C. M. da Fonseca, M. Ghebleh, A. Kanso and D. Stevanović, Counterexamples to a conjecture on Wiener index of common neighborhood graphs, *MATCH Commun. Math. Comput. Chem.* **72** (2014) 333−338.
2. R. C. Entringer, D. E. Jackson and D. A. Snyder, Distance in graphs, *Czechoslovak Math. J.* **26** (1976) 283−296.
3. A. Dobrynin, R. Entringer and I. Gutman, Wiener index of trees: theory and application, *Acta Appl. Math*. **66** (2001) 211−249.
4. M. Ghorbani, X. Li, H. R. Maimani, Y. Mao, S. Rahmani and M. Rajabi-Parsa, Steiner (revised) Szeged index of graphs, *MATCH Commun. Math. Comput. Chem.* **82** (2019) 733−742.
5. I. Gutman, S. Klavžar and B. Mohar, Fifty years of the Wiener index, *MATCH Commun. Math. Comput. Chem.* **35** (1997) 1−159.
6. I. Gutman and O. E. Polansky, Mathematical Concepts in Organic Chemistry, Springer, Berlin-Heidelberg-New York-Tokyo, 1986.
7. Y. L. Jin and X. D. Zhang, On two conjectures of the Wiener index, *MATCH Commun. Math. Comput. Chem*. **70** (2013) 583−589.
8. S. Klavžar and M. J. Nadjafi-Arani, Wiener index in weighted graphs via unification of θ^*- classes, *European J. Combin*. **36** (2014) 71−76.
9. M. Knor and R. Škrekovski, Wiener index of generalized 4-stars and of their quadratic line graphs, *Australas. J. Combin*. **58** (2014) 119−126.
10. X. Li and M. Zhang, Results on two kinds of Steiner distance-based indices for some classes of graphs, *MATCH Commun. Math. Comput. Chem.* **84** (2020) 567−578.
11. Y. Mao, Z. Wang and I. Gutman, Steiner Wiener index of graph products, *Trans. Comb*. **5** (2016) 39-50.
12. Y. Mao, Z. Wang, I. Gutman and A. Klobučar, Steiner degree distance, *MATCH Commun. Math. Comput. Chem.* **78** (2017) 221−230.
13. M. Masre, S. A. Fufa and T. Vetrík, Distance-based indices of complete m-ary trees, *Discrete Math. Algorithms Appl*. **12** (4) (2020) 2050041.

14. H. Wiener, Structural determination of paraffin boiling points, *J. Amer. Chem. Soc.* **69** (1947) 17–20.
15. K. Xu, M. Liu, K. C. Das, I. Gutman and B. Furtula, A survey on graphs extremal with respect to distance-based topological indices, *MATCH Commun. Math. Comput. Chem.* **71** (2014) 461–508.