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Sombor Index of Certain Graphs

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ABSTRACT

Let $G = (V,E)$ be a finite simple graph. The Sombor index $SO(G)$ of G is defined as $\sum_{uv\in E(G)}\sqrt{d_u^2+d_v^2}$, where d_u is the degree of vertex u in G. In this paper, we study this index for certain graphs and we examine the effects on $SO(G)$ when G is modified by operations on vertex and edge of G. Also we present bounds for the Sombor index of join and corona product of two graphs.

1. INTRODUCTION

Let $G = (V,E)$ be a finite, connected, simple graph. We denote the degree of a vertex v in G by d_v. A topological index of G is a real number related to G. It does not depend on the labeling or pictorial representation of a graph. The Wiener index $W(G)$ is the first distance based topological index defined as $W(G) = \sum_{\{u,v\}\subseteq G} d\,(u,v) = \frac{1}{2}\sum_{u,v\in V(G)} d\,(u,v)$ with the summation runs over all pairs of vertices of G [15]. The topological indices and graph invariants based on distances between vertices of a graph are widely used for characterizing molecular graphs, establishing relationships between structure and properties of molecules, predicting biological activity of chemical compounds, and making their chemical applications. The Wiener index is one of the most used topological indices with high correlation with many physical and chemical indices of molecular

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compounds [15]. Recently in [6] a new vertex-degree-based molecular structure descriptor was put forward, the Sombor index, defined as $SO(G) = \sum_{uv\in E(G)} \sqrt{d_u^2 + d_v^2}$.

Cruz, Gutman and Rada in [3] characterized the graphs extremal with respect to this index over the chemical graphs, chemical trees and hexagon systems. In [4] some novel lower and upper bounds on the Sombor index of graphs has presented by using some graph parameters, especially, maximum and minimum degree. Moreover, several relations on Sombor index with the first and second Zagreb indices of graphs obtained in [4]. In [5] the chemical importance of the Sombor index has investigated and it is shown that this index is useful in predicting physico- chemical properties with high accuracy compared to some well- established and often used indices. Also a sharp upper bound for the Sombor index among all molecular trees with fixed numbers of vertices has obtained, and those molecular trees achieving the extremal value has characterized. The mathematical relations between the Sombor index and some other well-known degree-based descriptors are investigated in [14].

The corona of two graphs G_1 and G_2, is the graph $G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where the ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2. The corona $G \circ K_1$, in particular, is the graph constructed from a copy of G, where for each vertex $v \in V(G)$, a new vertex v' and a pendant edge vv' are added. The join of two graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv | u \in V(G_1)$ and $v \in V(G_2)\}$. The Cartesian product $G\square H$ of graphs G and H is a graph such that the vertex set of $G\square H$ is the Cartesian product $V(G) \times V(H)$, and two vertices (u, u') and (v, v') are adjacent in $G\square H$ if and only if either $u = v$ and u' is adjacent to v' in H, or $u' = v'$ and u is adjacent to v in G. The dutch windmill graph D_n^k is the graph obtained by taking n, $(n \geq 2)$ copies of the cycle graph C_k, $(k \geq 3)$ with a vertex in common.

In the next section, we compute the Sombor index for special graphs, cactus chains and grid graphs. In Section 3, we examine the effects on $SO(G)$ when G is modified by operations on vertex and edge of G and finally in Section 4, we study the Sombor index of join and corona of two graphs.

2. SOMBOR INDEX OF CERTAIN GRAPHS

In this section, we compute the Sombor index for certain graphs, such as paths, cycles, friendship graph, grid graphs and cactus chains.

2.1 SOMBOR INDEX OF SPECIFIC GRAPHS

We begin with the following proposition:

Proposition 1. The following hold:

1. For every $n \in \mathbb{N} - \{1,2\}$, $SO(P_n) = 2\sqrt{5} + (2n-6)\sqrt{2}$, for $n \geq 2$, $SO(K_n) = \frac{n(n-1)^2}{2}\sqrt{2}$.

2. For the cycle graph C_n, $SO(C_n) = 2n\sqrt{2}$.

3. For every $\mathrm{m}, \mathrm{n} \in \mathbb{N}$, $\mathrm{SO}(\mathrm{K}_{1,\mathrm{n}}) = \mathrm{n}\sqrt{\mathrm{n}^2 + 1}$, and $\mathrm{SO}(\mathrm{K}_{\mathrm{m,n}}) = \mathrm{mn}\sqrt{\mathrm{m}^2 + \mathrm{n}^2}$.

4. For the wheel graph $W_n = C_{n-1} \vee K_1$,
$$SO(W_n) = (3n-3)\sqrt{2} + (n-1)\sqrt{9 + (n-1)^2}.$$

5. For the ladder graph $L_n = P_n \square K_2$, $(n \geq 3)$, $SO(L_n) = (9n-22)\sqrt{2} + 4\sqrt{13}$.

6. For the friendship graph $F_n = K_1 \vee nK_2$, $SO(F_n) = 2n\sqrt{2} + 4n\sqrt{n^2+1}$.

7. For the book graph $B_n = K_{1,n} \square K_2$, and for $n \geq 3$,
$$SO(B_n) = (3n+1)\sqrt{2} + 2n\sqrt{4 + (n+1)^2}.$$

8. For the Dutch windmill graph $D_n^{(m)}$, and for every $n \geq 3$ and $m \geq 2$,
$$SO(D_n^{(m)}) = 2m(n-2)\sqrt{2} + 4m\sqrt{m^2+1}.$$

Proof. The proof of all parts are easy and similar. For instance, we state the proof of Part 1. There are two edges with endpoints of degree 1 and 2. Also there are $n-3$ edges with endpoints of degree 2. Therefore $SO(P_n) = 2\sqrt{1+4} + (n-3)\sqrt{4+4}$, and we have the result. □

We have the following corollary:

Corollary 2. The natural numbers $1, 2, \ldots, 59$ cannot be the value of Sombor index of any graph. In other words, the smallest natural number as the Sombor index of a graph is 60.

Proof. The minimum value of $\sqrt{a^2+b^2}$ as a natural number (when a and b are natural numbers) is 5 and it is happen when $a = 4$ and $b = 3$. So we need at least 5 vertices to have such a graph. It is easy to see that the graphs of order 5 and 6 do not have natural Sombor index. On the other hand, all the values of $\sqrt{a^2+b^2}$ in the definition of Sombor index should be natural numbers (note that a and b are natural numbers) to have a natural value for the Sombor index. The complete bipartite graph $K_{3,4}$ has this condition and by the Part 3 of Theorem 1 we have the result. By an easy check, we observe that no other graph with 7 vertices has natural Sombor index and also there is no graph with natural Sombor index less than 60. □

Here, we consider the grid graph $(P_n \square P_m)$, and obtain its Sombor index.

Theorem 3. Let $P_m \square P_n$ be the gird graph (See Figure 1). Then,

$$SO(P_m \square P_n) = 2((4mn - 7m - 7n + 6)\sqrt{2} + 4\sqrt{13} + 5(m + n - 4)),$$

where $n, m \geq 3$.

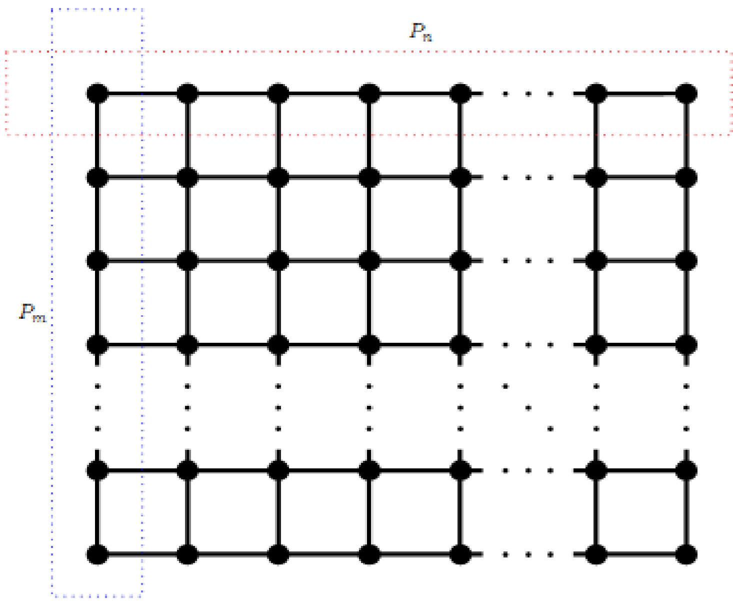

Figure 1: The graph $P_m \square P_n$.

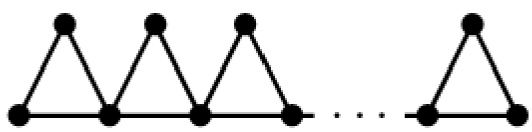

Figure 2: Chain triangular cactus T_n.

Proof. There are eight edges with endpoints of degree 2 and 3 and there are $2(m + n - 6)$ edges with endpoints of degree 3. Also there are $2(m + n - 4)$ edges with endpoints of degree 3 and 4 and there are $2nm - 5n - 5m + 12$ edges with endpoints of degree 4. Therefore, $SO(P_m \square P_n) = 8\sqrt{4+9} + 2(m + n - 6)\sqrt{9+9} + (2m + 2n - 8)\sqrt{9+16}$ $+(2nm - 5n - 5m + 12)\sqrt{16+16}$, and we have the result. □

2.2 SOMBOR INDEX OF CACTUS CHAINS

In this subsection, we consider a class of simple linear polymers called cactus chains. Cactus graphs were first known as Husimi tree, they appeared in the scientific literature some sixty years ago in papers by Husimi and Riddell concerned with cluster integrals in the theory of condensation in statistical mechanics [7,8,11]. We refer the reader to papers [2,9] for some aspects of parameters of cactus graphs.

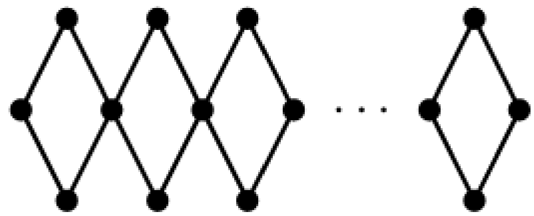

Figure 3: Para-chain square cactus Q_n.

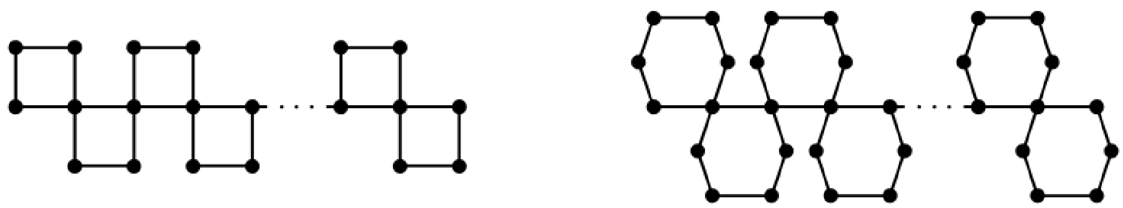

Figure 4: Para-chain square cactus O_n and ortho-chain O_n^h, respectively.

A cactus graph is a connected graph in which no edge lies in more than one cycle. Consequently, each block of a cactus graph is either an edge or a cycle. If all blocks of a cactus G are cycles of the same size i, the cactus is i-uniform. A triangular cactus is a graph whose blocks are triangles, i.e., a 3-uniform cactus. A vertex shared by two or more triangles is called a cut-vertex. If each triangle of a triangular cactus G has at most two cut-vertices, and each cut-vertex is shared by exactly two triangles, we say that G is a chain triangular cactus. By replacing triangles in this definitions with cycles of length 4 we obtain cacti whose every block is C_4. We call such cacti square cacti. Note that the internal squares may differ in the way they connect to their neighbors. If their cut-vertices are adjacent, we say that such a square is an ortho-square; if the cut-vertices are not adjacent, we call the square a para-square [1,12].

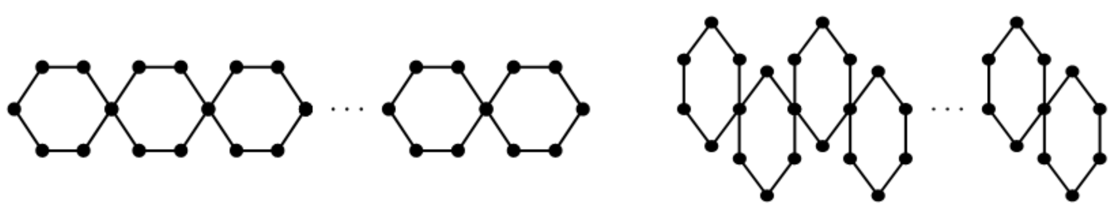

Figure 5: Para-chain L_n and meta-chain M_n, respectively.

Theorem 4. 1. Let T_n be the chain triangular graph of order n, Figure 2. Then for every $n \geq 2$, $SO(T_n) = (4n-4)\sqrt{2} + 4n\sqrt{5}$.

2. Let Q_n be the para-chain square cactus graph of order n, Figure 3. Then for every $n \geq 2$, $SO(Q_n) = 8\sqrt{2} + (8n-8)\sqrt{5}$.

3. Let O_n be the para-chain square cactus graph of order n, Figure 4. Then for every $n \geq 2$, $SO(O_n) = (6n-4)\sqrt{2} + 4n\sqrt{5}$.

4. Let O_n^h be the Ortho-chain graph of order n, Figure 4. Then for every $n \geq 2$, $SO(O_n^h) = (10n-4)\sqrt{2} + 4n\sqrt{5}$.

5. Let L_n be the para-chain hexagonal cactus graph of order n, Figure 5. Then for every $n \geq 2$, $SO(L_n) = (4n+8)\sqrt{2} + (8n-8)\sqrt{5}$.

6. Let M_n be the Meta-chain hexagonal cactus graph of order n, Figure 5. Then for every $n \geq 2$, $SO(M_n) = (4n+8)\sqrt{2} + (8n-8)\sqrt{5}$.

Proof. 1. There are two edges with endpoints of degree 2. Also there are $2n$ edges with endpoints of degree 2 and 4 and there are $n-2$ edges with endpoints of degree 4. Therefore, $SO(T_n) = 2\sqrt{4+4} + 2n\sqrt{4+16} + (n-2)\sqrt{16+16}$, and we have the result.

2. There are four edges with endpoints of degree 2. Also there are $4n-4$ edges with endpoints of degree 2 and 4. Therefore, $SO(Q_n) = 4\sqrt{4+4} + (4n-4)\sqrt{4+16}$, and the result follows.

3. There are $n+2$ edges with endpoints of degree 2. Also there are $2n$ edges with endpoints of degree 2 and 4 and there are $n-2$ edges with endpoints of degree 4. Therefore, $SO(O_n) = (n+2)\sqrt{4+4} + 2n\sqrt{4+16} + (n-2)\sqrt{16+16}$, and we have the result.

4. There are $3n+2$ edges with endpoints of degree 2. Also there are $2n$ edges with endpoints of degree 2 and 4 and there are $n-2$ edges with endpoints of degree 4. Therefore, $SO(O_n^h) = (3n+2)\sqrt{4+4} + 2n\sqrt{4+16} + (n-2)\sqrt{16+16}$, and the result follows.

5. There are $2n+4$ edges with endpoints of degree 2. Also, there are $4n-4$ edges with endpoints of degree 2 and 4. Therefore,
$$SO(L_n) = (2n+4)\sqrt{4+4} + (4n-4)\sqrt{4+16},$$
and we have the result.

6. There are $2n+4$ edges with endpoints of degree 2. Also there are $4n-4$ edges with endpoints of degree 2 and 4. Therefore,
$$SO(M_n) = (2n+4)\sqrt{4+4} + (4n-4)\sqrt{4+16},$$
and the result follows. □

Corollary 5. Meta-chain hexagonal cactus graphs and para-chain hexagonal cactus graphs of the same order, have the same Sombor index.

3 SOMBOR INDEX OF SOME OPERATIONS ON A GRAPH

In this section, we examine the effects on $SO(G)$ when G is modified by operations on vertex and edge of G.

Theorem 6. Let $G=(V,E)$ be a graph and $e=uv\in E$. Also let d_w be the degree of vertex w in G. Then, $SO(G-e) < SO(G) - \frac{|d_u-d_v|}{\sqrt{2}}$.

Proof. First we remove the edge e and consider $SO(G-e)$. Now obviously, by adding edge e to $G-e$ and $\sqrt{d_u^2+d_v^2}$ to $SO(G-e)$, the value of $SO(G)$ is greater than of sum of $SO(G-e)$ and $\sqrt{d_u^2+d_v^2}$. Since $\sqrt{a^2+b^2} \geq \frac{|a-b|}{\sqrt{2}}$, $SO(G) > SO(G-e) + \sqrt{d_u^2+d_v^2} \geq SO(G-e) + \frac{|d_u-d_v|}{\sqrt{2}}$, and therefore we have the result. □

Theorem 7. Let $G=(V,E)$ be a graph and $v\in V$. Also let d_u be the degree of vertex u in G. Then, $SO(G-v) < SO(G) - \sum_{uv\in E}\frac{|d_u-d_v|}{\sqrt{2}}$.

Proof. First we remove the vertex v and all edges related that and then we consider $SO(G-v)$. Now obviously, by adding the vertex v and all edges related that to $G-v$ and $\sum_{uv\in E}\sqrt{d_u^2+d_v^2}$ to $SO(G-v)$, the value of $SO(G)$ is greater than the sum of $SO(G-v)$ and $\sum_{uv\in E}\sqrt{d_u^2+d_v^2}$. Since $\sqrt{a^2+b^2} \geq \frac{|a-b|}{\sqrt{2}}$, $SO(G) > SO(G-v) + \sum_{uv\in E}\sqrt{d_u^2+d_v^2} \geq SO(G-v) + \sum_{uv\in E}\frac{|d_u-d_v|}{\sqrt{2}}$, and therefore we have the result. □

For any $k\in\mathbb{N}$, the k-subdivision of G is a simple graph $G^{\frac{1}{k}}$ which is constructed by replacing each edge of G with a path of length k. The following theorem is about the Sombor index of k-subdivision of graph G.

Theorem 8. Let $G=(V,E)$ be a graph with $|V|=n$ and $|E|=m$. For every $k\geq 2$, $SO(G^{\frac{1}{k}})=2m(k-2)\sqrt{2}+\sum_{u\in V} d_u\sqrt{d_u^2+4}$.

Proof. There are d_u edges incident $u\in V$ with endpoints of degree d_u and 2 in $G^{\frac{1}{k}}$. Also there are $m(k-2)$ edges with endpoints of degree 2 in that. So, $SO(G^{\frac{1}{k}})=m(k-2)\sqrt{4+4}+\sum_{u\in V} d_u\sqrt{d_u^2+4}$, and we have the result. □

As a result of the Theorem 8, we have:

Corollary 9. Let $G=(V,E)$ be a graph of order n and size m and. If Δ is the maximum degree and δ is the minimum degree of vertices in G, then for every $k\geq 2$,

$$2m(k-2)\sqrt{2}+n\delta\sqrt{\delta^2+4}\leq SO(G^{\frac{1}{k}})\leq 2m(k-2)\sqrt{2}+n\Delta\sqrt{\Delta^2+4}.$$

Remark. The bounds in the Corollary 9 are sharp. It suffices to consider cycle graph or complete graph.

For a given a graph parameter $f(G)$ and a positive integer n, the well-known Nordhaus-Gaddum problem is to determine sharp bounds for $f(G)+f(\overline{G})$ and $f(G)f(\overline{G})$ over the class of connected graph G, with order n, size m. Many Nordhaus-–Gaddum type relations have attracted considerable attention in graph theory. Comprehensive results regarding this topic can be found in e.g. [10,13].

Theorem 10. Let $G=(V,E)$ be a graph with $|V|=n$. Also let d_u be the degree of vertex u in G. Then, $SO(G)+SO(\overline{G})\geq\sum_{u,v\in V}\frac{|d_u-d_v|}{\sqrt{2}}$.

Proof. By the definition of Sombor index for the graph G, we have $SO(G)=\sum_{uv\in E}\sqrt{d_u^2+d_v^2}$, and $SO(\overline{G})=\sum_{uv\notin E}\sqrt{(n-1-d_u)^2+(n-1-d_v)^2}$. Since $\sqrt{a^2+b^2}\geq\frac{|a-b|}{\sqrt{2}}$,

$$\begin{aligned} SO(G)+SO(\overline{G}) &\geq \sum_{uv\in E}\frac{|d_u-d_v|}{\sqrt{2}}+\sum_{uv\notin E}\frac{|d_u-d_v|}{\sqrt{2}} \\ &= \sum_{u,v\in V}\frac{|d_u-d_v|}{\sqrt{2}}. \end{aligned}$$

Therefore, the result follows. □

4 SOMBOR INDEX OF JOIN AND CORONA OF TWO GRAPHS

In this section, we study the Sombor index of join product and corona product of two graphs.

Theorem 11. Let $G=(V_G,E_G)$ and $H=(V_H,E_H)$ be two graphs with $|V_G|=n$ and $|V_H|=m$. Also let d_u be the degree of vertex u in G and H before joining of two graphs. Then,

$$\begin{aligned} SO(G+H) &\geq \sum_{u\in G,v\in H}\frac{|d_u-d_v+m-n|}{\sqrt{2}} \\ &+\sum_{uv\in E_H}\frac{|d_u-d_v|}{\sqrt{2}}+\sum_{uv\in E_G}\frac{|d_u-d_v|}{\sqrt{2}} \end{aligned}$$

Proof. By the definition of join of two graphs and the definition of Sombor index, we have

$$\begin{aligned} SO(G+H) &= \sum_{u\in G,v\in H}\sqrt{(d_u+m)^2+(d_v+n)^2} \\ &+\sum_{uv\in E_H}\sqrt{(d_u+m)^2+(d_v+m)^2} \\ &+\sum_{uv\in E_G}\sqrt{(d_u+n)^2+(d_v+n)^2}, \end{aligned}$$

Since $\sqrt{a^2+b^2}\geq\frac{|a-b|}{\sqrt{2}}$, then we have the result. □

Theorem 12. The following Theorem gives the values of Sombor index for corona of P_n and C_n with K_1:

1. $SO(P_n\circ K_1)=(3n-9)\sqrt{2}+(n-2)\sqrt{10}+2\sqrt{5}+2\sqrt{13}$.
2. $SO(C_n\circ K_1)=3n\sqrt{2}+n\sqrt{10}$.

Proof. 1. There are two edges with endpoints of degrees 1 and 2 and two edges with end points of degree 2 and 3 . Also, there are $n-2$ edges with endpoints of degree 1 and 3 and there are $n-3$ edges with endpoints of degree 3. Therefore, we have $SO(P_n\circ K_1)=(n-3)\sqrt{9+9}+(n-2)\sqrt{1+9}+2\sqrt{1+4}+2\sqrt{4+9}$, and the result follows.

2. There are n edges with endpoints of degree 1 and 3 and there are n edges with endpoints of degree 3. Therefore, $SO(C_n\circ K_1)=n\sqrt{1+9}+n\sqrt{9+9}$, and we have the result. □

In the following result, we present a lower bound for the Sombor index of the corona of two graphs G and H.

Theorem 13. Let $G=(V_G,E_G)$ and $H=(V_H,E_H)$ be two graphs with $|V_H|=m$. Also let d_u be the degree of vertex u in G and H before corona of two graphs. Then,

$$\begin{aligned} SO(G\circ H) \quad &\geq \sum_{u\in G,v\in H}\frac{m|d_u-d_v+m-1|}{\sqrt{2}} \\ &+\sum_{uv\in E_H}\frac{|d_u-d_v|}{\sqrt{2}}+\sum_{uv\in E_G}\frac{|d_u-d_v|}{\sqrt{2}} \end{aligned}$$

Proof. By the definition of corona of two graphs and the definition of Sombor index, we have

$$\begin{aligned} SO(G\circ H) \quad &= \sum_{u\in G,v\in H} m\sqrt{(d_u+m)^2+(d_v+1)^2} \\ &+\sum_{uv\in E_H}\sqrt{(d_u+1)^2+(d_v+1)^2} \\ &+\sum_{uv\in E_G}\sqrt{(d_u+m)^2+(d_v+m)^2}. \end{aligned}$$

Since $\sqrt{a^2+b^2}\geq\frac{|a-b|}{\sqrt{2}}$, we have the result. □

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