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Non–Uniform Hypergraphs

G. H. SHIRDEL•, A. MORTEZAEE AND E. GOLPAR–RABOKY

Department of Mathematics, University of Qom, Qom, I. R. Iran

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ABSTRACT

Non-uniform hypergraphs are a generalization of hypergraphs in which not all edges need to have the same cardinality. It allows them to support a more complex data structure. In this paper, we extend some results for non-uniform hypergraphs and generalize the spectral results for uniform hypergraphs to non-uniform hypergraphs.

1. INTRODUCTION

Graph Theory is an important area of mathematics with many applications in computer science, engineering, social sciences, industry genetics, chemistry, industry, and business. Hypergraphs are a generalization of graphs, where an edge may contain more than two vertices and have found applications in social network analysis, image processing, machine learning.

In the k-uniform hypergraph, the number of vertices in every edge exactly is equal to k. But in non-uniform hypergraphs, there isn't such restriction and, then it allows the non-uniform hypergraph to support a more complex data structure [1], [46]. It appears in various domains of computer science as satisfiability problems and data analysis.

• Corresponding author (Email address: g.h.shirdel@qom.ac.ir)
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The spectral theory of uniform hypergraphs as determining several kinds of eigenvalues and eigenvectors of their associated tensors, namely adjacency tensor, Laplacian tensor, and signless Laplacian tensor, has studied in recent years, extremely and, in this respect, we have so many theorems. In 2005 eigenvalues and eigenvectors of a real tensor are defined in [28] and [35]. Qi in [35] introduced some basic definitions of spectral theory of supersymmetric real tensor. Later in [36], the spectral theory of uniform hypergraphs has been studied. Recently a number of papers appeared in different aspects contains, spectral uniform hypergraph theory [9, 10, 15, 22, 23, 26, 31, 33, 39, 40, 45, 53], eigenvalues [17, 25, 32, 41, 42, 43, 44, 47, 51], connectivity [16, 27], Laplacian tensor [4, 18, 20, 34, 36, 52], structured tensors related [7, 11], special uniform hypergraphs [5, 19, 21, 37, 48], uniform hypergraph properties [6, 12, 14, 29, 30].

Despite a lot of research in the spectral theory of uniform hypergraphs, there aren't so many scientific works in the spectral theory of non-uniform hypergraphs. Banerjee and others in 2016 [3] introduced the adjacency tensor of a non-uniform hypergraph and presented some of its properties. Then Banerjee and Char in 2017 [2] studied the non-uniform directed hypergraph. The spectral properties of non-uniform hypergraph are also analyzed in [24], [49], [50].

In this paper, we study the non-uniform hypergraph more precisely and then analyze some of its spectral properties. The significant point in the non-uniform hypergraph that there is not in uniformity, considering all permutations with the repetition of vertices in edge e. This point makes some theorems about non-uniform hypergraphs remain unproven. We try to compare the spectral properties of the non-uniform hypergraph with those of the uniform hypergraphs and, we will see that they haven't similar properties.

The rest of this paper is as follows: Some basic definitions of tensors and their H-eigenvalues are presented in the next section. In section 3 we analyze non-uniform hypergraphs precisely and propose some theorems about them and, then in section 4 the adjacency matrix of a non-uniform hypergraph is introduced and, we discuss some of its properties. Section 4 is concerning the comparing spectral properties of the uniform hypergraph and the non-uniform hypergraph and, finally, section 5 is the conclusion.

2. PRELIMINARIES

In this section we give some basic definitions of tensors and their eigenvalues. In this paper we only consider real tensors. A real tensor $\mathcal{T} = (t_{i_1 \cdots i_k})$ of order k and dimension n, for integers $k \geq 3$ and $n \geq 2$, is a multi-dimensional array with entries $t_{i_1 \cdots i_k}$ such that $t_{i_1 \cdots i_k} \in \mathbb{R}$ for all $i_j \in [n] := \{1,2,\cdots,n\}$ and $j \in [k]$, [35].

Definition 2.1 [38]. *Let $\mathcal{T}$ be an order k and n-dimensional tensor, and let P and Q are both matrices. Then $\mathcal{S} = P\mathcal{T}Q$ is an order k and n-dimensional tensor whose $s_{i_1\cdots i_k}$ entry is $s_{i_1\cdots i_k} = \sum_{j_1,\cdots,j_k=1}^{n} t_{j_1\cdots j_k} p_{i_1 j_1} q_{j_2 i_2} \cdots q_{j_k i_k}$.*

Now let $\mathbf{x} = (x_1, \cdots, x_n)^T \in \mathbb{C}^n$, the product $\mathcal{T}\mathbf{x}$ is a vector in $\mathbb{C}^n$ whose ith component as the following:

$$(\mathcal{T}\mathbf{x})_i = \sum_{i_2,\cdots,i_k=1}^{n} t_{i i_2 \cdots i_k} x_{i_2} \cdots x_{i_k}$$

The identity tensor of order k and dimension n, $\mathcal{I} = (i_{i_1\cdots i_k})$, is defined as $i_{i_1\cdots i_k} = 1$ if and only if $i_1 = \cdots = i_k \in [n]$ and zero otherwise.

Definition 2.2 [8, 35]. *Let $\mathcal{T}$ be an order k and n-dimensional tensor. Then a number $\lambda \in \mathbb{C}$ is called an eigenvalue of the tensor $\mathcal{T}$ if the polynomial system $(\lambda\mathcal{I} - \mathcal{T})\boldsymbol{x}^{[k-1]} = 0$ has a nonzero solution $\boldsymbol{x} \in \mathbb{C}^n$, where $\boldsymbol{x}^{[k-1]} = (x_1^{k-1}, \cdots, x_n^{k-1})^T$. In this case $\boldsymbol{x}$ is called an eigenvector of $\mathcal{T}$ corresponding to λ.*

Now if there exists a real eigenvector corresponding to λ, then λ is called an H-eigenvalue of $\mathcal{T}$ [35]. The set of all eigenvalues of $\mathcal{T}$, denoted by $Spec(\mathcal{T})$, is called the spectrum of $\mathcal{T}$. The H-spectrum of $\mathcal{T}$, denoted by $Hspec(\mathcal{T})$, is defined as $Hspec(\mathcal{T}) = \{\lambda \in \mathbb{R} | \lambda \text{ is an H} - \text{eigenvalue of } \mathcal{T}\}$.

The spectral radius of $\mathcal{T}$ is defined as the maximal absolute value of the eigenvalues of $\mathcal{T}$ and denoted by $\rho(\mathcal{T})$.

Definition 2.3 [38]. *Let $\mathcal{T}$ and $\mathcal{S}$ be two order k dimension n tensors. $\mathcal{T}$ and $\mathcal{S}$ are called diagonal similar if there exists a nonsingular diagonal matrix D of order n such that $\mathcal{S} = D^{-(k-1)}\mathcal{T}D$.*

The notation of weakly irreducible nonnegative tensors was introduced in [13].

Definition 2.4. *Let $\mathcal{T} = (t_{i_1\cdots i_k})$ be a k order n-dimensional nonnegative tensor. We associate a directed graph $G(\mathcal{T}) = (V, E(\mathcal{T}))$ with $\mathcal{T}$, where $V = [n]$,and a directed edge $(i, j) \in E(\mathcal{T})$ if there exists $\{i_2, \cdots, i_k\} \in [n]$ such that $j \in \{i_2, \cdots, i_k\}$ and $t_{i i_2 \dots i_k} > 0$. Now $\mathcal{T}$ is called weakly irreducible if $G(\mathcal{T})$ is strongly connected.*

3. The Non–Uniform Hypergraph and its Tensors

In this section first, we present some basic definitions of non-uniform hypergraphs and introduce its corresponding tensors by definitions in [3].

Definition 3.1. *A non-uniform hypergraph* $\mathcal{H}$ *is a pair* $\mathcal{H} = (V, E)$, *where* $V = \{1,2,\cdots,n\}$ *is the set of vertices and* $E = \{e_1, e_2, \cdots, e_m\}$ *is the set of edges that every edge is a nontrivial subset of* V.

Note that in the k-uniform hypergraph $|e_t| = k$ for $t = 1,\cdots,m$, but in the non-uniform hypergraph we don't have this restriction, then it is a useful tool for storing data. The degree of the vertex $i \in V$ is $d_i = |\{e \in E \mid i \in e\}|$. Let i,j be two different vertices. The set of all edges containing i is denoted by E_i, and E_{ij} is defined as follows:

$$E_{ij} = \{e \in E | i,j \in e\}.$$

We say i,j are adjacent, denoted by $i \leftrightarrow j$, if there exists an edge that contains them and i,j are connected if there exists a sequence of edges $e_{l_1},\cdots,e_{l_p}$ such that $i \in e_{l_1}$, $j \in e_{l_p}$ and $e_{l_t} \cap e_{l_{t+1}} \neq \phi$ for all $t \in \{1,\cdots,p-1\}$. The non-uniform hypergraph $\mathcal{H}$ is connected if every pair of different vertices of $\mathcal{H}$ is connected. Let $k = \max_{e\in E}|e|$ be the maximum cardinality of edges, m.c($\mathcal{H}$), in this paper, we denote every non-uniform hypergraph $\mathcal{H}$ by non-uniform k-hypergraph if m.c($\mathcal{H}$)=k.

Definition 3.2. *Let* $\mathcal{H} = (V, E)$ *be a non-uniform* k*-hypergraph and* k *be even.* $\mathcal{H}$ *is called odd-bipartite if it is trivial (i.e.* $E = \phi$ *) or, there is a nontrivial subset of* V *as* V_1 *such that every edge in* E *contains an exactly odd number of vertices in* V_1.

Definition 3.3. *Let* $e = \{l_1, l_2, \cdots, l_s\}$ *be a set of* s *distinct objects and* $k \geq s$. *We denote the number of all permutations with repetition of* $l_1, l_2, \cdots, l_s$ *of length* k *with at least once for each element of the set e by* α_e^k *and we have:*

$$\alpha_e^k = \sum_{\substack{k_1,k_2,\cdots,k_s\geq 1\\ \sum k_t = k}} \frac{k!}{k_1!k_2!\cdots k_s!}$$

It's trivial that if $s = k$, then $\alpha_e^k = k!$.

Definition 3.4. *Let* $e = \{i, l_2, \cdots, l_s\}$ *be a set of* s *distinct objects and* $k \geq s$. *We denote the number of all permutations with repetition of* $i, l_2, \cdots, l_s$ *of length* k *with at least once for each element of the set* e, *in which* i *is in first place, by* $\alpha_e^k(i)$.

Definition 3.5. *Let* $e = \{i, j, l_3, \cdots, l_s\}$ *be a set of* s *distinct objects and* $k \geq s$. *We denote the number of all permutations with repetition of* $i, j, l_3, \cdots, l_s$ *of length* k *with at least once for each element of the set* e *in which* i *is in the first place and* j *is in the second place, by* $\alpha_e^k(i,j)$.

In the following theorems, we explain some essential concepts of non-uniform hypergraphs by $\alpha_e^k(i)$ and $\alpha_e^k(i,j)$.

Theorem 1. *Let $\mathcal{H} = (V,E)$ be a non-uniform k-hypergraph and $i \in V$. Suppose that $e = \{i, l_2, \cdots, l_s\} \in E_i$ then we have that:*

$$\alpha_e^k(i) = \sum_{\substack{k_1,k_2,\cdots,k_s \geq 1 \\ \sum k_t = k}} \frac{(k-1)!}{(k_1-1)!k_2!\cdots k_s!} = \frac{\alpha_e^k}{s}$$

Proof. Let $e = \{i, l_2, \cdots, l_s\} \in E_i$, we want to determine the number of $a_{ii_2\cdots i_k}$ corresponding to the e. So, first, we should transform these s objects into k objects by creating copies of them, and then determine the number of permutations with repetition of these k objects. Now suppose that create k objects in which there are k_1 copies of i, k_2 copies of l_2, $\cdots$ and k_s copies of l_s such that $k_t \geq 1$ for $t = 1, \cdots, s$. The number of permutations with repetition of these k objects in which i is in the first place is equal to $\frac{(k-1)!}{(k_1-1)!k_2!\cdots k_s!}$ and thus the number of all permutations with repetition of $i, l_2, \cdots, l_s$ in the form of $i, i_2, \cdots, i_k$ is

$$\sum_{\substack{k_1,k_2,\cdots,k_s \geq 1 \\ \sum k_t = k}} \frac{(k-1)!}{(k_1-1)!k_2!\cdots k_s!} \tag{3.1}$$

Now we show that (3.1) equals to α_e^k/s. Let A be the set of all permutations with repetition of $i, l_2, \cdots, l_s$ of the form $i_1, i_2, \cdots, i_k$, we have $|A| = \alpha_e^k$. Now we partition A into $A_1, A_2, \cdots, A_s$ in which $A_1 = \{O \in A \mid O = i, i_2, \cdots, i_k\}$ and $A_t = \{O \in A \mid O = l_t, i_2, \cdots, i_k\}$ for $t = 2,3, \cdots, s$. It is trivial that $c := |A_1| = |A_2| = \cdots = |A_s|$. Now we have:

$$\alpha_e^k = |A| = \sum_{t=1}^{s} |A_t| = sc \;\;\Rightarrow\;\; c = \frac{\alpha_e^k}{s}$$

On the other hand, $\alpha_e^k(i) = |A_1|$, then we have :

$$\alpha_e^k(i) = \sum_{\substack{k_1,k_2,\cdots,k_s \geq 1 \\ \sum k_t = k}} \frac{(k-1)!}{(k_1-1)!k_2!\cdots k_s!} = \frac{\alpha_e^k}{s} \qquad \blacksquare$$

Similar to Theorem 1, we have the following Theorem.

Theorem 2. *Let $\mathcal{H} = (V,E)$ be a non-uniform k-hypergraph and $i,j \in V$. Suppose that $e = \{i, j, l_3, \cdots, l_s\} \in E_{ij}$, then the number of permutations with repetition of $i, j, l_3, \cdots, l_s$ of the form $i, j, i_3, \cdots, i_k$ with at least once for each element of the set $\{i, j, l_3, \cdots, l_s\}$ is:*

$$\alpha_e^k(i,j) = \sum_{\substack{k_1,k_2,\cdots,k_s \geq 1 \\ \sum k_t = k}} \frac{(k-2)!}{(k_1-1)!(k_2-1)!\cdots k_s!} = \frac{\alpha_e^k}{s(s-1)} - \frac{\alpha_e^{k-1}}{s(s-1)}$$

Proof. By a similar proof as Theorem 1, we have that:

$$\alpha_e^k(i,j) = \sum_{\substack{k_1,k_2,\cdots,k_s\geq 1\\ \sum k_t = k}} \frac{(k-2)!}{(k_1-1)!(k_2-1)!\cdots k_s!}.$$

Now we show that $\alpha_e^k(i,j) = \frac{\alpha_e^k}{s(s-1)} - \frac{\alpha_e^{k-1}}{s(s-1)}$. Suppose that A is the set of all permutations with repetition of $i, j, l_3, \cdots, l_s$ of the form $i, i_2, i_3, \cdots, i_k$ with at least once for each element of the set $\{i, j, l_3, \cdots, l_s\}$. By Theorem(1) we have $|A| = \frac{\alpha_e^k}{s}$. Now we partition $|A|$ into $A_1, A_2, \cdots, A_s$ where $A_1 = \{O \in A | O = i, i, i_3, \cdots, i_k\}$, $A_2 = \{O \in A | O = i, j, i_3, \cdots, i_k\}$ and $A_t = \{O \in A | O = i, l_t, i_2, \cdots, i_k\}$ for $t = 3, \cdots, s$. It is trivial that $c := |A_2| = |A_3| = \cdots = |A_s|$. It is easy to see that the number of all permutations with repetition of $i, j, l_3, \cdots, l_s$ with at least once for each element of the set $\{i, j, l_3, \cdots, l_s\}$ in which the first and second objects are identical, is $\alpha_e^{k-1} = \sum_{\substack{k_1,k_2,\cdots,k_s\geq 1\\ \sum k_t = k-1}} \frac{(k-1)!}{k_1!k_2!\cdots k_s!}$.

Therefore, $|A_1| = \frac{\alpha_e^{k-1}}{s}$. Now we have

$$\frac{\alpha_e^k}{s} = |A| = |A_1| + (s-1)c = \frac{\alpha_e^k}{s(s-1)} - \frac{\alpha_e^{k-1}}{s(s-1)}.$$

Since $\alpha_e^k(i,j) = |A_2| = c$, the result follows. ■

The following definition for the adjacency tensor of a non-uniform hypergraph has been proposed by Banerjee [3]:

Definition 3.6. *Let $\mathcal{H} = (V, E)$ be a non-uniform k-hypergraph. The adjacency tensor of $\mathcal{H}$ is defined as a k order n-dimensional tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_k})$, $(1 \leq i_1, i_2, \cdots, i_k \leq n)$, in which $a_{i_1 i_2 \cdots i_k} = \frac{s}{\alpha_e^k}$, if there is $e = \{l_1, l_2, \cdots, l_s\} \in E$ of cardinality $s \leq k$ such that $i_1, i_2, \cdots, i_k$ are chosen in all possible way from $\{l_1, \cdots, l_s\}$ with at least once for each element of the set.*

By Theorem 1, it is easy to see that $\sum_{i_2,i_3,\cdots,i_k=1}^{n} a_{i i_2 \cdots i_k} = d_i$. The degree tensor, $\mathcal{D}$, is a k-th order n-dimensional diagonal tensor with its diagonal element $d_{i\cdots i}$ being d_i. Then $\mathcal{L} := \mathcal{D} - \mathcal{A}$ is the Laplacian of the hypergraph $\mathcal{H}$, and $\mathcal{Q} := \mathcal{D} + \mathcal{A}$ is the signless Laplacian of the hypergraph $\mathcal{H}$. By Definition 2.2, if (λ, x) is an eigenpair of $\mathcal{A}$ then for $i \in V$ we have:

$$\lambda x_i^{k-1} = (\mathcal{A}\mathrm{x})_i = \sum_{i_2 \cdots i_k = 1}^{n} a_{i i_2 \cdots i_k} x_{i_2} \cdots x_{i_k}$$

$$= \sum_{e=\{i,l_2,\cdots,l_{s_e}\}\in E_i} \frac{s_e}{\alpha_e^k}\left(\sum_{\substack{k_1,k_2,\cdots,k_{s_e}\geq 1\\ \sum k_t=k}} \frac{(k-1)!}{(k_1-1)!\,k_2!\cdots k_{s_e}!} x_i^{k_1-1}x_{l_2}^{k_2}\cdots x_{l_{s_e}}^{k_{s_e}}\right)$$

Similarly if (λ, x) is an eigenpair of $\mathcal{L}$, then we have:

$$\lambda x_i^{k-1} = d_i x_i^{k-1} - \sum_{e=\{i,l_2,\cdots,l_{s_e}\}\in E_i} \frac{s_e}{\alpha_e^k}\Big(\sum_{\substack{k_1,k_2,\cdots,k_{s_e}\geq 1\\ \sum k_t=k}} \frac{(k-1)!}{(k_1-1)!\,k_2!\cdots k_{s_e}!} x_i^{k_1-1}x_{l_2}^{k_2}\cdots x_{l_{s_e}}^{k_{s_e}}\Big)$$

and if (λ, x) is an eigenpair of $\mathcal{Q}$, then we have:

$$\lambda x_i^{k-1} = d_i x_i^{k-1} + \sum_{e=\{i,l_2,\cdots,l_{s_e}\}\in E_i} \frac{s_e}{\alpha_e^k}\left(\sum_{\substack{k_1,k_2,\cdots,k_{s_e}\geq 1\\ \sum k_t=k}} \frac{(k-1)!}{(k_1-1)!\,k_2!\cdots k_{s_e}!} x_i^{k_1-1}x_{l_2}^{k_2}\cdots x_{l_{s_e}}^{k_{s_e}}\right)$$

The last equalities follow from the fact that we should consider all possible permutations with repetition of $i, l_2, \cdots, l_{s_e}$ such that vertex i occurs in the first place.

4. The Adjacency Matrix of a Non–Uniform k–Hypergraph

In this section we introduce the adjacency matrix of a non-uniform k-hypergraph. This matrix has similar properties to those in the adjacency matrix of a graph.

Definition 4.1. *Let $\mathcal{H} = (V, E)$ be a non-uniform k-hypergraph. The adjacency matrix of $\mathcal{H}$ is $(A_{\mathcal{H}})_{n\times n}$ where $(A_{\mathcal{H}})_{ij} = \sum_{i_3,\cdots,i_k=1}^{n} a_{iji_3\cdots i_k}$, $1 \leq i,j \leq n$,* in which $\mathcal{A} = (a_{i_1 i_2\cdots i_k})$ for $(1 \leq i_1, i_2, \cdots, i_k \leq n)$ is the adjacency tensor of $\mathcal{H}$.

It is easy to see that $d_i = \sum_{j=1}^{n} (A_{\mathcal{H}})_{ij}$ for $i \in V$.

Theorem 3. *Let $\mathcal{H} = (V, E)$ be a non-uniform k-hypergraph and $A_{\mathcal{H}}$ be its adjacency matrix, then*

$$(A_{\mathcal{H}})_{ij} = \sum_{e\in E_{ij}} \Big(\frac{1}{s_e-1} - \frac{\alpha_e^{k-1}}{\alpha_e^k(s_e-1)}\Big)$$

Proof. By Definition 4.1 and Theorem 2, we have:

$$(A_{\mathcal{H}})_{ij} = \sum_{i_3,\cdots,i_k=1}^{n} a_{iji_3\cdots i_k}$$

$$= \sum_{e\in E_{ij}} \alpha_e^k(i,j)\,\frac{s_e}{\alpha_e^k} = \sum_{e\in E_{ij}} \left(\frac{\alpha_e^k}{s_e(s_e-1)} - \frac{\alpha_e^{k-1}}{s_e(s_e-1)}\right)\frac{s_e}{\alpha_e^k}$$

$$= \sum_{e\in E_{ij}} \left(\frac{1}{s_e-1} - \frac{\alpha_e^{k-1}}{\alpha_e^k(s_e-1)}\right) \qquad \blacksquare$$

By Definition 4.1 and the proof of Theorem 2, we have:

$$(A_{\mathcal{H}})_{ii} = \sum_{i_3,\cdots,i_k=1}^{n} a_{iii_3\cdots i_k} = \sum_{e\in E_i} \frac{\alpha_e^{k-1}}{s_e}\frac{s_e}{\alpha_e^k} = \sum_{e\in E_i} \frac{\alpha_e^{k-1}}{\alpha_e^k}$$

Let $\mathcal{H}$ be a uniform hypergraph. Since in uniform hypergraphs $\alpha_e^{k-1} = 0$ then $(A_{\mathcal{H}})_{ii} = 0$ that agrees the result in uniform hypergraphs. Also we have:

$$\begin{aligned}
d_i &= \sum_{j=1}^{n} (A_{\mathcal{H}})_{ij} \\
&= \sum_{\substack{j=1\\ j\neq i}}^{n} \sum_{e\in E_{ij}} \left(\frac{1}{s_e-1} - \frac{\alpha_e^{k-1}}{\alpha_e^k(s-1)}\right) + \sum_{e\in E_i} \frac{\alpha_e^{k-1}}{\alpha_e^k} \\
&= \sum_{\substack{j=1\\ j\neq i}}^{n} \sum_{e\in E_{ij}} \frac{1}{s_e-1} + \sum_{e\in E_i} \left(\frac{\alpha_e^{k-1}}{\alpha_e^k} - \frac{(s-1)\alpha_e^{k-1}}{\alpha_e^k(s-1)}\right) \qquad (4.1) \\
&= \sum_{\substack{j=1\\ j\neq i}}^{n} \sum_{e\in E_{ij}} \frac{1}{s_e-1}
\end{aligned}$$

Equality (4.1) is correct since every $e \in E_i$ occurs exactly in $s_e - 1$ edges in E_{ij}, for $j \neq i$. Therefore, $A_{\mathcal{H}}$ can be defined as follows that agrees with Definition 4.1:

$$(A_{\mathcal{H}})_{ij} = \begin{cases} \sum_{e\in E_{ij}} \frac{1}{s_e-1} & i \neq j \\ 0 & i = j \end{cases}$$

In the following, we define the Cartesian product of two non-uniform k-hypergraphs.

Definition 4.2. *Let $\mathcal{H}_1 = (V_1, E_1)$ and $\mathcal{H}_2 = (V_2, E_2)$ be two non-uniform k-hypergraphs. The Cartesian product of $\mathcal{H}_1$ and $\mathcal{H}_2$ is a non-uniform k-hypergraph $\mathcal{H} = (V, E)$ where $V = V_1 \times V_2$ and*

$$E = \{\{v\} \times e \mid v \in V_1 \, and \;\; e \in E_2\} \cup \{e \times \{v\} \mid e \in E_1 \, and \;\; v \in V_2\}.$$

Theorem 4. *Let $\mathcal{H} = (V, E)$ be the Cartesian product of two non-uniform k-hypergraphs, $\mathcal{H}_1 = (V_1, E_1)$ and $\mathcal{H}_2 = (V_2, E_2)$ where $V_1 = [n_1]$ and $V_2 = [n_2]$. If λ and μ are eigenvalues of $A_{\mathcal{H}_1}$ and $A_{\mathcal{H}_2}$, respectively, then $\lambda + \mu$ is the eigenvalue of $A_{\mathcal{H}}$.*

Proof. Let $A_{\mathcal{H}}$ be the adjacency matrix of $\mathcal{H}$, then by Definition(4.1), $(A_{\mathcal{H}})_{(a_1,b_1),(a_2,b_2)} = 0$ if (a_1, b_1) and (a_2, b_2) are not adjacent in $\mathcal{H}$. Now suppose that (a_1, b_1) and (a_2, b_2) are adjacent, we consider two cases:

1. $(a_1, b_1) = (a_2, b_2) = (a, b)$, then we have:

$$(A_{\mathcal{H}})_{(a,b),(a,b)} = \sum_{e\in E_{(a,b)}} \frac{\alpha_e^{k-1}}{\alpha_e^k} = \sum_{e\in (E_1)_a} \frac{\alpha_e^{k-1}}{\alpha_e^k} + \sum_{e\in (E_2)_b} \frac{\alpha_e^{k-1}}{\alpha_e^k}$$
$$= (A_{\mathcal{H}_1})_{(a,a)} + (A_{\mathcal{H}_2})_{(b,b)}$$

2. $(a_1, b_1) \neq (a_2, b_2)$, there are two subcases:

I. $a_1 = a_2 = a$ and b_1 and b_2 are adjacent in $\mathcal{H}_2$, then we have:

$$(A_{\mathcal{H}})_{(a,b_1),(a,b_2)} = \sum_{e\in E_{(a,b_1)(a,b_2)}} \left(\frac{1}{s_e-1} - \frac{\alpha_e^{k-1}}{\alpha_e^k(s-1)}\right)$$
$$= \sum_{e\in (E_2)_{b_1b_2}} \left(\frac{1}{s_e-1} - \frac{\alpha_e^{k-1}}{\alpha_e^k(s-1)}\right) = (A_{\mathcal{H}_2})_{(b_1,b_2)}$$

II. $b_1 = b_2 = b$ and a_1 and a_2 are adjacent in $\mathcal{H}_1$, then we have:

$$(A_{\mathcal{H}})_{(a_1,b),(a_2,b)} = \sum_{e\in E_{(a_1,b)(a_2,b)}} \left(\frac{1}{s_e-1} - \frac{\alpha_e^{k-1}}{\alpha_e^k(s-1)}\right)$$
$$= \sum_{e\in (E_1)_{a_1a_2}} \left(\frac{1}{s_e-1} - \frac{\alpha_e^{k-1}}{\alpha_e^k(s-1)}\right) = (A_{\mathcal{H}_1})_{(a_1,a_2)}$$

Now suppose that $X \in \mathbb{R}^{n_1}$ and $Y \in \mathbb{R}^{n_2}$ are the eigenvectors corresponding to the eigenvalues λ and μ, respectively. Let $Z \in \mathbb{R}^{n_1+n_2}$ is a vector in which $z_{(a,b)} = x_a y_b$ for $(a,b) \in [n_1] \times [n_2]$, we show that Z is an eigenvector of $A_{\mathcal{H}}$ corresponding to the $\lambda + \mu$. Let $(a_1, b_1) \in [n_1] \times [n_2]$, then we have:

$$(A_{\mathcal{H}}Z)(a_1,b_1) = \sum_{(a_2,b_2)\in[n_1]\times[n_2]} (A_{\mathcal{H}})_{(a_1,b_1)(a_2,b_2)} z(a_2,b_2)$$
$$= \sum_{(a_2,b_2)\leftrightarrow(a_1,b_1)} (A_{\mathcal{H}})_{(a_1,b_1)(a_2,b_2)} z(a_2,b_2)$$
$$= (A_{\mathcal{H}})_{(a_1,b_1)(a_1,b_1)} z(a_1,b_1) + \sum_{\substack{a_2\leftrightarrow a_1\\ a_2\neq a_1}} (A_{\mathcal{H}})_{(a_1,b_1)(a_2,b_2)} z(a_2,b_2) + \sum_{\substack{b_2\leftrightarrow b_1\\ b_2\neq b_1}} (A_{\mathcal{H}})_{(a_1,b_1)(a_2,b_2)} z(a_2,b_2)$$
$$= \left[(A_{\mathcal{H}_1})_{a_1a_1} + (A_{\mathcal{H}_2})_{b_1b_1}\right] x(a_1)y(b_1) + \sum_{\substack{a_2\leftrightarrow a_1\\ a_2\neq a_1}} (A_{\mathcal{H}_1})_{a_1a_2} x(a_2)y(b_1) + \sum_{\substack{b_2\leftrightarrow b_1\\ b_2\neq b_1}} (A_{\mathcal{H}_2})_{b_1b_2} x(a_1)y(b_2)$$
$$= \sum_{a_2\leftrightarrow a_1} (A_{\mathcal{H}_1})_{a_1a_2} x(a_2)y(b_1) + \sum_{b_2\leftrightarrow b_1} (A_{\mathcal{H}_2})_{b_1b_2} x(a_1)y(b_2)$$
$$= y(b_1)\sum_{a_2\leftrightarrow a_1} (A_{\mathcal{H}_1})_{a_1a_2} x(a_2) + x(a_1)\sum_{b_2\leftrightarrow b_1} (A_{\mathcal{H}_2})_{b_1b_2} y(b_2)$$
$$= y(b_1)\lambda x(a_1) + x(a_1)\mu y(b_1)$$
$$= (\lambda+\mu)x(a_1)y(b_1) = (\lambda+\mu)z(a_1,b_1)$$

This completes the proof. ∎

5. SOME SPECTRAL PROPERTIES OF NON-UNIFORM HYPERGRAPHS

Unlike hypergraphs, there are not many results on the spectral properties of non-uniform hypergraphs. Consider the following definition.

Definition 5.1. *Let $Y \in \mathbb{R}^n$ be a vector and $e = \{i, l_2, \cdots, l_s\}$ be an edge in the non-uniform k-hypergraph $\mathcal{H} = (V, E)$. The $\beta_e(y_i)$ is defined as follows:*

$$\beta_e(y_i) = \sum_{\substack{k_1, k_2, \cdots, k_{s_e} \geq 1 \\ \sum k_t = k}} \frac{(k-1)!}{(k_1-1)!\, k_2! \cdots k_s!} y_i^{k_1-1} y_{l_2}^{k_2} \cdots y_{l_s}^{k_s}$$

Now we have the following theorem:

Theorem 5. *Let $Y \in \mathbb{R}^n$ be a vector and $e = \{i, j, l_3, \cdots, l_s\}$ be an edge in the non-uniform k-hypergraph $\mathcal{H} = (V, E)$ where i, j are two distinct vertices in V. Then $\beta_e(y_i) = \beta_e(y_j)$ if and only if $y_i = y_j$.*

Proof. By Definition 5.1, we have:

$$\begin{aligned}
\beta_e(y_i) &= \sum_{\substack{k_1, k_2, \cdots, k_s \geq 1 \\ \sum k_t = k}} \frac{(k-1)!}{(k_1-1)! k_2! \cdots k_s!} y_i^{k_1-1} y_j^{k_2} \cdots y_{l_s}^{k_s} \\
&= \sum_{\substack{k_1, k_2, \cdots, k_s \geq 1, k_1 = 1 \\ \sum k_t = k}} \frac{(k-1)!}{k_2! k_3! \cdots k_s!} y_j^{k_2} y_{l_3}^{k_3} \cdots y_{l_s}^{k_s} \\
&+ \sum_{\substack{k_1, k_2, \cdots, k_s \geq 1, k_1 \geq 2 \\ \sum k_t = k}} \frac{(k-1)!}{(k_1-1)! k_2! \cdots k_s!} y_i^{k_1-1} y_j^{k_2} \cdots y_{l_s}^{k_s} \\
&= \beta_e^{(1)}(y_i) + \beta_e^{(2)}(y_i)
\end{aligned}$$

Similarly, we have for $\beta_e(y_j)$:

$$\begin{aligned}
\beta_e(y_j) &= \sum_{\substack{k_1, k_2, \cdots, k_s \geq 1, k_2 = 1 \\ \sum k_t = k}} \frac{(k-1)!}{k_1! k_3! \cdots k_s!} y_i^{k_1} y_{l_3}^{k_3} \cdots y_{l_s}^{k_s} \\
&+ \sum_{\substack{k_1, k_2, \cdots, k_s \geq 1, k_2 \geq 2 \\ \sum k_t = k}} \frac{(k-1)!}{k_1! (k_2-1)! \cdots k_s!} y_i^{k_1} y_j^{(k_2-1)} \cdots y_{l_s}^{k_s} \\
&= \beta_e^{(1)}(y_j) + \beta_e^{(2)}(y_j)
\end{aligned}$$

Next, we show that $\beta_e^{(2)}(y_i) = \beta_e^{(2)}(y_j)$. Suppose that $k_1, k_2, \cdots, k_s$ are a choice of permissible k_t's in $\beta_e^{(2)}(y_i)$, then its corresponding term in $\beta_e^{(2)}(y_i)$ is:

$$\frac{(k-1)!}{(k_1-1)!\, k_2! \cdots k_s!} y_i^{k_1-1} y_j^{k_2} \cdots y_{l_s}^{k_s}$$

Take $k'_1 = k_1 - 1$, $k'_2 = k_2 + 1$, $k'_t = k_t$ for $3 \leq t \leq s$. It's trivial that $k'_1, k'_2, \cdots, k'_s$ are permissible in $\beta_e^{(2)}(y_j)$ and its corresponding term in $\beta_e^{(2)}(y_j)$ is:

$$\frac{(k-1)!}{k_1'!\,(k_2'-1)!\cdots k_s'!}y_i^{k_1'}y_j^{(k_2'-1)}\cdots y_{l_s}^{k_s'}=\frac{(k-1)!}{(k_1-1)!\,k_2!\cdots k_s!}y_i^{k_1-1}y_j^{k_2}\cdots y_{l_s}^{k_s}$$

Consequently, every term in $\beta_e^{(2)}(y_i)$ is equal to one term in $\beta_e^{(2)}(y_j)$ and since the number of terms in $\beta_e^{(2)}(y_i)$ and $\beta_e^{(2)}(y_j)$ is identical then $\beta_e^{(2)}(y_i)=\beta_e^{(2)}(y_j)$. Necessity is trivial. Now conversely suppose that $\beta_e(y_i)=\beta_e(y_j)$, then $\beta_e^{(1)}(y_i)+\beta_e^{(2)}(y_i)=\beta_e^{(1)}(y_j)+\beta_e^{(2)}(y_j)$ and so $\beta_e^{(1)}(y_i)=\beta_e^{(1)}(y_j)$. Therefore, $y_i=y_j$. ■

Let i,j be two vertices in the non-uniform k-hypergraph $\mathcal{H}=(V,E)$ such that $E_i=E_j=E'$,and then $d_i=d_j=d$. Now if (λ,X) is an H-eigenpair of the Laplacian tensor of $\mathcal{H}$, then for $i,j\in V$, we have:

$$(d-\lambda)x_i^{k-1}=\sum_{e\in E'}\frac{s_e}{\alpha_e^k}\beta_e(x_i)$$

$$(d-\lambda)x_j^{k-1}=\sum_{e\in E'}\frac{s_e}{\alpha_e^k}\beta_e(x_j)$$

and thus:

$$(d-\lambda)x_i^{k}=\sum_{e\in E'}\frac{s_e}{\alpha_e^k}x_i\beta_e(x_i)=$$

$$\sum_{e\in E'}\frac{s_e}{\alpha_e^k}\Big(\sum_{\substack{k_1,k_2,\cdots,k_{s_e}\ge 1\\ \sum k_t=k}}\frac{(k-1)!}{(k_1-1)!k_2!\cdots k_{s_e}!}x_i^{k_1}x_j^{k_2}\cdots x_{l_{s_e}}^{k_{s_e}}\Big) \tag{5.1}$$

$$(d-\lambda)x_j^{k}=\sum_{e\in E'}\frac{s_e}{\alpha_e^k}x_j\beta_e(x_j)=$$

$$\sum_{e\in E'}\frac{s_e}{\alpha_e^k}\Big(\sum_{\substack{k_1,k_2,\cdots,k_{s_e}\ge 1\\ \sum k_t=k}}\frac{(k-1)!}{k_1!(k_2-1)!\cdots k_{s_e}!}x_i^{k_1}x_j^{k_2}\cdots x_{l_{s_e}}^{k_{s_e}}\Big) \tag{5.2}$$

But with a little care, we understand the coefficients of terms in the right sides of (5.1) and (5.2) are not identical. Then we can't deduce that the left sides of (5.1) and (5.2) are equal. Generally, it seems that because of existence of different choices of $k_1,k_2,\cdots,k_s$ in $\beta_e(x_i)$ and $\beta_e(x_j)$ that there are not in uniformity, many results are not provable here.

Theorem 6. *Let $\mathcal{H}=(V,E)$ be a non-uniform k-hypergraph and $\mathcal{H}_i=(V_i,E_i)$ for $i\in[s]$ be its connected components. If $(0,\boldsymbol{X})$ is an eigenpair of $\mathcal{L}(\mathcal{H})$ or $\mathcal{Q}(\mathcal{H})$, in which $|x_t|\le 1$ for every $t\in V$, then $(0,\boldsymbol{X}(V_i))$ is an eigenpair of $\mathcal{L}(\mathcal{H}_i)$ or $\mathcal{Q}(\mathcal{H}_i)$, whenever $\boldsymbol{X}(V_i)\neq 0$ and in this situation we have $|x_t|=1$ for every $t\in V_i$.*

Proof. We first assume that $(0,\mathbf{X})$ is an eigenpair of $\mathcal{Q}(\mathcal{H})$, the proof for the $\mathcal{L}(\mathcal{H})$ is similar. It is easy that $(0,\mathbf{X}(V_i))$ is an eigenpair of $\mathcal{Q}(\mathcal{H}_i)$, whenever $\mathbf{X}(V_i)\neq 0$.

Without loss of generality, suppose that $\mathbf{X}(V_i)$ is an eigenvector of $\mathcal{Q}(\mathcal{H}_i)$ and $x_j = 1$ for some $j \in V_i$ and $|x_t| \leq 1$ for $t \in V_i$. Suppose that $e = \{j, l_2, \cdots, l_{s_e}\} \in E_j$:

$$|\beta_e(y_j)| = \Big|\sum_{\substack{k_1,k_2,\cdots,k_{s_e}\geq 1\\ \sum k_t = k}} \frac{(k-1)!}{(k_1-1)!k_2!\cdots k_s!} y_j^{k_1-1} y_{l_2}^{k_2} \cdots y_{l_s}^{k_s}\Big|$$

$$\leq \sum_{\substack{k_1,k_2,\cdots,k_{s_e}\geq 1\\ \sum k_t = k}} \frac{(k-1)!}{(k_1-1)!k_2!\cdots k_s!} |y_j|^{k_1-1} |y_{l_2}|^{k_2} \cdots |y_{l_s}|^{k_s}$$

$$\leq \sum_{\substack{k_1,k_2,\cdots,k_{s_e}\geq 1\\ \sum k_t = k}} \frac{(k-1)!}{(k_1-1)!k_2!\cdots k_s!} = \frac{\alpha_e^k}{s_e}$$

Then $\frac{s_e}{\alpha_e^k}|\beta_e(y_j)| \leq 1$. By eigen equations in the vertex $j \in V_i$, we have:

$$0 = (\mathcal{Q}(\mathcal{H})\mathbf{X}^{k-1})_j = d_j + \sum_{e\in E_j} \frac{s_e}{\alpha_e^k}\beta_e(x_j)$$

$$\Rightarrow \sum_{e\in E_j} \frac{s_e}{\alpha_e^k}\beta_e(x_j) = -d_j.$$

We show that $\frac{s_e}{\alpha_e^k}\beta_e(y_j) = -1$ for all $e \in E_j$. First we show that $\frac{s_e}{\alpha_e^k}|\beta_e(y_j)| = 1 \quad \forall$. By contradiction suppose that there exists $e^* \in E_j$ such that for all $e \in E_j$, $\frac{s_e}{\alpha_e^k}|\beta_e(y_j)| < 1$. Thus,

$$d_j = |-d_j| = \Big|\sum_{e\in E_j} \frac{s_e}{\alpha_e^k}\beta_e(x_j)\Big|$$

$$\leq \sum_{e\in E_j} \frac{s_e}{\alpha_e^k}|\beta_e(x_j)| < d_j,$$

that is a contradiction. Now suppose that $\frac{s_e}{\alpha_e^k}\beta_e(y_j) = a_e + ib_e$ for arbitrary $e \in E_j$, we have $\sum_{e\in E_j} \frac{s_e}{\alpha_e^k}\beta_e(x_j) = -d_j$ which shows that $\sum_{e\in E_j} a_e = -d_j$ and $\sum_{e\in E_j} b_e = 0$. On the other hand $a_e^2 + b_e^2 = 1$, then $-1 \leq a_e \leq 1$ and $-1 \leq b_e \leq 1$. We show that $a_e = -1$ for all $e \in E_j$. Suppose that there exists $e^* \in E_j$ such that $a_{e^*} > -1$, then $-d_j < \sum_{e\in E_j} a_e = -d_j$, and therefore $\sum_{e\in E_j} b_e = 0$. Thus $\frac{s_e}{\alpha_e^k}\beta_e(y_j) = -1$ for all $e \in E_j$. Since $|x_t| \leq 1$ for all $t \in V$ and $\frac{s_e}{\alpha_e^k}|\beta_e(y_j)| = 1$, for all $e \in E_j$, we have $|x_t| = 1$, where $t \in e$. Now the result follows from the connectivity of $\mathcal{H}_i$. ■

The situation for odd-bipartite non-uniform hypergraphs is principally different. Unlike odd-bipartite uniform hypergraphs [40, 20, 37], because of considering permutations with repetition of vertices of an edge in non-uniform hypergraphs, it can't be obtained similar results to uniformity case. In the following, we only propose one theorem that gives a necessary condition for equality of $Hspec(\mathcal{L})$ and $Hspec(\mathcal{Q})$. First, consider the following lemma:

Lemma 1. *Let $\mathcal{H} = (V, E)$ be a connected non-uniform k-hypergraph, then $\mathcal{A}$, the adjacency tensor of $\mathcal{H}$, is weakly irreducible.*

Proof. Suppose that $\mathcal{H}$ is connected, take matrix $M(\mathcal{A})$, such that

$$m_{ij} = \sum_{j\in\{i_2,i_3,\cdots,i_k\}} a_{ii_2\cdots i_k} = \sum_{e=\{i,j,l_3,\cdots,l_{s_e}\}} \frac{s_e}{\alpha_e^k}\frac{\alpha_e^k}{s_e} = |E_{ij}|$$

By Definition(2.4), we should show that $M(\mathcal{A})$ is irreducible. By contradiction, suppose that $M(\mathcal{A})$ is reducible, then by definition there exists $I \subsetneqq \{1, \cdots, n\}$ such that $m_{ij} = 0$ for $i \in I$ and $j \notin I$. This shows that:

$$\nexists e \in E_{ij} \text{ such that } \quad i \in I \quad , \quad j \in V\setminus I$$

On the other word, there are not $e \in E$ containing an element of I and an element of $V\setminus I$ that contradicts the connectivity of $\mathcal{H}$. Thus $\mathcal{A}$ is weakly irreducible. ∎

Theorem 7. *Let $\mathcal{H} = (V, E)$ be a connected non-uniform k-hypergraph where $V = [n]$ and let $\mathcal{L}$ and $\mathcal{Q}$ be the Laplacian and signless Laplacian tensors of $\mathcal{H}$, respectively. If $Hspec(\mathcal{L}) = Hspec(\mathcal{Q})$, then k is even and $\mathcal{H}$ is odd-bipartite.*

Proof. Since $\mathcal{H}$ is connected then by Lemma 1, $\mathcal{A}$ and thus $\mathcal{Q}$ are weakly irreducible. Therefore by proof of Theorem 2.2 and Lemma 2.1 in [40], there exists a diagonal matrix ,P, of order n with all the diagonal entries ± 1 and $P \neq -I_n$ such that $\mathcal{L} = P^{-(k-1)}\mathcal{Q}P$. Now we have:

$$\mathcal{L} = P^{-(k-1)}\mathcal{Q}P \Leftrightarrow \mathcal{D} - \mathcal{A} = P^{-(k-1)}(\mathcal{D} + \mathcal{A})P \quad \Leftrightarrow -\mathcal{A} = P^{-(k-1)}\mathcal{A}P \tag{5.3}$$

Now let $V_1 = \{i \in V | p_{ii} = -1\}$. $V_1 \neq \phi$ for otherwise $P = I_n$ and thus $\mathcal{A} = 0$ that is a contradiction. On the other hand, since $P \neq -I_n$ then $V_1 \subsetneqq V$. Now by (5.3) we have:

$$-a_{i_1i_2\cdots i_k} = p_{i_1i_1}^{-(k-1)} p_{i_2i_2} \cdots p_{i_ki_k} a_{i_1i_2\cdots i_k} \tag{5.4}$$

Suppose that $e = \{i, j, l_3, \cdots, l_s\}$ is an edge in $\mathcal{H}$, then $a_{i_1i_2\cdots i_k} \neq 0$ for every permutation with repetition of $i, j, l_3, \cdots, l_s$ with at least once for each element of the set $\{i, j, l_3, \cdots, l_s\}$. Suppose that $i, j, i_3 \cdots i_k$ is a permissible permutation with repetition of $i, j, l_3, \cdots, l_s$, then $a_{iji_3\cdots i_k} \neq 0$ and therefore by (5.4) we have $-p_{ii}^k = p_{ii}p_{jj} \cdots p_{i_ki_k}a_{i_1i_2\cdots i_k}$. Similarly for the permissible permutation with repetition $j, i, i_3 \cdots i_k$ of $i, j, l_3, \cdots, l_s$, we have:

$$-p_{jj}^k = p_{jj}p_{ii} \cdots p_{i_ki_k}a_{i_1i_2\cdots i_k}$$

Thus $p_{ii}^k = p_{jj}^k$. Now if k is odd, then $p_{ii} = p_{jj}$ and since $\mathcal{H}$ is connected, then all diagonal entries of P are identital, therefore $P = I_n$ or $P = -I_n$ that is a contradiction. Thus k is even.

Now suppose that $e = \{l_1, l_2, \cdots, l_s\}$ is an edge in $\mathcal{H}$, without loss of generality assume that $e \cap V_1 = \{l_1, l_2, \cdots, l_r\}$, we show that r is odd. We consider a permissible permutation with repetition in which $k_1 = k_2 = \cdots = k_r = 1$ and $\sum_{t=r+1}^{s} k_t = k - r$. We have $i_1, i_2, \cdots, i_k = l_1, l_2, \cdots, l_r, l_{r+1}^{k_{r+1}}, \cdots, l_s^{k_s}$. On the other hand, for this permutation, $-1 = p_{i_1 i_1}^{k} = p_{l_1 l_1} \cdots p_{l_r l_r}$. Consequently, r is odd. ∎

6. CONCLUSION

In this paper, we analyze the non-uniform hypergraph. Because of the existence of edges with different cardinality in the general hypergraph, we should consider all permutations with repetition of vertices in each edge. Then some theorems in uniform hypergraphs are fundamentally different here. We try to identify some of their properties that lead to different results in non-uniform hypergraphs.

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