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On Edge Mostar Index of Graphs

HECHAO LIU[1, 2], LING SONG[1], QIQI XIAO[1] AND ZIKAI TANG[1,•]

[1]School of Mathematics and Statistics, Hunan Normal University, Changsha, Hunan 410081, P. R. China
[2]School of Mathematical Sciences, South China Normal University, Guangzhou, 510631, P. R. China

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ABSTRACT

The edge Mostar index $Mo_e(G)$ of a connected graph G is defined as $Mo_e(G) = \sum_{e=uv\in E(G)} |m_u(e|G) - m_v(e|G)|$, where $m_u(e|G)$ and $m_v(e|G)$ are, respectively, the number of edges of G lying closer to vertex u than to vertex v and the number of edges of G lying closer to vertex v than to vertex u. In this paper, we determine the extremal values of edge Mostar index of some graphs. We characterize extremal trees, unicyclic graphs and determine the extremal graphs with maximum and second maximum edge Mostar index among cacti with size m and t cycles. At last, we give some open problems.

1. INTRODUCTION

In this paper, all graphs we consider are finite, undirected, and simple. Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. Let $|G|$ and $|E(G)|$ be the number of vertices and edges of G, respectively. For a vertex $u \in V(G)$, the degree of u, denoted by $d_G(u)$ (or simply $d(u)$), is the number of vertices which are adjacent to u. Call a vertex u a pendent vertex of G, if $d(u) = 1$ and call an edge uv a pendent edge of G, if $d(u) = 1$ or $d(v) = 1$. C_n, S_n and P_n denote the cycle, star, and path with n vertices, respectively. For $v \in V(G)$, let $G - v$ be a subgraph of G obtained by deleting vertex v and adjacent edges. For $e \in E(G)$, let $G - e$ be a subgraph of G obtained by deleting edge e.

Among all the topological indices, the most well-known is the Wiener index [8], which is defined as the sum of distances over all unordered vertex pairs in G, namely

•Corresponding Author (Email address: zikaitang@163.com)
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$W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v)$. A long time known property of the Wiener index is the formula [8]

$$W(G) = \sum_{e=uv\in E(G)} n_u(e|G)n_v(e|G),$$

where $n_u(e|G)$ and $n_v(e|G)$ are, respectively, the number of vertices of G lying closer to vertex u than to vertex v and the number of vertices of G lying closer to vertex v than to vertex u. It is applicable for trees. Using the above formula, another topological index named the Szeged index [3], was introduced by Gutman, which is an extension of the Wiener index and defined by

$$Sz(G) = \sum_{e=uv\in E(G)} n_u(e|G)n_v(e|G).$$

Given an edge $e = uv \in E(G)$, the distance between the vertex x and the edge e, denoted by $d(x,e)$, is defined as $d(x,e) = \min\{d(x,u), d(x,v)\}$. Denote $M_u(e|G) = \{e \in E(G): d(u,e) < d(v,e)\}$ and $M_v(e|G) = \{e \in E(G): d(v,e) < d(u,e)\}$. Let $m_u(e|G) = |M_u(e|G)|$ and $m_v(e|G) = |M_v(e|G)|$. Then, the edge Szeged index [4] of G is defined as

$$Sz_e(G) = \sum_{e=uv\in E(G)} m_u(e|G)m_v(e|G).$$

Szeged index and edge Szeged index belongs to the class of bond-additive indices. Recently, another bond-additive topological index, named the Mostar index, has been introduced [2]. The Mostar index of a graph G is defined as

$$Mo(G) = \sum_{e=uv\in E(G)} |n_u(e|G) - n_v(e|G)|.$$

In [2], Došlić et al. proposed and investigated the Mostar index as a measure of peripherality in graphs. They determined its extremal values and characterized extremal trees and unicyclic graphs and gave a cut method for computing the Mostar index of benzenoid systems. In [6], Tepeh proved a conjecture of [2] on a characterization of bicyclic graphs with given number of vertices. One can refer [1,5,7] for more and some other details on the Mostar index.

The edge Mostar index [1] of a graph G is defined as

$$Mo_e(G) = \sum_{e=uv\in E(G)} |m_u(e|G) - m_v(e|G)|.$$

For the sake of simplicity, we consider the contribution $\phi(e)$ of an edge $e = uv$ defined as $\phi(e) = |m_u(e|G) - m_v(e|G)|$. The edge Mostar index is also one of the bond-additive indices. Edge Mostar index has also been introduced recently as a quantitative refinement of the distance nonbalancedness, and it can also measure peripherality of every edge and consider the contributions of all edges into a global measure of peripherality for a given chemical graph.

A connected graph is a cactus if any two cycles have at most one common vertex. A cycle in a cactus is called end-block if all but one vertex of the cycle have degree two. A bundle is a cactus that all cycles in the cactus have exactly one common vertex. Denoted by $C(m,t)$ the class of all cactus with m edges in cycle and t cycles.

In this paper, we determine the extremal values of edge Mostar index of some graphs. We characterize extremal trees, unicyclic graphs and determine the extremal graphs with maximum and second maximum edge Mostar index among cacti with size m and t cycles. At last, we give some open problems.

2. PRELIMINARY RESULTS

Lemma 2.1. *Let $e = uv$ be a cut edge of connected graph G. Then*

$$\phi(e) = |m_u(e|G) - m_v(e|G)| \leq m - 1$$

with equality if and only if e is a pendent edge.

Lemma 2.2. (The edge-lifting transformation) *Let G be a graph with a cut, not pendent edge $e = uv$. G' is the graph obtained by contracting the edge e and adding a pendent edge $e' = wz$ at the contracting vertex w, see Figure 1. Then $Mo_e(G) < Mo_e(G')$.*

Proof. From the definition of edge Mostar index, we known that $\phi_G(e) \leq m - 3$ and $\phi_{G'}(e) = m - 1$. The contribution of other edges stays unchanged. Then $Mo_e(G) - Mo_e(G') \leq -2 < 0$. So, $Mo_e(G) < Mo_e(G')$. ■

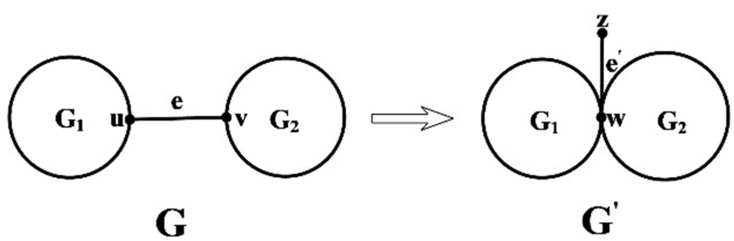

Figure 1. The edge-lifting transformation.

3. THE EXTREMAL TREES AND UNICYCLIC GRAPHS

Theorem 3.1. *Let G be a tree with m $(m \geq 4)$ edges. Then*

$$Mo_e(P_{m+1}) < Mo_e(L_2) \leq Mo_e(G) \leq Mo_e(L_1) < Mo_e(S_{m+1}),$$

for graphs L_1 and L_2 presented in Figure 2.

Proof. Using the edge-lifting transformation of Lemma 2.2 repeatedly, we have that

$$Mo_e(G) \leq Mo_e(L_1) = m^2 - m - 2 < Mo_e(S_{m+1}) = m^2 - m.$$

Suppose that G is a tree with m edges and G is not a path. Then, there exists a vertex z of degree at least three such that at least two components of $G - z$ are paths. Denote the two paths are $P_s = u_1u_2\cdots u_s$ and $P_t = v_1v_2\cdots v_t$ $(1 \le s \le t)$. Let $G' = G - \{u_{s-1}u_s\} + \{v_tu_s\}$. Then

$$\begin{aligned} Mo_e(G) - Mo_e(G') = &[(m-1) + (m-3) + \cdots + (m-2s+3) + (m-2s+1)] \\ &+ [(m-1) + (m-3) + \cdots + (m-2t+3) + (m-2t+1)] \\ &- [(m-1) + (m-3) + \cdots + (m-2s+5) + (m-2s+3)] \\ &- [(m-1) + (m-3) + \cdots + (m-2t+1) + (m-2t-1)] \\ &= 2(t-s) + 2 > 0. \end{aligned}$$

By computation, we have that $Mo_e(P_{m+1}) = \frac{1}{2}m^2$ for $m \equiv 0\ (mod\ 2)$; $Mo_e(P_{m+1}) = \frac{1}{2}(m^2-1)$ for $m \equiv 1\ (mod\ 2)$. It means that $Mo_e(P_{m+1}) = \lfloor\frac{1}{2}m^2\rfloor$. $Mo_e(L_2) = \frac{1}{2}m^2 + 2$ for $m \equiv 0\ (mod\ 2)$; $Mo_e(L_2) = \frac{1}{2}(m^2+3)$ for $m \equiv 1\ (mod\ 2)$. It means that $Mo_e(L_2) = \lfloor\frac{1}{2}m^2\rfloor + 2$. Such that $Mo_e(G) \ge Mo_e(L_2) = \lfloor\frac{1}{2}m^2\rfloor + 2 > Mo_e(P_{m+1}) = \lfloor\frac{1}{2}m^2\rfloor$.

The proof is completed. ■

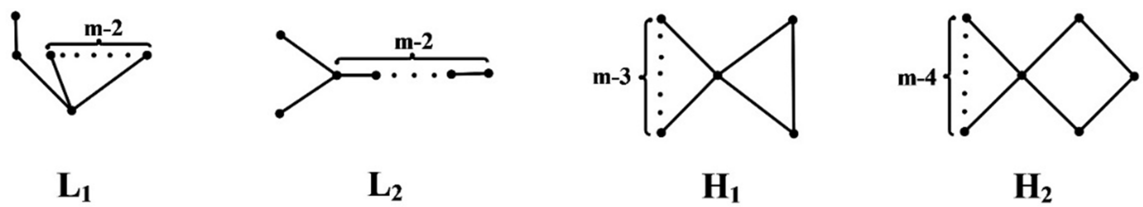

Figure 2. The extremal trees and unicyclic graphs.

If G is a unicyclic graph, It is obvious that $Mo_e(G) \ge Mo_e(C_m) = 0$.

Lemma 3.2. *Let G be a unicyclic graphs with m edges, and the unique cycle C_g. Then*

$$Mo_e(G) \le \begin{cases} (m-g)(m+g-1), & g \equiv 0\ (mod\ 2) \\ (m-g)(m+g+2), & g \equiv 1\ (mod\ 2) \end{cases},$$

with equality if and only if G is obtained from C_g by attaching $m-g$ pendent edges at the same one vertex of C_g.

Proof. Suppose that G is a unicyclic graph with the unique cycle $C_g = v_1v_2\cdots v_gv_1$. Repeating the edge-lifting transformation of Lemma 2.2, the edge of $E(G)\backslash E(C_g)$ are all pendent edge. Denote m_j $(1 \le j \le g)$ the number of pendent edges attached at v_j, then $\sum_{j=1}^{g} m_j = m - g$.

i. $g \equiv 0\ (mod\ 2)$. For $j \equiv 0\ (mod\ g)$, $\phi(v_jv_{j+1}) = |\sum_{k=1}^{\frac{g}{2}} m_{j+k} - \sum_{k=\frac{g}{2}+1}^{g} m_{j+k}| \le \sum_{j=1}^{g} m_j = m - g$. As the arbitrariness of j, the equality holds if and only if all

$m-g$ pendent edges attached at the same one vertex of C_g. Such $\sum_{e\in E(G)}\phi(e) \le (m-1)(m-g)+(m-g)g=(m-g)(m+g-1)$, the equality holds if and only if all cut edges are pendent edges and all pendent edges attached at the same one vertex of C_g.

ii. $g \equiv 1\ (mod\ 2)$. For $j \equiv 1\ (mod\ g)$, $\phi(v_j v_{j+1}) = |\sum_{k=1}^{\frac{g}{2}-1} m_{j+k} - \sum_{k=\frac{g+3}{2}}^{g} m_{j+k}| \le m-g-m_j$. As the arbitrariness of j, the equality holds if and only if all $m-g$ pendent edges attached at the same one vertex of C_g. Such $\sum_{e\in E(G)}\phi(e) \le (m-1)(m-g)+(m-g)(g-1)=(m-g)(m+g-2)$, the equality holds if and only if all cut edges are pendent edges and all pendent edges attached at the same one vertex of C_g.

The proof is completed. ■

Theorem 3.3. *Let G be a unicyclic graphs with m edges, then*

$$Mo_e(G) \le \begin{cases} m^2-2m-3, & 3\le m\le 8 \\ 60, & m=9 \\ m^2-m-12, & m\ge 10 \end{cases},$$

with equality if and only if $G \cong H_1$ (see Figure 2) for $3\le m\le 8$; $G\cong H_1$ or $G\cong H_2$ for $m=9$; $G\cong H_2$ (Figure 2) for $m\ge 10$.

Proof. By Lemma 3.2, if $g\equiv 0\ (mod\ 2)$, then $Mo_e(G)\le (m-g)(m+g-1)\le (m-4)(m+3)=m^2-m-12$, with equality if and only if $g=4$ and all $m-4$ pendent edges attached at the same one vertex of C_4, i.e. $G\cong H_2$. If $g\equiv 1\ (mod\ 2)$, then $Mo_e(G)\le (m-g)(m+g-2)\le (m-3)(m+1)=m^2-2m-3$, with equality if and only if $g=3$ and all $m-3$ pendent edges attached at the same one vertex of C_3, i.e. $G\cong H_1$.

Comparing the edge Mostar index of H_1 and H_2, $Mo_e(H_1)-Mo_e(H_2)=9-m$. Such that, if $3\le m\le 8$, then $Mo_e(G)\le m^2-2m-3$ with equality if and only if $G\cong H_1$; if $m=9$, then $Mo_e(G)\le 60$ with equality if and only if $G\cong H_1$ or $G\cong H_2$; if $m\ge 10$, then $Mo_e(G)\le m^2-m-12$ with equality if and only if $G\cong H_2$.

The proof is completed. ■

4. THE MAXIMUM VALUE OF EDGE MOSTAR INDEX AMONG CACTI

In the following, we give the sharp upper bounds of edge Mostar index among cacti.

Lemma 4.1. *Let G be a connected graph with a cycle C_g and $G-E(C_g)$ has g connected components. Then*

$$\sum_{e=uv\in E(C_g)}\phi(e)\leq\begin{cases}g(m-g), & g\equiv 0\ (mod\ 2)\\ (g-1)(m-g), & g\equiv 1\ (mod\ 2)\end{cases},$$

with equality if and only if C_g is an end-block.

Proof. Let $C_g = v_1v_2\cdots v_gv_1$. Denoted by G_j the components of $G-E(C_g)$ that contains v_j for $1\leq j\leq g$. Let $e_g = v_gv_1$ and $e_{gj} = v_jv_{j+1}$ $(1\leq j\leq g-1)$. Denote $m_j = E(G_j)$, then $\sum_{j=1}^{g} m_j = m-g$.

(i) $g\equiv 0\ (mod\ 2)$.

For $e_g = v_gv_1\in E(C_g)$, we have that $M_{v_1}(e) = E(G_1)\cup E(G_2)\cup\cdots\cup E\left(G_{\frac{g}{2}}\right)\cup\{e_1,e_2,\dots,e_{\frac{g}{2}-1}\}$ and $M_{v_g}(e) = E(G_{\frac{g}{2}+1})\cup E(G_{\frac{g}{2}+2})\cup\cdots\cup E(G_g)\cup\{e_{\frac{g}{2}+1},e_{\frac{g}{2}+2},\dots,e_{g-1}\}$. If $\sum_{j=1}^{\frac{g}{2}} m_j\geq\sum_{j=\frac{g}{2}+1}^{g} m_j$, then

$$\phi(e) = |\sum_{j=1}^{\frac{g}{2}} m_j-\sum_{j=\frac{g}{2}+1}^{g} m_j| = \sum_{j=1}^{g} m_j-2\sum_{j=\frac{g}{2}+1}^{g} m_j\leq m-g,$$

equality holds if and only if $m_j = 0$ for $j = \frac{g}{2}+1,\frac{g}{2}+2,\dots,g$. If $\sum_{j=1}^{\frac{g}{2}} m_j\leq\sum_{j=\frac{g}{2}+1}^{g} m_j$, then

$$\phi(e) = |\sum_{j=1}^{\frac{g}{2}} m_j-\sum_{j=\frac{g}{2}+1}^{g} m_j| = \sum_{j=1}^{g} m_j-2\sum_{j=1}^{\frac{g}{2}} m_j\leq m-g,$$

equality holds if and only if $m_j = 0$ for $j = 1,2,\dots,\frac{g}{2}$.

Similarly, we have that $\phi(e_k) = |m_{v_k}(e_k)-m_{v_k+1}(e_k)|\leq m-g$ $(1\leq k\leq g-1)$, equality holds if and only if $m_j = 0$ for $j = k-\frac{g}{2},k-\frac{g}{2}+1,\dots,k$ or $m_j = 0$ for $j = k+1,k+2,\dots,k+\frac{g}{2}$, where $j\equiv 0\ (mod\ g)$. Thus, $\sum_{e=uv\in E(C_g)}\phi(e)\leq g(m-g)$, with equality if and only if C_g is an end-block.

(ii) $g\equiv 1\ (mod\ 2)$.

For $e_g = v_gv_1\in E(C_g)$, we have that $M_{v_1}(e) = E(G_1)\cup E(G_2)\cup\cdots\cup E(G_{\frac{g-1}{2}})\cup\{e_1,e_2,\dots,e_{\frac{g-1}{2}}\}$ and $M_{v_g}(e) = E(G_{\frac{g+3}{2}})\cup E(G_{\frac{g+5}{2}})\cup\cdots\cup E(G_g)\cup\{e_{\frac{g+1}{2}},e_{\frac{g+3}{2}},\dots,e_{g-1}\}$.If $\sum_{j=1}^{\frac{g-1}{2}} m_j\geq\sum_{j=\frac{g+3}{2}}^{g} m_j$, then

$$\phi(e) = |\sum_{j=1}^{\frac{g-1}{2}} m_j-\sum_{j=\frac{g+3}{2}}^{g} m_j| = \sum_{j=1}^{g} m_j-m_{\frac{g+1}{2}}-2\sum_{j=\frac{g+3}{2}}^{g} m_j\leq m-g-m_{\frac{g+1}{2}},$$

equality holds if and only if $m_j = 0$ for $j = \frac{g+3}{2},\frac{g+5}{2},\dots,g$. If $\sum_{j=1}^{\frac{g-1}{2}} m_j\leq\sum_{j=\frac{g+3}{2}}^{g} m_j$, then

$$\phi(e) = |\sum_{j=1}^{\frac{g-1}{2}} m_j - \sum_{j=\frac{g+3}{2}}^{g} m_j| = \sum_{j=1}^{g} m_j - m_{\frac{g+1}{2}} - 2\sum_{j=1}^{\frac{g-1}{2}} m_j \le m - g - m_{\frac{g+1}{2}},$$

equality holds if and only if $m_j = 0$ for $j = 1, 2, \dots, \frac{g-1}{2}$.

Similarly, we have that $\phi(e_k) = |m_{v_k}(e_k) - m_{v_{k+1}}(e_k)| \le m - g$ $(1 \le k \le g - 1)$, equality holds if and only if $m_j = 0$ for $j = k - \frac{g-3}{2}, k - \frac{g-5}{2}, \dots, k$ or $m_j = 0$ for $j = k + 1, k + 2, \dots, k + \frac{g-1}{2}$, where $j \equiv 0\ (mod\ g)$. Thus,

$$\sum_{e=uv\in E(C_g)} \phi(e) \le \sum_{j=1}^{g} (m - g - m_j) \le (g-1)(m-g),$$

with equality if and only if C_g is an end-block.

So, the proof is completed. ■

Denote $G^m(g_1, g_2, \dots, g_t)$ a bundle of t cycles with lengths $g_1, g_2, \dots, g_t$ and $m - \sum_{j=1}^{t} g_j$ pendent edges attached to the unique common vertices of all cycles. Let $\mathcal{G}_m$ be the set of $G^m(g_1, g_2, \dots, g_t)$ with $g_j = 3$ or $g_j = 4$ for $j = 1, 2, \dots, t$.

Lemma 4.2. *For any graph $G \in \mathcal{C}(m, t)$, suppose that $C_1, C_2, \dots, C_t$ be the edge-disjoint cycles. Denote $g_j = |C_j|$ for $j = 1, 2, \dots, t$,* where $g_j \equiv 1\ (mod\ 2)$ $(j = 1, 2, \dots, r)$ and $g_j \equiv 0\ (mod\ 2)$ $(j = r + 1, r + 2, \dots, t)$. Then

$$Mo_e(G) \le m^2 - m(r+1) - \sum_{j=1}^{r} g_j (g_j - 2) - \sum_{j=r+1}^{t} g_j (g_j - 1),$$

with equality if and only if $G \cong G^m(g_1, g_2, \dots, g_t)$.

Proof. Denote E^* the set of all cut edge of G. Then $E^* = E(G)\{\cup_{j=1}^{t} E(C_j)\}$ and $|E^*| = m - \sum_{j=1}^{t} g_j$. By Lemma 2.1, we have that $\sum_{e\in E^*} \phi(e) \le (m-1)(m - \sum_{j=1}^{t} g_j)$, with equality if and only if all cut edges are pendent edges.

By Lemma 4.1, we have that (1) If $j = 1, 2, \dots, r$, then $\sum_{e\in E(C_j)} \phi(e) \le (g_j - 1)(m - \sum_{j=1}^{t} g_j)$, with equality if and only if C_j is an end-block. (2) If $j = r + 1, r + 2, \dots, t$, then $\sum_{e\in E(C_j)} \phi(e) \le g_j(m - \sum_{j=1}^{t} g_j)$, with equality if and only if C_j is an end-block. With the definition of edge Mostar index, we have that

$$\begin{aligned} Mo_e(G) &\le (m-1)\left(m - \sum_{j=1}^{t} g_j\right) - \sum_{j=1}^{r} (g_j - 1)\left(m - g_j\right) - \sum_{j=r+1}^{t} g_j \left(m - g_j\right) \\ &= m(m-1) - \sum_{j=1}^{t} g_j (m-1) + \sum_{j=1}^{t} g_j \left(m - g_j\right) - \sum_{j=1}^{r} \left(m - g_j\right) \\ &= m(m-1) - \sum_{j=1}^{t} g_j \left(g_j - 1\right) - \sum_{j=1}^{r} \left(m - g_j\right) \\ &= m^2 - m(r+1) - \sum_{j=1}^{r} g_j (g_j - 2) - \sum_{j=r+1}^{t} g_j (g_j - 1), \end{aligned}$$

with equality if and only if all cut edges are pendent edges and all cycles are end-blocks, i.e. $G \cong G^m(g_1, g_2, \cdots, g_t)$.

Hence, the proof is completed. ■

Theorem 4.3. *Let $G \in \mathcal{C}(m,t)$ be a connected graph. Then*

(1) *If $m \geq 10$ and $m < 4t$, then $Mo_e(G) \leq 2m^2 - 8m + (24-4m)t$ with equality if and only if $G \cong G^m(\underbrace{3,3,\dots,3}_{4t-m}, \underbrace{4,4,\dots,4}_{m-3t})$.*

(2) *If $m \geq 10$ and $m \geq 4t$, then $Mo_e(G) \leq m^2 - m - 12t$ with equality if and only if $G \cong G^m(4,4,\dots,4)$.*

(3) *If $m = 9$, then $Mo_e(G) = 72 - 12t$ with equality if and only if $G \cong \mathcal{G}_9$.*

(4) *If $m < 9$, then $Mo_e(G) \leq m^2 - m - (m+3)t$ with equality if and only if $G \cong G^m(3,3,\dots,3)$.*

Proof. Suppose that $C_1, C_2, \dots, C_t$ are t edge-disjoint cycles of G and $g_j = |C_j|$ *for* $j = 1,2,\dots,t$, where $g_j \equiv 1\ (mod\ 2)$ $(j = 1,2,\cdots,r)$ and $g_j \equiv 0\ (mod\ 2)$ $(j = r+1, r+2,\dots,t)$. By Lemma 4.2, we have that $Mo_e(G) \leq Mo_e(G^m(g_1, g_2, \dots, g_t))$. Let

$$f(g_1, g_2, \dots, g_t) = Mo_e(G^m(g_1, g_2, \dots, g_t))$$
$$= m^2 - m(r+1) - \sum_{j=1}^{r} g_j(g_j - 2) - \sum_{j=r+1}^{t} g_j(g_j - 1).$$

Then $\frac{\partial f(g_1, g_2, \dots, g_t)}{\partial g_j} = -4g_j - 1 < 0$. So, $f(g_1, g_2, \dots, g_t)$ is decreased for g_j $(1 \leq j \leq t)$. Hence, $f(g_1, g_2, \dots, g_t) \leq f(\underbrace{3,3,\dots,3}_{r}, \underbrace{4,4,\cdots,4}_{t-r}) = m^2 - m - 12t - r(m-9)$.

Denote $H(r) = m^2 - m - 12t - r(m-9)$, $H'(r) = 9 - m$. Note that if $m \geq 10$ and $m - 4t < 0$, then there are at least s triangles, where $3s + 4(t-s) = m$, i.e., $s = 4t - m > 0$. So we have that

If $m \geq 10$ and $m < 4t$, then $H'(r) < 0$ and $Mo_e(G) \leq H(4t-m) = 2m^2 - 8m + (24-4m)t$ with equality if and only if $G \cong G^m(\underbrace{3,3,\dots,3}_{4t-m}, \underbrace{4,4,\dots,4}_{m-3t})$.

If $m \geq 10$ and $m \geq 4t$, then $H'(r) < 0$ and $Mo_e(G) \leq H(0) = m^2 - m - 12t$ with equality if and only if $G \cong G^m(4,4,\dots,4)$.

If $m = 9$, then $H'(r) = 0$ and $Mo_e(G) \leq f(g_1, g_2, \dots, g_t) \leq H(r) = 72 - 12t$ with equality if and only if $G \cong \mathcal{G}_9$.

If $m < 9$, then $H'(r) > 0$ and $Mo_e(G) \leq f(g_1, g_2, \dots, g_t) \leq H(r) \leq H(t) = m^2 - m - (m+3)t$ *with equality if and only if* $G \cong G^m(3,3,\dots,3)$.

The proof is completed. ■

If $t = 1$, $\mathcal{C}(n,1)$ is the set of unicyclic graphs. The maximum edge Mostar index among $\mathcal{C}(n,1)$ are determined, which is consistent with the result of the Theorem 3.3.

5. THE SECOND MAXIMUM VALUE OF EDGE MOSTAR INDEX AMONG CACTI

In the following, we will determine the unique graph in $\mathcal{C}(m,t)$with second maximum edge Mostar index. We assume that $m \geq 10$ *and* $m \geq 4t$. Let

$$\mathcal{C}_0(m,t) \triangleq G^m(\underbrace{4,4,\cdots,4}_{t}).$$

Denote $\mathcal{C}_1(m,t)$ the graph that is obtained from $\mathcal{C}_0(m-1,t)$ by adding a pendent edge at a pendent vertex. If $G \in \mathcal{C}(m,t)\backslash\{\mathcal{C}_0(m,t)\}$, there are three possibilities:

(1) There exists a cycle that is not C_4;

(2) There exists a cycle that is not an end-block;

(3) There exists a cut edge that is not a pendent edge.

Lemma 5.1. *Let* $G \in \mathcal{C}(m,t)\backslash\{\mathcal{C}_0(m,t)\}$ *with* $m \geq 10$, $m \geq 4t$ *and there exists a cycle* that is *not* C_4. *Then*

(1) *If* G *has odd cycle, then* $Mo_e(G) \leq m^2 - 2m - 12t + 9$ *with equality if and only if* $G \cong G^m(3,\underbrace{4,4,\ldots,4}_{t-1})$;

(2) *If all cycle of* G *are even, then* $Mo_e(G) \leq m^2 - m - 12t - 18$ *with equality if and only if* $G \cong G^m(6,\underbrace{4,4,\ldots,4}_{t-1})$.

Proof. (1) If G has odd cycle, then $r \geq 1$. By Lemma 4.2 and Thereom 4.3, we have that

$$Mo_e(G) \leq f(g_1,g_2,\ldots,g_t) \leq (\underbrace{3,3,\ldots,3}_{r},\underbrace{4,4,\ldots,4}_{t-r}) \leq (3,\underbrace{4,4,\ldots,4}_{t-1})$$
$$= m^2 - 2m - 12t + 9,$$

with equality if and only if $G \cong G^m(3,\underbrace{4,4,\ldots,4}_{t-1})$.

(2) If all cycle of G are even, then $r = 0$. By Lemma 4.2 and Thereom 4.3, we have that $Mo_e(G) \leq f(g_1,g_2,\ldots,g_t) \leq f(6,\underbrace{4,4,\ldots,4}_{t-1}) = m^2 - m - 12t - 18$ with equality if and only if $G \cong G^m(6,\underbrace{4,4,\ldots,4}_{t-1})$. The proof is completed. ■

Lemma 5.2. *Let* $G \in \mathcal{C}(m,t)\backslash\{\mathcal{C}_0(m,t)\}$ *with* $m \geq 10$, $m \geq 4t$ *and there exists a cycle that is not an end-block. Then* $Mo_e(G) \leq m^2 - 2m - 12t + 9$ *or* $Mo_e(G) \leq m^2 - m - 12t - 2$.

Proof. If there exists a cycle that is not C_4, then by Lemma 5.1, one knowns that $Mo_e(G) \leq m^2 - 2m - 12t + 9$ or $Mo_e(G) \leq m^2 - m - 12t - 18$. In the following, we assume that all cycles are C_4 and $C = v_1v_2v_3v_4v_1$is not an end-block.

(1) If $d(v_1) \geq 3$ and $d(v_2) \geq 3$, then

$$\sum_{e\in E(C)} \phi(e) \leq 2(m-4) + 2(m-6) = 4m - 20.$$

(2) If $d(v_1) \geq 3$ and $d(v_3) \geq 3$, then $\sum_{e\in E(C)} \phi(e) \leq 4(m-6) = 4m - 24 < 4m - 20$.

Then

$$\begin{aligned} Mo_e(G) &\leq (m-1)(m-4t) + 4(m-4)(t-1) + 4m - 20 \\ &= m^2 - m - 12t - 4 \\ &< m^2 - m - 12t - 2. \end{aligned}$$

The proof is completed. ■

Lemma 5.3. *Let $G \in \mathcal{C}(m,t)\backslash\{\mathcal{C}_0(m,t)\}$ with $m \geq 10$, $m \geq 4t$ and there exists a cut edge that is not a pendent edge. Then $Mo_e(G) \leq m^2 - m - 12t - 2$ with equality if and only if $G \cong \mathcal{C}_1(m,t)$.*

Proof. Suppose that $e = uv$ is a cut edge that is not a pendent edge. Then $1 \leq m_u(e), m_v(e) \leq m - 2$, such $\phi(e) \leq m - 3$ with equality if and only if one component of $G - e$ contains a single edge.By Lemma 4.2 and Theorem 4.3, we have that

$$\begin{aligned} Mo_e(G) &\leq m - 3 + (m-1)\left(m - \textstyle\sum_{j=1}^{t} g_j - 1\right) \\ &\quad + \textstyle\sum_{j=1}^{r}(g_j - 1)(m - g_j) + \sum_{j=r+1}^{t} g_j\,(m - g_j) \\ &= f(g_1, g_2, \dots, g_t) - 2 \\ &\leq f(\underbrace{3,3,\dots,3}_{r}, \underbrace{4,4,\dots,4}_{t-r}) - 2 \\ &= m^2 - m - 12t - r(m-9) - 2 \\ &= H(r) - 2 \leq H(0) - 2 \\ &= m^2 - m - 12t - 2 \end{aligned}$$

with equality if and only if all cycles are C_4 and end-block, $e = uv$ is the only cut edge that is not a pendent edge, one component of $G - e$ containing a single edge, i.e. $G \cong \mathcal{C}_1(m,t)$. The proof is completed. ■

By Lemma 5.1, 5.2, 5.3, we have the main result.

Theorem 5.4. *Let $G \in \mathcal{C}(m,t)\backslash\{\mathcal{C}_0(m,t)\}$ with $m \geq 10$, $m \geq 4t > 0$.Then*

(1) $Mo_e(G) \leq 89 - 12t$ *for* $m = 10$ *with equality if and only if* $G \cong G(3, \underbrace{4,4,\dots,4}_{t-1})$.

(2) $Mo_e(G) \leq 108 - 12t$ *for* $m = 11$ *with equality if and only if* $G \cong G(3, \underbrace{4,4,\dots,4}_{t-1})$ *or* $G \cong \mathcal{C}_1(m,t)$.

(3) $Mo_e(G) \leq m^2 - m - 12t - 2$ *for* $m \geq 12$ *with equality if and only if* $G \cong \mathcal{C}_1(m,t)$.

Let $\Theta_{a,b,c}$ be the Theta graph which is consisted by the three internally disjoint paths P_a, P_b, P_c of lengths a, b, c, respctively. If the bicyclic graphs are cacti, then through the results of Theorem 4.3 and 5.4, we can get the extremal graph. If there exists the Theta graph among bicyclic graphs, then we have the following conjectures.

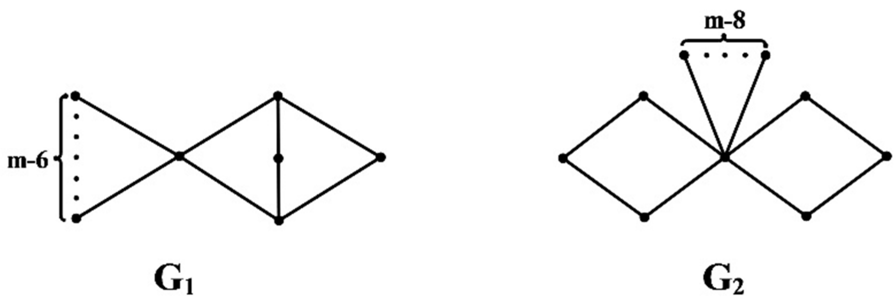

Figure 3. The extremal bicyclic graphs G_1 and G_2.

Conjecture 5.5. *If the size* m *of bicyclic graphs is large enough, then* $\Theta_{m-4,2,2}$ *has the minimum edge Mostar index.*

Conjecture 5.6. *If the size* m *of bicyclic graphs is large enough, then the bicyclic graphs* G_1 *and* G_2 (*see Figure* 3) *have the maximum edge Mostar index.*

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