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Some Topological Indices of Edge Corona of Two Graphs

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ABSTRACT

In this paper, we compute the Wiener index, first Zagreb index, second Zagreb index, degree distance index and Gutman index of edge corona of two graphs.

1. INTRODUCTION

Throughout this paper we consider only simple connected graphs, i.e., connected graphs without loops and multiple edges. Let $G=(V,E)$ be a graph with vertex set $V(G)=\{v_1,v_2,\ldots,v_n\}$ and edge set $E(G)=\{e_1,e_2,\ldots,e_m\}$. We denote the shortest distance between two vertices u and v in G by $d(u,v)$ and the degree of a vertex v in G by $d(v)$. A topological index is a real number derived from the structure of a graph, which is invariant under graph isomorphism. The Wiener index $W(G)$ of a graph G is defined as

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$W(G)=\sum_{\{u,v\}\subseteq V(G)} d(u,v)$. The Wiener index is a classical distance based topological index. It was first introduced by H. Wiener [26] with an application to chemistry. The Wiener index has been extensively studied by many chemists and mathematicians. For more details about Wiener index the reader may refer to [5, 6, 12, 16, 20]. The first and second Zagreb indices of a graph denoted by $M_1(G)$ and $M_2(G)$, respectively, are degree based topological indices introduced by Gutman and Trinajstić [15]. These two indices are defined as

$$M_1(G)=\sum_{e=uv\in E(G)} (d(u)+d(v))=\sum_{v\in V(G)} d(v)^2,$$

and $M_2(G)=\sum_{e=uv\in E(G)} d(u)d(v)$. More information about Zagreb indices can be found in [13, 14, 19]. The degree distance, $DD(G)$, and Gutman index $Gut(G)$ of a graph G are defined, respectively, as

$$DD(G)=\sum_{\{u,v\}\subseteq V(G)} (d_G(u)+d_G(v))d_G(u,v),$$
$$Gut(G)=\sum_{\{u,v\}\subseteq V(G)} d_G(u)d_G(v)d_G(u,v).$$

The degree distance was introduced independently by A. A. Dobrynin, A. A. Kochetova [7] and Gutman [11]. The Gutman index was introduced by Gutman [11] which was earlier known as modified Schultz index. Studies on degree distance and Gutman index can be found in [2, 3, 8, 21, 25] and the references cited therein.

The corona [9] of two graphs G and H is the graph obtained by taking one copy of G, $|V(G)|$ copies of H and joining i-th vertex of G to every vertex in the i-th copy of H. The edge corona [17] of two graphs G and H denoted by $G\Diamond H$ is obtained by taking one copy of G and $|E(G)|$ copies of H and joining end vertices of i-th edge of G to every vertex in the i-th copy of H. Various researchers studied the topological indices of the corona of two graphs, for example see [23, 27]. Motivated by these works, in this paper we compute some topological indices of edge corona of two graphs. In Section 2, we derive formulas for the Wiener index and Zagreb indices of edge corona of two graphs. As special cases of these formulas we express Wiener index and Zagreb indices of edge corona $P_n\Diamond H$, $C_n\Diamond H$ and $K_n\Diamond H$, where H is an arbitrary graph, in terms of vertices and edges. In Section 3, we compute the degree distance and Gutman index of edge corona of two graphs and obtain formula in terms of vertices and edges for degree distance and Gutman index of edge corona of two graphs G and H, when $G = C_n$, P_n or K_n and $H = C_m$, P_m , K_m or $\overline{K}_m$.

2. WIENER INDEX, FIRST ZAGREB INDEX AND SECOND ZAGREB INDEX OF EDGE CORONA OF TWO GRAPHS

Let G_1 and G_2 be graphs with vertex sets $V(G_1)=\{v_1, v_2, \ldots, v_{n_1}\}$, $V(G_2)=\{u_1, u_2, \ldots, u_{n_2}\}$, and edge sets $E(G_1)=\{e_1, e_2, \ldots, e_{m_1}\}$ and $E(G_2)=\{e_1', e_2', \ldots, e_{m_2}'\}$. Let the vertex set of i-th copy of G_2 be denoted by $V(G_2)=\{u_{i1}, u_{i2}, \ldots, u_{in_2}\}$. In this section, we compute the Wiener index and Zagreb indices of edge corona of two graphs G_1 and G_2.

Let $e = uv$ and $f = xy$ be two distinct edges in G. The distance $d_G(e,f)$ between two edges e and f is given by $min\{d_G(x,u), d_G(x,v), d_G(y,u), d_G(y,v)\}$. The edge-Wiener index $W_e(G)$ [4] of a graph G, is the sum of all distances between unordered pairs of vertices in the line graph $L(G)$. Equivalently, edge-Wiener index of G is the Wiener index of the line graph $L(G)$ of G i.e.,

$$W_e(G) = W(L(G)) = \sum_{\{xy,uv\}\in E(G)} (min\{d_G(x,u), d_G(x,v), d_G(y,u), d_G(y,v)\} + 1).$$

Note that in literature different definitions of the edge-Wiener index and distance between two edges are proposed; for example see [18]. The vertex PI index [1] of a graph G is defined as

$$PI(G) = \sum_{e_i = v_l v_m \in E(G)} [n_{e_i}(v_l \mid G) + n_{e_i}(v_m \mid G)],$$

where $n_{e=uv}(u|G)$ is the number of vertices in G that are closer to u than v in G. It may be noted that if G is a bipartite graph, then $PI(G)=|V(G)||E(G)|$.

We need the following two lemmas to prove our main results. As the proofs directly follow from the definition of the edge corona, we omit the details.

Lemma 2.1 *We have*

1. $d_{G_1 \lozenge G_2}(v_i) = (n_2 + 1)\, d_{G_1}(v_i)$, $v_i \in V(G_1)$.
2. $d_{G_1 \lozenge G_2}(u_{ij}) = d_{G_2}(u_j) + 2$, $u_{ij} \in V_i(G_2)$.

Lemma 2.2 *We have*

1. $d_{G_1 \lozenge G_2}(v_i, v_j) = d_{G_1}(v_i, v_j)$, $v_i, v_j \in V(G_1)$.
2. $d_{G_1 \lozenge G_2}(u_{ij}, u_{ik}) = \begin{cases} 1, & u_j u_k \in E(G_2) \\ 2, & u_j u_k \notin E(G_2) \end{cases}$, $u_{ij}, u_{ik} \in V_i(G_2)$.
3. $d_{G_1 \lozenge G_2}(u_{ij}, u_{km}) = d_{G_1}(e_i, e_k) + 2$, $u_{ij} \in V_i(G_2)$ *and* $u_{km} \in V_k(G_2)$.
4. $$d_{G_1 \lozenge G_2}(u_{ij}, v_k) = \begin{cases} d_{G_1}(v_l, v_k) + 1, & d_{G_1}(v_l, v_k) = d_{G_1}(v_m, v_k), \\ & u_{ij} \in V_i(G_2), v_k \in V(G_1), \\ (d_{G_1}(v_l, v_k) + d_{G_1}(v_m, v_k) + 1)/2, & d_{G_1}(v_l, v_k) \neq d_{G_1}(v_m, v_k), \\ & u_{ij} \in V_i(G_2), v_k \in V(G_1), \end{cases}$$

where $v_l v_m = e_i$.

Theorem 2.3 *The Wiener index of* $G_1 \lozenge G_2$ *is given by*

$$W(G_1 \Diamond G_2) = W(G_1) + n_2^2 W_e(G_1) + \frac{n_2}{2}(DD(G_1) - PI(G_1)) + m_1 n_2 \left(\frac{n_2}{2}(m_1+1) + n_1 - 1 \right) - m_1 m_2.$$

Proof. We have

$$W(G_1 \Diamond G_2) = A_1 + A_2 + A_3 + A_4, \tag{2.1}$$

where

$$A_1 = \sum_{\{v_i, v_j\} \subseteq V(G_1)} d_{G_1 \Diamond G_2}(v_i, v_j),$$
$$A_2 = \sum_{e_i \in E(G_1)} \sum_{\{u_{ij}, u_{ik}\} \subseteq V_i(G_2)} d_{G_1 \Diamond G_2}(u_{ij}, u_{ik}),$$
$$A_3 = \sum_{\{e_i, e_k\} \subseteq E(G_1)} \sum_{u_{ij} \in V_i(G_2), u_{km} \in V_k(G_2)} d_{G_1 \Diamond G_2}(u_{ij}, u_{km}),$$
$$A_4 = \sum_{e_i \in E(G_1)} \sum_{u_{ij} \in V_i(G_2), v_k \in V(G_1)} d_{G_1 \Diamond G_2}(u_{ij}, u_k).$$

We now compute A_i for $i = 1,2,3,4$ .By Lemma 2.2, we have

$$A_1 = \sum_{\{v_i, v_j\} \subseteq V(G_1)} d_{G_1 \Diamond G_2}(v_i, v_j) = \sum_{\{v_i, v_j\} \subseteq V(G_1)} d_{G_1}(v_i, v_j) = W(G_1), \tag{2.2}$$

$$\begin{aligned} A_2 &= \sum_{e_i \in E(G_1)} \sum_{\{u_{ij}, u_{ik}\} \subseteq V_i(G_2)} d_{G_1 \Diamond G_2}(u_{ij}, u_{ik}), \\ &= \sum_{e_i \in E(G_1)} \left(\sum_{u_{ij} u_{ik} \notin E_i(G_2)} 2 + \sum_{u_{ij} u_{ik} \in E_i((G_2)} 1 \right) \\ &= \sum_{e_i \in E(G_1)} \left(\sum_{u_{ij} u_{ik} \subseteq V_i(G_2)} 2 - \sum_{u_{ij} u_{ik} \in E_i(G_2)} 1 \right) \\ &= \sum_{e_i \in E(G_1)} (n_2(n_2 - 1) - m_2) \\ &= m_1(n_2(n_2-1) - m_2), \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} A_3 &= \sum_{\{e_i, e_k\} \subseteq E(G_1)} \sum_{u_{ij} \in V_i(G_2), u_{km} \in V_k(G_2)} d_{G_1 \Diamond G_2}(u_{ij}, u_{km}), \\ &= \sum_{\{e_i, e_k\} \subseteq E(G_1)} \sum_{u_{ij} \in V_i(G_2), u_{km} \in V_k(G_2)} (d_{G_1}(e_i, e_k) + 2), \\ &= n_2^2 \sum_{\{e_i, e_k\} \subseteq E(G_1)} (d_{G_1}(e_i, e_k) + 2) \\ &= n_2^2 (W_e(G_1) + m_1(m_1 - 1)/2). \end{aligned} \tag{2.4}$$

We have

$$A_4 = \sum_{e_i \in E(G_1)} \sum_{u_{ij} \in V_i(G_2),\, v_k \in V(G_1)} d_{G_1 \Diamond G_2}(u_{ij}, u_k).$$

Using Lemma 2.2, the above equation can be rewritten as follows:

$$\begin{aligned} A_4 &= n_2 \sum_{v_l v_m \in E(G_1)} \Big(\sum_{v_k \in V(G_1), d_{G_1}(v_m, v_k) = d_{G_1}(v_l, v_k)} (d_{G_1}(v_l, v_k) + d_{G_1}(v_m, v_k) + 2)/2 \\ &\quad + \sum_{v_k \in V(G_1), d_{G_1}(v_m, v_k) \neq d_{G_1}(v_l, v_k)} (d_{G_1}(v_l, v_k) + d_{G_1}(v_m, v_k) + 1)/2 \\ &= n_2 \Big(\sum_{e_i = v_l v_m \in E(G_1)} \sum_{v_k \in V(G_1)} (d_{G_1}(v_l, v_k) + d_{G_1}(v_m, v_k)) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{e_i=v_lv_m\in E(G_1)} 2n_1-(n_{e_i}(v_l\,|\,G_1)+n_{e_i}(v_m\,|\,G_1)))/2\\
&= n_2(\sum_{v_j\in V(G_1)}\sum_{v_iv_j\in V(G_1)}\sum_{v_k\in V(G_1)} d_{G_1}(v_i,v_k)+2n_1m_1-PI(G_1))/2\\
&= n_2(\sum_{v_i\in V(G_1)}\sum_{v_k\in V(G_1)} d(v_i)d_{G_1}(v_i,v_k)+2n_1m_1-PI(G_1))/2\\
&= n_2(\sum_{\{v_i,v_k\}\subseteq V(G_1)} (d(v_i)+d(v_k))d_{G_1}(v_i,v_k)+2n_1m_1-PI(G_1))/2\\
&= n_2(DD(G_1)+2n_1m_1-PI(G_1))/2. \qquad (2.5)
\end{aligned}
$$

Using (2.2), (2.3), (2.4) and (2.5) in (2.1) the result follows.

Corollary 2.4 *Let G_1 be a bipartite graph. Then the Wiener index of $G_1\Diamond G_2$ is given by* $W(G_1\Diamond G_2)=W(G_1)+n_2^2W_e(G_1)+n_2(DD(G_1)-n_1m_1)/2+m_1n_2\left(n_2(m_1+1)/2+n_1-1\right)-m_1m_2.$

We need the following lemmas for future references.

Lemma 2.5 [22] *Let P_n and C_n denote the path and cycle on n vertices, respectively. Then*

$$W(P_n)=n(n^2-1)/6,$$

$$W(C_n)=\begin{cases} n^3/8, & \text{if } n \text{ is even}\\ n(n^2-1)/8, & \text{if } n \text{ is odd}.\end{cases}$$

Lemma 2.6 [8, 24] *Let P_n and C_n denote the path and cycle on n vertices, respectively. Then*

$$DD(P_n)=n(n-1)(2n-1)/3,$$

$$DD(C_n)=\begin{cases} n^3/2, & \text{if } n \text{ is even}\\ n(n^2-1)/2, & \text{if } n \text{ is odd}.\end{cases}$$

It can be easily verified that $W(K_n)=n(n-1)/2$, $DD(K_n)=n(n-1)^2$, $W_e(K_n)$ $=n(n-1)(n^2-3n+2)/4$, $PI(K_n)=n(n-1)$, $PI(P_n)=n(n-1)$ and $PI(C_{2n+1})=2n\,(2n+1)$. Using these facts and also by applying above two lemmas in Theorem 2.3, we obtain the following corollary.

Corollary 2.7 *Let G be a graph with n_1 vertices and m_1 edges. Then*

1. $W(P_n\Diamond G)=(n-1)((n_1+1)^2n^2+(n_1+1)^2n-6n_1-6m_1)/6.$
2. $W(C_{2n}\Diamond G)=n((n+1)^2n_1^2+(n^2+n-1)2n_1+n^2-2m_1).$
3. $W(C_{2n+1}\Diamond G)=(n+1/2)((n_1+1)^2n^2+(3n_1^2+4n_1+1)n+2n_1^2-2m_1).$
4. $W(K_n\Diamond G)=(1/8)n(n-1)(3n_1^2n^2-7n_1^2n+6n_1^2+8n_1n-12n_1-4m_1+4).$

Theorem 2.8 *The first Zagreb index of* $G_1 \Diamond G_2$ *is given by*

$$M_1(G_1 \Diamond G_2) = (n_2+1)^2 M_1(G_1) + m_1(M_1(G_2) + 4n_2 + 8m_2).$$

Proof. By Lemma 2.1, we have

$$\begin{aligned}
M_1(G_1 \Diamond G_2) &= \sum_{x \in V(G_1 \Diamond G_2)} d^2_{G_1 \Diamond G_2}(x) \\
&= \sum_{v_i \in V(G_1)} d^2_{G_1 \Diamond G_2}(v_i) + \sum_{e_i \in E(G_1)} \sum_{u_{ij} \in V_i(G_2)} d^2_{G_1 \Diamond G_2}(u_{ij}) \\
&= (n_2+1)^2 \sum_{v_i \in V(G_1)} d^2_{G_1}(v_i) + \sum_{e_i \in E(G_1)} \sum_{u_{ij} \in V_i(G_2)} (d_{G_2}(u_j) + 2)^2 \\
&= (n_2+1)^2 M_1(G_1) + \sum_{e_i \in E(G_1)} \sum_{u_{ij} \in V_i(G_2)} (d^2_{G_2}(u_j) + 4d_{G_2}(u_j) + 4) \\
&= (n_2+1)^2 M_1(G_1) + \sum_{e_i \in E(G_1)} (M_1(G_2) + 8m_2 + 4n_2) \\
&= (n_2+1)^2 M_1(G_1) + m_1 M_1(G_2) + 8m_1 m_2 + 4n_2 m_1.
\end{aligned}$$

This completes the proof.

Theorem 2.9 *The second Zagreb index of* $G_1 \Diamond G_2$ *is given by*

$$M_2(G_1 \Diamond G_2) = (n_2+1)^2 M_2(G_1) + m_1 M_2(G_2) + 2m_1 M_1(G_2) + 4m_1 m_2 + 2(n_2+1)(m_2+n_2) M_1(G_1).$$

Proof. By Lemma 2.1, we have

$$\begin{aligned}
M_2(G_1 \Diamond G_2) &= \sum_{xy \in E(G_1 \Diamond G_2)} d_{G_1 \Diamond G_2}(x)\, d_{G_1 \Diamond G_2}(y) \\
&= \sum_{v_i v_j \in E(G_1)} d_{G_1 \Diamond G_2}(v_i)\, d_{G_1 \Diamond G_2}(v_j) + \sum_{e_i \in E(G_1)} \sum_{u_{ij} u_{ik} \in E_i(G_2)} d_{G_1 \Diamond G_2}(u_{ij})\, d_{G_1 \Diamond G_2}(u_{ik}) \\
&\quad + \sum_{e_i = v_l v_m \in E(G_1)} \sum_{u_{ij} \in V_i(G_2)} (d_{G_1 \Diamond G_2}(v_l) + d_{G_1 \Diamond G_2}(v_m))\, d_{G_1 \Diamond G_2}(u_{ij}) \\
&= (n_2+1)^2 \sum_{v_i v_j \in E(G_1)} d_{G_1}(v_i)\, d_{G_1}(v_j) \\
&\quad + \sum_{e_i \in E(G_1)} \sum_{u_{ij} u_{ik} \in E_i(G_2)} (d_{G_2}(u_{ij}) + 2)(d_{G_2}(u_{ik}) + 2) \\
&\quad + (n_2+1) \sum_{e_i = v_l v_m \in E(G_1)} \sum_{u_{ij} \in V_i(G_2)} (d_{G_1}(v_l) + d_{G_1}(v_m))\, (d_{G_2}(u_{ij}) + 2) \\
&= (n_2+1)^2 M_2(G_1) + \sum_{e_i \in E(G_1)} (M_2(G_2) + 2M_1(G_2) + 4m_2) \\
&\quad + (n_2+1) \sum_{e_i = v_l v_m \in E(G_1)} (d_{G_1}(v_l) + d_{G_1}(v_m))\, (2m_2 + 2n_2) \\
&= (n_2+1)^2 M_2(G_1) + m_1(M_2(G_2) + 2M_1(G_2) + 4m_2) \\
&\quad + 2(n_2+1)(m_2+n_2) M_1(G_1).
\end{aligned}$$

Hence the proof.

Using the facts that $M_1(P_n) = 4n-6\ (n \ge 2)$, $M_2(P_n) = 4n-8\ (n \ge 3)$, $M_1(C_n) = M_2(C_n) = 4n$, $M_1(K_n) = n(n-1)^2$, $M_2(K_n) = n(n-1)^3/2$, in Theorems 2.8 and 2.9, we obtain the following corollaries.

Corollary 2.10 *We have*

1. $M_1(P_n \lozenge P_m) = 4m^2n - 6m^2 + 24mn - 28m - 10n + 8.$
2. $M_1(C_n \lozenge C_m) = 4n(m^2 + 6m + 1).$
3. $M_1(P_n \lozenge C_m) = 4m^2n - 6m^2 + 24mn - 28m + 4n - 6.$
4. $M_1(C_n \lozenge P_m) = 4m^2n + 24mn - 10n.$
5. $M_1(K_n \lozenge K_m) = (1/2)n(n-1)(m+1)^2(m + 2n - 2).$
6. $M_1(K_n \lozenge P_m) = n(n-1)((n-1)(m+1)^2 + 8m - 7).$
7. $M_1(P_n \lozenge K_m) = (m+1)^2(mn - m + 4n - 6).$
8. $M_1(K_n \lozenge C_m) = n(n-1)((m+1)^2 n - m^2 + 6m - 1).$
9. $M_1(C_n \lozenge K_m) = m^3n + 6m^2n + 9mn + 4n.$
10. $M_1(P_n \lozenge \overline{K_m}) = 4m^2n - 6m^2 + 12mn - 16m + 4n - 6.$
11. $M_1(C_n \lozenge \overline{K_m}) = 4n(m^2 + 3m + 1).$
12. $M_1(K_n \lozenge \overline{K_m}) = n(n-1)((m+1)^2(n-1) + 2m).$

Corollary 2.11 *We have*

1. $M_2(P_n \lozenge P_m) = (20n - 32)m^2 + (32n - 44)m - 28n + 28.$
2. $M_2(C_n \lozenge C_m) = 20m^2n + 40mn + 4n.$
3. $M_2(P_n \lozenge C_m) = (20n - 32)m^2 + (40n - 56)m + 4n - 8.$
4. $M_2(C_n \lozenge P_m) = 20m^2n + 32mn - 28n.$
5. $M_2(K_n \lozenge K_m) = (1/4)n(n-1)(m^2 + (4n-5)m + 2(n-1)^2)(m+1)^2.$
6. $M_2(K_n \lozenge P_m) = (1/2)n(n-1)(m^2n^2 + 6m^2n + 2mn^2 - 7m^2 + n^2 + 14m - 6n - 19).$
7. $M_2(P_n \lozenge K_m) = (1/2((n-1)m^2 + (7n - 11)m + 8n - 16))(m+1)^2.$
8. $M_2(K_n \lozenge C_m) = (1/2)n(n-1)((m+1)^2 n^2 + (6m^2 + 4m - 2)n - 7m^2 + 10m + 1).$
9. $M_2(C_n \lozenge K_m) = (1/2)n(m^2 + 7m + 8)(m+1)^2.$
10. $M_2(P_n \lozenge \overline{K_m}) = (12n - 20)m^2 + (16n - 28)m + 4n - 8.$
11. $M_2(C_n \lozenge \overline{K_m}) = 12m^2n + 16mn + 4n.$
12. $M_2(K_n \lozenge \overline{K_m}) = (1/2)n(m+1)(n-1)^2((m+1)n + 3m - 1).$

3. Degree distance and Gutman index of edge corona of two graphs

For a graph G, we define

$$C(G) := \sum_{\substack{w \in V(G), e = uv \in E(G) \\ d_G(w,v) = d_G(u,w)}} d_G(w).$$

It is easy to see that $C(C_{2n+1}) = 4n+2$ and $C(K_n) = n(n-1)^2(n-2)/2$.

In this section, we compute the degree distance and Gutman index of edge corona of two graphs.

Theorem 3.1 *The degree distance index of* $G_1 \Diamond G_2$ *is given by*

$$\begin{aligned} DD(G_1 \Diamond G_2) &= (m_2+2n_2+1)DD(G_1) + 4n_2(m_2+n_2)W_e(G_1) \\ &\quad + (1/2)n_2(n_2+1)(2Gut(G_1) + C(G_1)) - (m_2+n_2)PI(G_1) \\ &\quad - m_1M_1(G_2) + 2m_1(m_2+n_2)(n_1+(m_1+1)n_2-2) - 4m_1m_2 + m_1^2n_2(n_2+1). \end{aligned}$$

Proof. We have

$$DD(G_1 \Diamond G_2) = A_1 + A_2 + A_3 + A_4, \tag{3.1}$$

where

$$\begin{aligned} A_1 &= \sum_{\{v_i,v_j\} \subseteq V(G_1)} (d_{G_1 \Diamond G_2}(v_i) + d_{G_1 \Diamond G_2}(v_j))\, d_{G_1 \Diamond G_2}(v_i, v_j), \\ A_2 &= \sum_{e_i \in E(G_1)} \sum_{\{u_{ij},u_{ik}\} \subseteq V_i(G_2)} (d_{G_1 \Diamond G_2}(u_{ij}) + d_{G_1 \Diamond G_2}(u_{ik}))\, d_{G_1 \Diamond G_2}(u_{ij}, u_{ik}), \\ A_3 &= \sum_{\{e_i,e_k\} \subseteq E(G_1)} \sum_{u_{ij} \in V_i(G_2), u_{km} \in V_k(G_2)} (d_{G_1 \Diamond G_2}(u_{ij}) + d_{G_1 \Diamond G_2}(u_{km}))\, d_{G_1 \Diamond G_2}(u_{ij}, u_{km}), \\ A_4 &= \sum_{e_i \in E(G_1)} \sum_{u_{ij} \in V_i(G_2), v_k \in V(G_1)} (d_{G_1 \Diamond G_2}(u_{ij}) + d_{G_1 \Diamond G_2}(v_k))\, d_{G_1 \Diamond G_2}(u_{ij}, v_k). \end{aligned}$$

Using Lemmas 2.1 and 2.2, A_1, A_2, A_3 and A_4 can be computed as follows:

$$\begin{aligned} A_1 &= \sum_{\{v_i,v_j\} \subseteq V(G_1)} (d_{G_1 \Diamond G_2}(v_i) + d_{G_1 \Diamond G_2}(v_j))\, d_{G_1 \Diamond G_2}(v_i, v_j) \\ &= (n_2+1) \sum_{\{v_i,v_j\} \subseteq V(G_1)} (d_{G_1}(v_i) + d_{G_1}(v_j))\, d_{G_1}(v_i, v_j) \\ &= (n_2+1)DD(G_1). \end{aligned} \tag{3.2}$$

$$\begin{aligned} A_2 &= \sum_{e_i \in E(G_1)} \sum_{\{u_{ij},u_{ik}\} \subseteq V_i(G_2)} (d_{G_1 \Diamond G_2}(u_{ij}) + d_{G_1 \Diamond G_2}(u_{ik}))\, d_{G_1 \Diamond G_2}(u_{ij}, u_{ik}) \\ &= \sum_{e_i \in E(G_1)} (2\sum_{\{u_j,u_k\} \subseteq V(G_2)} (d_{G_2}(u_j) + d_{G_2}(u_k) + 4) \\ &\quad - \sum_{u_ju_k \in E(G_2)} (d_{G_2}(u_j) + d_{G_2}(u_k) + 4)) \\ &= \sum_{e_i \in E(G_1)} (2(n_2-1)\sum_{u_j \in V(G_2)} (d_{G_2}(u_j)) + 4n_2(n_2-1) - M_1(G_2) - 4m_2) \\ &= \sum_{e_i \in E(G_1)} \left(4(n_2-1)m_2 + 4n_2(n_2-1) - M_1(G_2) - 4m_2\right) \\ &= 4(n_2-2)m_1m_2 + 4m_1n_2(n_2-1) - m_1M_1(G_2). \end{aligned} \tag{3.3}$$

$$A_3 = \sum_{\{e_i,e_k\}\subseteq E(G_1)} \sum_{u_{ij}\in V_i(G_2),u_{km}\in V_k(G_2)} (d_{G_1\Diamond G_2}(u_{ij}) + d_{G_1\Diamond G_2}(u_{km}))\, d_{G_1\Diamond G_2}(u_{ij},u_{km})$$
$$= 2n_2 \sum_{\{e_i,e_k\}\subseteq E(G_1)} \sum_{u_j\in V(G_2)} (d_{G_2}(u_i)+2)(d_{G_1}(e_i,e_k)+2)$$
$$= 2n_2 \sum_{\{e_i,e_k\}\subseteq E(G_1)} \left((d_{G_1}(e_i,e_k)+2)(2m_2+2n_2)\right)$$
$$= 4n_2(m_2+n_2)(W_e(G_1)+m_1(m_1-1)/2). \tag{3.4}$$

$$A_4 = \sum_{e_i\in E(G_1)} \sum_{u_{ij}\in V_i(G_2),v_k\in V(G_1)} (d_{G_1\Diamond G_2}(u_{ij}) + d_{G_1\Diamond G_2}(v_k)) d_{G_1\Diamond G_2}(u_{ij},v_k)$$
$$= \sum_{v_lv_m\in E(G_1)} ((1/2)\sum_{v_k\in V(G_1)} (d_{G_1}(v_l,v_k)+d_{G_1}(v_m,v_k))(2(m_2+n_2)$$
$$+ n_2(n_2+1)d_{G_1}(v_k))$$
$$+(m_2+n_2)(2\,n_1-(n_{e_i}(v_l\,|\,G_1)+n_{e_i}(v_m\,|\,G_1)))$$
$$+m_1n_2(n_2+1)+(1/2)\;n_2(n_2+1)\sum_{v_k\in V(G_1),d_{G_1}(v_m,v_k)=d_{G_1}(v_l,v_k)} d_{G_1}(v_k).$$

Thus,

$$A_4 = (m_2+n_2)(DD(G_1)-PI(G_1))+(1/2)n_2(n_2+1)(2Gut(G_1)+C(G_1))$$
$$+(m_2+n_2)(2m_1n_1)+m_1^2n_2(n_2+1). \tag{3.5}$$

Employing (3.2), (3.3), (3.4) and (3.5) in (3.1), we get the required result.

Corollary 3.2 *Let G_1 be a bipartite graph. Then*

$$DD(G_1\Diamond G_2) = (m_2+2n_2+1)DD(G_1)+4n_2(m_2+n_2)W_e(G_1)$$
$$+n_2(n_2+1)Gut(G_1)-(m_2+n_2)n_1m_1-m_1M_1(G_2)$$
$$+2m_1(m_2+n_2)(n_1+(m_1+1)n_2-2)-4m_1m_2+m_1^2n_2(n_2+1).$$

Theorem 3.3 *The Gutman index of $G_1\Diamond G_2$ is given by*

$$Gut\,(G_1\Diamond G_2) = (n_2+1)(2m_2+3n_2+1)Gut\,(G_1)+(4m_2^2+8n_2m_2+4n_2^2)W_e(G_1)$$
$$-3m_1M_1(G_2)-m_1M_2(G_2)+(m_2+n_2)(n_2+1)\,C(G_1)$$
$$+2m_1n_2(m_1(2\,n_2+3m_2+1)+n_2+2m_2-2)+2m_1m_2(m_1(m_2+1)+m_2-6).$$

***Proof*.**

$$Gut(G_1\Diamond G_2) = A_1+A_2+A_3+A_4, \tag{3.6}$$

where

$$A_1 = \sum_{\{v_i,v_j\}\subseteq V(G_1)} d_{G_1\Diamond G_2}(v_i)\, d_{G_1\Diamond G_2}(v_j)\, d_{G_1\Diamond G_2}(v_i,v_j),$$
$$A_2 = \sum_{e_i\in E(G_1)} \sum_{\{u_{ij},u_{ik}\}\subseteq V_i(G_2)} d_{G_1\Diamond G_2}(u_{ij})\, d_{G_1\Diamond G_2}(u_{ik})\, d_{G_1\Diamond G_2}(u_{ij},u_{ik}),$$
$$A_3 = \sum_{\{e_i,e_k\}\subseteq E(G_1)} \sum_{u_{ij}\in V_i(G_2),u_{km}\in V_k(G_2)} d_{G_1\Diamond G_2}(u_{ij})\, d_{G_1\Diamond G_2}(u_{km})\, d_{G_1\Diamond G_2}(u_{ij},u_{km}),$$
$$A_4 = \sum_{e_i\in E(G_1)} \sum_{u_{ij}\in V_i(G_2),u_k\in V(G_1)} d_{G_1\Diamond G_2}(u_{ij})\, d_{G_1\Diamond G_2}(v_k)\, d_{G_1\Diamond G_2}(u_{ij},v_k).$$

Using Lemmas 2.1 and 2.2, A_1, A_2, A_3 and A_4 can be computed as follows:

$$\begin{aligned}
A_1 &= \sum_{\{v_i,v_j\}\subseteq V(G_1)} d_{G_1\Diamond G_2}(v_i)\, d_{G_1\Diamond G_2}(v_j)\, d_{G_1\Diamond G_2}(v_i,v_j)\\
&= (n_2+1)^2 \sum_{\{v_i,v_j\}\subseteq V(G_1)} d_{G_1}(v_i)\, d_{G_1}(v_j)\, d_{G_1}(v_i,v_j)\\
&= (n_2+1)^2 Gut(G_1).
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
A_2 &= \sum_{e_i\in E(G_1)} \sum_{\{u_{ij},u_{ik}\}\subseteq V_i(G_2)} d_{G_1\Diamond G_2}(u_{ij})\, d_{G_1\Diamond G_2}(u_{ik})\, d_{G_1\Diamond G_2}(u_{ij},u_{ik})\\
&= \sum_{e_i\in E(G_1)} (2\sum_{\{u_j,u_k\}\subseteq V(G_2)} (d_{G_2}(u_j) d_{G_2}(u_k) + 2(d_{G_2}(u_j)+d_{G_2}(u_k))+4)\\
&- \sum_{u_j,u_k\in E(G_2)} (d_{G_2}(u_j) d_{G_2}(u_k) + 2(d_{G_2}(u_j)+d_{G_2}(u_k))+4)\\
&= \sum_{e_i\in E(G_1)} \Big((4m_2^2 - M_1(G_2)) + 8(n_2-1)m_2 + 4n_2(n_2-1)\\
&- M_2(G_2) - 2M_1(G_2) - 4m_2)\Big)\\
&= m_1(4m_2^2 - 3M_1(G_2) - M_2(G_2) + 4(n_2-1)(2m_2+n_2) - 4m_2).
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
A_3 &= \sum_{\{e_i,e_k\}\subseteq E(G_1)} \sum_{u_{ij}\in V_i(G_2),u_{km}\in V_k(G_2)} d_{G_1\Diamond G_2}(u_{ij})\, d_{G_1\Diamond G_2}(u_{km})\, d_{G_1\Diamond G_2}(u_{ij},u_{km})\\
&= \sum_{\{e_i,e_k\}\subseteq E(G_1)} \sum_{u_{ij}\in V_i(G_2),u_{km}\in V_k(G_2)} (d_{G_2}(u_{ij})+2)(d_{G_2}(u_{km})+2)(d_{G_1}(e_i,e_k)+2)\\
&= \sum_{\{e_i,e_k\}\subseteq E(G_1)} (d_{G_1}(e_i,e_k)+2)\, (\sum_{u_{ij}\in V_i(G_2),u_{km}\in V_k(G_2)} d_{G_2}(u_{ij}) d_{G_2}(u_{km})\\
&+ 2\sum_{u_{ij}\in V_i(G_2),u_{km}\in V_k(G_2)} (d_{G_2}(u_{ij}) + d_{G_2}(u_{km})) + 4n_2^2)\\
&= \sum_{\{e_i,e_k\}\subseteq E(G_1)} (d_{G_1}(e_i,e_k)+2)(4m_2^2 + 8n_2m_2 + 4n_2^2)\\
&= (4m_2^2 + 8n_2m_2 + 4n_2^2)(W_e(G_1) + m_1(m_1-1)/2).
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
A_4 &= \sum_{e_i\in E(G_1)} \sum_{u_{ij}\in V_i(G_2),v_k\in V(G_1)} d_{G_1\Diamond G_2}(u_{ij})\, d_{G_1\Diamond G_2}(v_k)\, d_{G_1\Diamond G_2}(u_{ij},v_k)\\
&= \sum_{v_lv_m\in E(G_1)} \sum_{v_k\in V(G_1)} ((m_2+n_2)(n_2+1)(d_{G_1}(v_l,v_k) + d_{G_1}(v_m,v_k)) d_{G_1}(v_k)\\
&+ (m_2+n_2)(n_2+1)(2m_1 + \sum_{v_k\in V(G_1), d_{G_1}(v_m,v_k)=d_{G_1}(v_l,v_k)} d_{G_1}(v_k)))\\
&= (m_2+n_2)(n_2+1)[2Gut(G_1) + C(G_1) + 2m_1^2].
\end{aligned} \tag{3.10}$$

Now, the result follows using (3.7), (3.8), (3.9) and (3.10) in (3.6).

Corollary 3.4 *Let G_1 be a bipartite graph. Then*

$$\begin{aligned}
Gut(G_1\Diamond G_2) &= (n_2+1)(2m_2+3n_2+1)Gut(G_1) + (4m_2^2+8n_2m_2+4n_2^2)W_e(G_1)\\
&- 3m_1M_1(G_2) - m_1M_2(G_2) + 2m_1n_2(m_1(2n_2+3m_2+1) + n_2\\
&+ 2m_2 - 2) + 2m_1m_2(m_1(m_2+1) + m_2 - 6).
\end{aligned}$$

Lemma 3.5 [10] *Let P_n and C_n denote the path and the cycle on n vertices, respectively. Then*

$$Gut(P_n) = (n-1)(2n^2-4n+3)/3,$$

$$Gut(C_n) = \begin{cases} n^3/2, & \text{if } n \text{ is even}, \\ n(n^2-1)/2, & \text{if } n \text{ is odd}. \end{cases}$$

Using Lemmas 2.5, 2.6 and 3.5 in Theorems 3.1 and 3.3, we obtain the following corollaries.

Corollary 3.6 *We have*

1. $DD(P_n \Diamond P_m) = (n-1)(2m^2n^2 + m^2n + 2mn^2 - 16m - n + 14).$
2. $DD(C_{2n} \Diamond C_m) = (12m^2 + 16m + 4)n^3 + (20m^2 + 12m)n^2 + (8m^2 - 32m)n.$
3. $DD(C_{2n+1} \Diamond C_m) = 2(2n+1)(3m^2n^2 + 8m^2n + 4mn^2 + 5m^2 + 7mn + n^2 - 5m + n).$
4. $DD(P_n \Diamond C_m) = (1/3)(n-1)(6m^2n^2 + 3m^2n + 8mn^2 + 2mn + 2n^2 - 48m - n).$
5. $DD(C_{2n} \Diamond P_m) = (12m^2 + 12m)n^3 + (20m^2 + 4m - 4)n^2 + (8m^2 - 36m + 28)n.$
6. $DD(C_{2n+1} \Diamond P_m) = 2(2n+1)(3m^2n^2 + 8m^2n + 3mn^2 + 5m^2 + 4mn - 7m - n + 6).$
7. $DD(K_n \Diamond P_m) = (1/2)n(n-1)(8m^2n^2 - 18m^2n - mn^2 + 14m^2 + 13mn$
   $-30m - 2n + 16).$
8. $DD(P_n \Diamond K_m) = (1/3)(n-1)(m+1)^2(mn^2 + mn + 2n^2 - 3m - n).$
9. $DD(C_{2n} \Diamond K_m) = 2n((m^2 + 3m + 2)n^2 + (2m^2 + 3m)n - m)(m+1).$
10. $DD(C_{2n+1} \Diamond K_m) = ((m^2 + 3m + 2)n^2 + (3m^2 + 6m + 2)n + m^2 + 2m)(m+1)(2n+1).$
11. $DD(K_n \Diamond C_m) = (1/4)(n-1)(m+1)(m^2n^3 - 5m^2n^2 + 2mn^3 + 14m^2n - 2mn^2$
    $-12m^2 + 6mn + 4n^2 - 8m - 4n).$
12. $DD(P_n \Diamond \overline{K_m}) = (1/3)(n-1)(4m^2n^2 + m^2n + 6mn^2 + 2n^2 - 12m - n).$
13. $DD(C_{2n} \Diamond \overline{K_m}) = 4n(2m^2n^2 + 3m^2n + 3mn^2 + m^2 + 2mn + n^2 - 2m).$
14. $DD(C_{2n+1} \Diamond \overline{K_m}) = 2(2n+1)(2m^2n^2 + 5m^2n + 3mn^2 + 3m^2 + 5mn + n^2 + n).$
15. $DD(K_n \Diamond \overline{K_m}) = (1/2)n(n-1)(5m^2n^2 - 11m^2n + 2mn^2 + 8m^2 + 2mn - 8m + 2n - 2).$

Corollary 3.7 *We have*

1. $Gut(P_n \Diamond P_m) = (n-1)(6m^2n^2 + m^2 - 6mn - 30m + 39).$
2. $Gut(C_{2n} \Diamond C_m) = 4n(9m^2n^2 + 12m^2n + 6mn^2 + 4m^2 + 4mn + n^2 - 16m).$
3. $Gut(C_{2n+1} \Diamond C_m) = 2(2n+1)(9m^2n^2 + 21m^2n + 6mn^2 + 12m^2 + 10mn + n^2 - 12m + n).$

4. $Gut(C_{2n} \Diamond P_m) = 4n(9m^2n^2 + 12m^2n + 4m^2 - 6mn - 20m + 20)$.
5. $Gut(C_{2n+1} \Diamond P_m) = 2(2n+1)(9m^2n^2 + 21m^2n + 12m^2 - 6mn - 22m + 19)$.
6. $Gut(P_n \Diamond C_m) = (1/3)(n-1)(18m^2n^2 + 12mn^2 + 3m^2 - 12mn + 2n^2 - 90m - 4n + 3)$.
7. $Gut(K_n \Diamond P_m) = (1/2)n(n-1)(21m^2n^2 - 46m^2n - 6mn^2 + 33m^2 + 16mn - 50m - n + 41)$.
8. $Gut(P_n \Diamond K_m) = (1/6)(n-1)(m+2)(m+1)^2(mn^2 + mn + 2n^2 - 3m - 4n + 3)$.
9. $Gut(K_n \Diamond C_m) = (1/2)n(21m^2n^3 - 67m^2n^2 + 10mn^3 + 79m^2n - 30mn^2 + n^3 - 33m^2$
   $-2mn + 22m + 3n - 1) - (3/2)n^3$.
10. $Gut(C_{2n} \Diamond K_m) = n(m+1)^2(m^2n^2 + 2m^2n + 4mn^2 + 4mn + 4n^2 - m)$.
11. $Gut(C_{2n+1} \Diamond K_m) = (n+1/2)(m+1)^2((n^2+3n+1)m^2 + (4n^2+8n+3)m + 4n^2 + 4n)$.
12. $Gut(C_{2n+1} \Diamond \overline{K_m}) = 2(2n+1)(4m^2n^2 + 8m^2n + 4mn^2 + 4m^2 + 6mn + n^2 + n)$.
13. $Gut(C_{2n} \Diamond \overline{K_m}) = 4n(4m^2n^2 + 4m^2n + 4mn^2 + m^2 + 2mn + n^2 - 2m)$.
14. $Gut(P_n \Diamond \overline{K_m}) = (1/3)(n-1)(8m^2n^2 - 4m^2n + 8mn^2 + 3m^2 - 10mn + 2n^2$
    $-6m - 4n + 3)$.
15. $Gut(K_n \Diamond \overline{K_m}) = (1/2)n(n-1)(8m^2n^2 - 17m^2n + 6mn^2 + 11m^2 - 12mn + n^2$
    $+2m - 2n + 1)$.

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