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M–Polynomial of some Graph Operations and Cycle Related Graphs

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ABSTRACT

In this paper, we obtain M-polynomial of some graph operations and cycle related graphs. As an application, we compute M-polynomial of some nanostructures viz., $TUC_4C_8[p,q]$ nanotube, $TUC_4C_8[p,q]$ nanotorus, line graph of subdivision graph of $TUC_4C_8[p,q]$ nanotube and $TUC_4C_8[p,q]$ nanotorus, V-tetracenic nanotube and V-tetracenic nanotorus. Further, we derive some degree based topological indices from the obtained polynomials.

1. INTRODUCTION

Let G be a simple, connected, undirected graph of order n and size m with vertex set $V(G)$ and edge set $E(G)$. The *degree* $d_G(v)$ of a vertex $v \in V(G)$ is the number of edges incident to it in G. An isolated vertex or singleton graph is a vertex with degree zero. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertices of G and let $d_i = d_G(v_i)$. The *subdivision graph* $S(G)$ [24] of a graph G is the graph obtained by inserting a new vertex onto each edge of G. Let G_1 and G_2 be two graphs of order n_1, n_2 and size m_1, m_2 respectively. The *union* [24] of G_1 and G_2 is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$ is denoted by $G_1 \cup G_2$ and $|V(G_1 \cup G_2)| = n_1 + n_2$, $|E(G_1 \cup G_2)| = m_1 + m_2$. The *join* [24] $G_1 + G_2$ of G_1 and G_2 is the graph obtained from $G_1 \cup G_2$ by joining each vertex of G_1 with every vertex of G_2 by an edge. Order and size of $G_1 + G_2$ are $n_1 + n_2$ and $m_1 + m_2 + n_1 n_2$, respectively. The

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corona [24] $G_1 \circ G_2$ of two graphs G_1 and G_2 of order n_1 and n_2 respectively, is defined as the graph obtained by taking one copy of G_1 and n_1 copies of G_2 and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2. For undefined graph theoretic terminologies and notions refer [24].

Several topological indices have been defined in the literature. Among them some standard topological indices are first Zagreb index [22], second Zagreb index [23], modified second Zagreb index [10], Randic′ index [36], harmonic index [16], symmetric division index [10] and inverse sum index [10]. The general form of these degree-based topological indices of a graph is given by

$$TI(G) = \sum_{e=uv\in E(G)} f(d_G(u), d_G(v)),$$

where $f = f(x, y)$ is a function appropriately chosen for the computation. Table 1 gives the standard topological indices defined by $f(x, y)$. For more details on degree-based and distance based topological indices refer [1–7,12,13,18,19,21,32,39–41,43,45].

It would be interesting that, if all these topological indices are obtained from a single expression. This role is played by polynomials. In fact there are several graph polynomials like PI polynomial [3], Tutte polynomial [14], matching polynomial [15,20], Schultz polynomial [25], Zang-Zang polynomial [46], etc., Among them, the Hosoya polynomial [26] is the best and well-known polynomial which plays a vital role in determining distance-based topological indices such as Wiener index [44], hyper Wiener index [9] of graphs. Similarly, M-polynomial which was introduced in 2015 by Deutsch and Klavžar in [10], which is useful in determining many degree-based topological indices (listed in Tables 1 and 2). This motivates us to study M-polynomial of some graph operations and some cycle related graphs. Recently, the study of M-polynomial are reported in [8,11,28,33–35,37].

Table 1. [10] Operators to derive degree-based topological indices from M-polynomial.

Notation	**Topological Index**	$\boldsymbol{f(x, y)}$	**Derivation from $\boldsymbol{M(G; x, y)}$**
$M_1(G)$	First Zagreb	$x + y$	$(D_x + D_y)(M(G; x, y))\|_{x=y=1}$
$M_2(G)$	Second Zagreb	xy	$(D_x D_y)(M(G; x, y))\|_{x=y=1}$
$M_2^m(G)$	Second modified Zagreb	$\frac{1}{xy}$	$(S_x S_y)(M(G; x, y))\|_{x=y=1}$
$S_D(G)$	Symmetric division	$\frac{x^2 + y^2}{xy}$	$(D_x S_y + D_y S_x)(M(G; x, y))\|_{x=y=1}$
$H(G)$	Harmonic	$\frac{2}{x + y}$	$2S_x J(M(G; x, y))\|_{x=1}$
$I_n(G)$	Inverse sum	$\frac{xy}{x + y}$	$S_x J D_x D_y(M(G; x, y))\|_{x=1}$

where, $D_x = x\frac{\partial f(x,y)}{\partial x}$, $D_y = y\frac{\partial f(x,y)}{\partial y}$, $S_x = \int_0^x \frac{f(t,y)}{t}dt$, $S_y = \int_0^y \frac{f(x,t)}{t}dt$ and $J(f(x,y)) = f(x,x)$ are the operators. Along with these operators, we also mention two more operators in Table 2 to calculate general sum connectivity index and first general Zagreb index.

Definition 1. [10] *Let G be a graph. Then M-polynomial of G is defined as*

$$M(G;x,y) = \sum_{i\le j} m_{ij}(G)x^i y^j,$$

where $m_{ij}, i,j \ge 1$, is the number [19] of edges uv of G such that $\{d_G(u), d_G(v)\} = \{i,j\}$.

Table 2: New operators to derive degree-based topological indices from M-polynomial.

Notation	**Topological Index**	$f(x,y)$	**Derivation from** $\mathbf{M(G;x,y)}$
$\chi_\alpha(G)$	General sum connectivity [21]	$(x+y)^\alpha$	$D_x^\alpha(J(M(G;x,y)))\vert_{x=1}$
$M_1^\alpha(G)$	First general Zagreb [31]	$x^{\alpha-1}+y^{\alpha-1}$	$(D_x^{\alpha-1}+D_y^{\alpha-1})(M(G;x,y))\vert_{x=y=1}$

Note 1: Hyper Zagreb index is obtained by taking $\alpha = 2$ in general sum connectivity index.
Note 2: Taking $\alpha = 2{,}3$ in first general Zagreb index, first Zagreb and forgotten topological indices are obtained respectively.

2. M–POLYNOMIAL OF SOME GRAPH OPERATIONS

In this section, we obtain M-polynomial of some graph operations.

Lemma 2.1. *For any r-regular graph G of order n and size m, the M-polynomial of G is given by $M(G;x,y) = mx^r y^r$.*

Proof. Since G is a r-regular graph with m edges and every edge is incident on vertex of degree r, the proof follows. □

The *product* [24] $G \times H$ of graphs G and H has the vertex set $V(G\times H) = V(G)\times V(H)$ and $(a,x)(b,y)$ is an edge of $G\times H$ if and only if [$a = b$ and $xy \in E(H)$] or [$x = y$ and $ab \in E(G)$].

Theorem 2.2. *Let G be an r_1-regular graph of order n_1 and H be an r_2-regular graph of order n_2. Then $M(G\times H;x,y) = n_1n_2x^{r_1+r_2}y^{r_1+r_2}$.*

Proof. Since the graphs G and H are regular graphs of degree r_1 and r_2 respectively. Therefore the graph obtained by product of G and H is a regular graph of degree r_1+r_2 with n_1n_2 vertices. Hence the result follows from Lemma 2.1. □

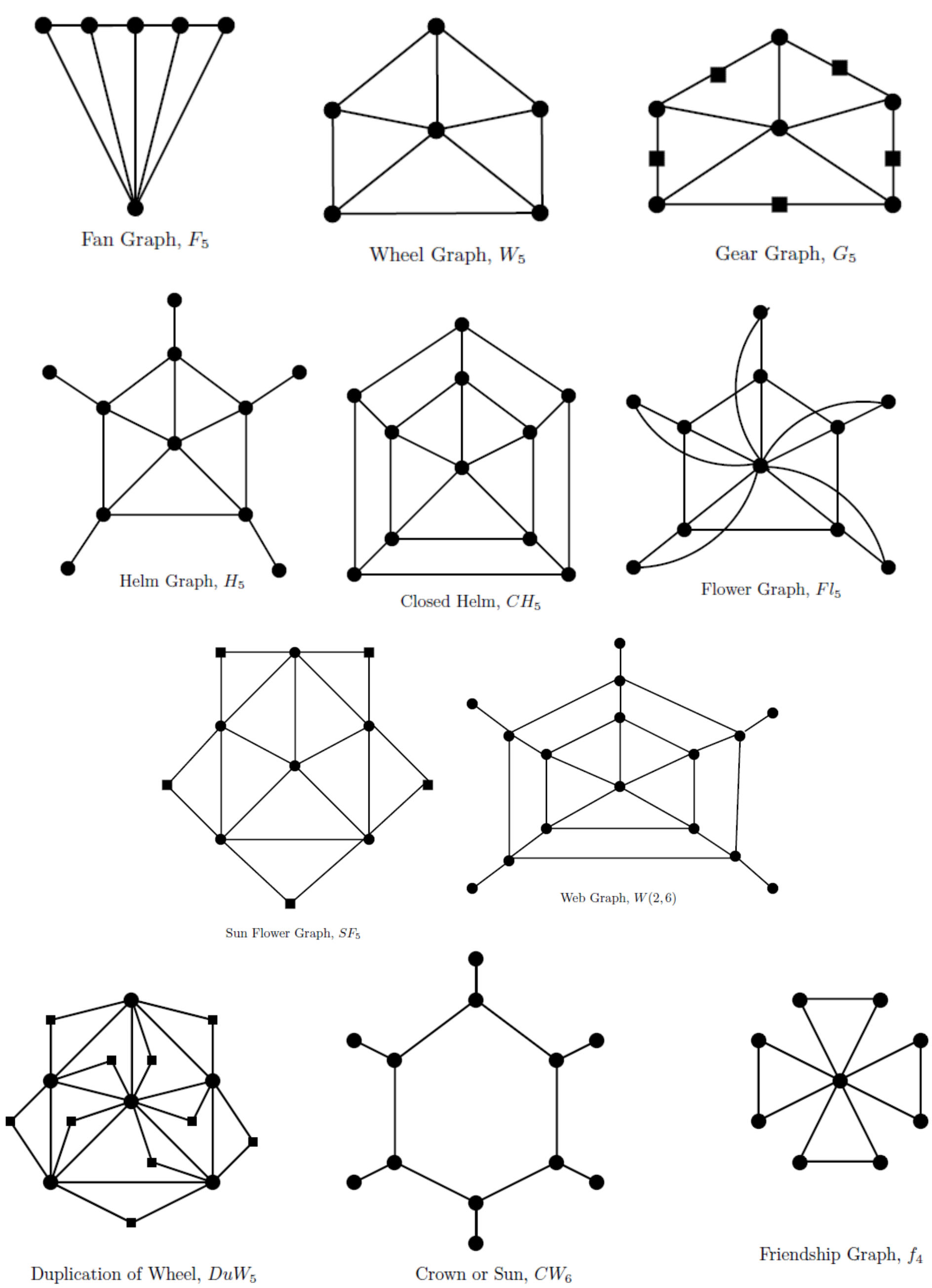


Figure 1. Some cycle related graphs.

The *composition* [24] $G[H]$ of graphs G and H with disjoint vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$ is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ and $(a, x)(b, y)$ is an edge of $G[H]$ if and only if [a is adjacent to b in G] or [$a = b$ and x is adjacent to y in H].

Theorem 2.3. *Let G be an r_1-regular graph of order n_1 and H be an r_2-regular graph of order n_2. Then,* $M(G[H];x,y) = n_1 n_2 x^{n_2 r_1 + r_2} y^{n_2 r_1 + r_2}$.

Proof. Since G and H are regular graphs of degree r_1 and r_2 respectively. The graph obtained by the composition of two graphs G and H is a regular graph of degree $n_2 r_1 + r_2$ with $n_1 n_2$ vertices. Hence the result follows from Lemma 2.1. □

3. M−POLYNOMIAL OF CYCLE RELATED GRAPHS

In this section, we obtain M-polynomial of some cycle related graphs, Figure 1. Definitions 2-10 can be found in [17], definition 11 is in [42] and definitions 12-16 can be found in [30, 38]. We also derive some topological indices (mentioned in Tables 1 and 2) of these graphs from the respective M-polynomials. For more details on wheel related graphs refer [17,27,38,42] and references cited there in.

Definition 2. *The fan graph F_n, $(n \geq 3)$ is defined as the graph $K_1 + P_n$, where K_1 is singleton graph and P_n is the path on n vertices.*

Theorem 3.1. *Let F_n be a fan of order $n+1$ and size $2n-1$. Then,*

$$M(F_n;x,y) = 2x^2y^3 + 2x^2y^n + (n-3)x^3y^3 + (n-2)x^3y^n.$$

Proof. The fan F_n has $n+1$ vertices and $2n-1$ edges. It is easy to see that $|m_{\{2,3\}}| = 2$, $|m_{\{2,n\}}| = 2$ and the remaining edge partition of F_n is as follows:

$$|E_{\{3,3\}}| = |uc \in E(F_n): d_u = 3 \quad and \quad d_c = 3| = (n-3),$$
$$|E_{\{3,n\}}| = |uc \in E(F_n): d_u = 3 \quad and \quad d_c = n| = (n-2),$$

proving the result. □

Corollary 3.2. *If F_n is a Fan, then*

1. $M_1(F_n) = n^2 + 9n - 10$,
2. $M_2(F_n) = 3n^2 + 7n - 15$,
3. $\mathrm{M}_2^m(F_n) = \frac{n^2+3n+3}{9n}$,
4. $S_D(F_n) = \frac{n^3+7n^2+4n-6}{3n}$,
5. $H(F_n) = \frac{n^2+2n+12}{3(n+2)} + \frac{9n-23}{5(n+3)}$,
6. $I_n(F_n) = \frac{3n(n-2)}{n+3} + \frac{3(5n-7)}{10} + \frac{4n}{n+2}$,
7. $\chi_\alpha(F_n) = 2 \cdot 5^\alpha + 2(n+2)^\alpha + (n-3) \cdot 6^\alpha + (n-2)(n-3)^\alpha$,
8. $M_1^\alpha(F_n) = 2^{\alpha+2} + 3^\alpha(2n-5) + 3^\alpha(n-1) + n^{\alpha+1}$.

Proof. The M-polynomial for fan F_n is given by

$$M(F_n; x, y) = 2x^2y^3 + 2x^2y^n + (n-3)x^3y^3 + (n-2)x^3y^n.$$

Using the expressions from Tables 1 and 2, we have

$$D_x = x\frac{\partial f(x,y)}{\partial x} = 4x^2y^n + 4x^2y^3 + 3(n-3)x^3y^3 + 3(n-2)x^3y^n$$

$$D_y = y\frac{\partial f(x,y)}{\partial y} = 2nx^2y^n + 6x^2y^3 + 3(n-3)x^3y^3 + n(n-2)x^3y^n$$

$$S_x = \int_0^x \frac{f(t,y)}{t}dt = x^2y^n + x^2y^3 + \frac{(n-3)}{3}x^3y^3 + \frac{(n-2)}{3}x^3y^n$$

$$S_y = \int_0^y \frac{f(x,t)}{t}dt = \frac{2}{n}x^2y^n + \frac{2}{3}x^2y^3 + \frac{(n-3)}{3}x^3y^3 + \frac{(n-2)}{n}x^3y^n.$$

Therefore,

$$M_1(F_n) = (D_x + D_y)(M(F_n; x, y))|_{x=y=1} = n^2 + 9n - 10,$$

$$M_2(F_n) = (D_xD_y)(M(F_n; x, y))|_{x=y=1} = 3n^2 + 7n - 15,$$

$$\mathrm{M}_2^m(F_n) = (S_xS_y)(M(F_n; x, y))|_{x=y=1} = \frac{1}{3n} + \frac{n+3}{9},$$

$$S_D(F_n) = (D_xS_y + D_yS_x)(M(F_n; x, y))|_{x=y=1} = \frac{n^3+7n^2+4n-6}{3n},$$

$$H(F_n) = 2S_x J(M(F_n; x, y))|_{x=1} = \frac{n^2+2n+12}{3(n+2)} + \frac{9n-23}{5(n+3)},$$

$$I_n(F_n) = S_xJD_xD_y(M(Fn; x, y))|_{x=1} = \frac{3n(n-2)}{n+3} + \frac{3(5n-7)}{10} + \frac{4n}{n+2},$$

$$\chi_\alpha(F_n) = D_x^\alpha\left(J(M(F_n; x, y))\right)|_{x=1} = 2\cdot 5^\alpha + 2(n+2)^\alpha + (n-3)\cdot 6^\alpha + (n-2)(n-3)^\alpha,$$

$$M_\alpha^1(F_n) = (D_x^\alpha + D_y^\alpha)(M(F_n; x, y))|_{x=y=1} = 2^{\alpha+2} + 3^\alpha(2n-5) + 3^\alpha(n-1) + n^{\alpha+1}.$$

□

Definition 3. *The wheel $W_n = C_n + K_1$ is a graph with $n + 1$ vertices and $2n$ edges, where the vertex c with degree n is called the central vertex while the vertices on the cycle C_n are called rim vertices.*

Theorem 3.3. *Let W_n be a wheel of order $n + 1$ and size $2n$. Then,*

$$M(W_n; x, y) = nx^3y^3(1 + y^{n-3}).$$

Proof. The wheel W_n has $n + 1$ vertices and $2n$ edges. The edge set of W_n can be partitioned as,

$$|E_{\{3,3\}}| = |uv \in E(W_n): d_u = 3 \quad and \quad d_v = 3| = n,$$

$$|E_{\{3,n\}}| = |uc \in E(W_n): d_u = 3 \quad and \quad d_c = n|$$

$$= |E(W_n) - |E_{\{3,3\}}| = n.$$

□

Corollary 3.4. *If W_n is a wheel, then*

1. $M_1(W_n) = n^2 + 9n,$
2. $M_2(W_n) = 3n^2 + 9n$,

3. $M_2^m(W_n) = \frac{n+3}{9}$,
4. $S_D(W_n) = \frac{n^2+6n+9}{3}$,
5. $H(W_n) = \frac{n^2+9n}{3\,(n+3)}$,
6. $I_n(W_n) = \frac{3n}{2} + \frac{3n^2}{n+3}$,
7. $\chi_\alpha(W_n) = n(6^\alpha + (n+3)^\alpha)$,
8. $M_1^\alpha(W_n) = 3^{\alpha+1} + n^\alpha$.

Proof. Let $M(W_n; x, y) = \sum_{i \le j} m_{ij}(W_n)x^i y^j = nx^3y^3(1 + y^{n-3})$. Using the expressions from Tables 1 and 2, we have

$$D_x = x\frac{\partial f(x,y)}{\partial x} = 3nx^3y^3 + 3nx^3y^n$$
$$D_y = y\frac{\partial f(x,y)}{\partial y} = 3nx^3y^3 + n^2x^3y^n$$
$$S_x = \int_0^x \frac{f(t,y)}{t}dt = \frac{nx^3y^3}{3} + \frac{nx^3y^n}{3}$$
$$S_y = \int_0^y \frac{f(x,t)}{t}dt = \frac{nx^3y^3}{3} + x^3y^n.$$

Thus we get,

$M_1(W_n) = (D_x + D_y)(M(W_n; x, y))|_{x=y=1} = n^2 + 9n$,
$M_2(W_n) = (D_xD_y)(M(W_n; x, y))|_{x=y=1} = 3n^2 + 9n$,
$M_2^m(W_n) = (S_xS_y)(M(W_n; x, y))|_{x=y=1} = \frac{n}{9} + \frac{1}{3}$,
$S_D(W_n) = (D_xS_y + D_yS_x)(M(W_n; x, y))|_{x=y=1} = \frac{n^2+6n+9}{3}$,
$H(W_n) = 2S_xJ(M(W_n; x, y))|_{x=1} = \frac{n}{3} + \frac{2n}{n+3}$,
$I_n(W_n) = S_xJD_xD_y(M(W_n; x, y))|_{x=1} = \frac{3n}{2} + \frac{3n^2}{n+3}$,
$\chi_\alpha(W_n) = D_x^\alpha\left(J(M(W_n; x, y))\right)|_{x=1} = n(6^\alpha + (n+3)^\alpha)$,
$M_1^\alpha(W_n) = (D_x^\alpha + D_y^\alpha)(M(W_n; x, y))|_{x=y=1} = 3^{\alpha+1} + n^\alpha$.

Definition 4. *The gear graph G_n is a wheel graph with a vertex added between each pair adjacent vertices of the outer circle.*

Theorem 3.5. *Let G_n be a gear graph. Then $M(G_n; x, y) = 2nx^2y^3 + nx^3y^n$.*

Proof. Let G_n is a graph having $(2n+1)$ vertices and $3n$ edges. The edge partition of G_n is given by,

$$\begin{aligned} |E_{\{2,3\}}| &= |uv \in E(G_n): d_u = 2 \quad and \quad d_v = 3| = 2n, \\ |E_{\{3,n\}}| &= |uv \in E(G_n): d_u = 3 \quad and \quad d_v = n| \\ &= |E(G_n)| - |E_{\{2,3\}}| = n. \end{aligned}$$

Using definition of M-polynomial and above edge partitions, we get the desired result. □

Corollary 3.6. *If G_n is a gear graph, then*

1. $M_1(G_n) = n^2 + 13n$,
2. $M_2(G_n) = 3n^2 + 12n$,
3. $M_2^m(G_n) = \frac{n+1}{3}$,
4. $S_D(G_n) = \frac{n^2}{3} + \frac{13n}{3} + 3$,
5. $H(G_n) = \frac{4n}{5} + \frac{n}{n+3}$,
6. $I_n(G_n) = \frac{12n}{5} + \frac{3n^2}{n+3}$,
7. $\chi_\alpha(G_n) = 2n5^\alpha + n(n+3)^\alpha$,
8. $M_1^\alpha(G_n) = n(2^{\alpha+1} + 3^{\alpha+1} + n^\alpha)$.

Definition 5. *The helm H_n is a graph obtained from a wheel W_n with central vertex c, by attaching a pendant edge to each rim vertex of W_n. A closed helm CH_n is the graph with central vertex c, obtained from a helm by joining each pendant vertex to form a cycle.*

Theorem 3.7. *Let H_n be a helm. Then $M(H_n; x, y) = nxy^4 + nx^4y^4 + nx^4y^n$.*

Proof. Let H_n is a graph having $(2n+1)$ vertices and $3n$ edges. The edge partition of H_n is given by,

$$\begin{aligned} |E_{\{1,4\}}| &= |uv \in E(H_n): d_u = 1 \quad and \quad d_v = 4| = n, \\ |E_{\{4,4\}}| &= |uv \in E(H_n): d_u = 4 \quad and \quad d_v = 4| = n, \\ |E_{\{4,n\}}| &= |uv \in E(H_n): d_u = 4 \quad and \quad d_v = n| \\ &= |E(H_n)| - |E_{\{1,4\}}| - |E_{\{4,4\}}| = n. \end{aligned}$$

□

Corollary 3.8. *If H_n is a helm graph, then*

1. $M_1(H_n) = n^2 + 17n$,
2. $M_2(H_n) = 4n^2 + 20n$,
3. $M_2^m(H_n) = \frac{5n+4}{16}$,
4. $S_D(H_n) = \frac{n(n+1)}{4} + 6n + 4$,
5. $H(H_n) = \frac{2n}{5} + \frac{n}{4} + \frac{2n}{n+4}$,
6. $I_n(H_n) = \frac{n^2}{n+4} + \frac{14n}{5}$,

7. $\chi_\alpha(H_n) \quad = n(5^\alpha + 8^\alpha + (n+4)^\alpha,$
8. $M_1^\alpha(H_n) \quad = n(4^{\alpha+1} + n^\alpha).$

Theorem 3.9. *Let CH_n be a closed helm. Then*

$$M(CH_n; x, y) = nx^3y^3 + nx^3y^4 + nx^4y^4 + nx^4y^n.$$

Proof. Let CH_n is a graph having $(2n+1)$ vertices and $4n$ edges. The edge partition of CH_n is given by,

$$\begin{aligned} |E_{\{3,3\}}| &= |uv \in E(CH_n): d_u = 3 \quad and \quad d_v = 3| = n, \\ |E_{\{3,4\}}| &= |uv \in E(CH_n): d_u = 3 \quad and \quad d_v = 4| = n, \\ |E_{\{4,4\}}| &= |uv \in E(CH_n): d_u = 4 \quad and \quad d_v = 4| = n, \\ |E_{\{4,n\}}| &= |uv \in E(CH_n): d_u = 4 \quad and \quad d_v = n| = n. \end{aligned}$$

□

Corollary 3.10. *If CH_n is a gear graph, then*

1. $M_1(CH_n) \quad = \quad n^2 + 25n,$
2. $M_2(CH_n) \quad = \quad 4n^2 + 37n \;,$
3. $M_2^m(CH_n) \quad = \frac{37n+36}{144},$
4. $S_D(CH_n) \quad = \frac{73n+3}{12},$
5. $H(CH_n) \quad = \frac{n}{3} + \frac{n}{4} + \frac{2n}{7} + \frac{2n}{n+4},$
6. $I_n(CH_n) \quad = \frac{3n}{2} + \frac{12n}{7} + \frac{4n^2}{n+4} + 2n,$
7. $\chi_\alpha(CH_n) \quad = n(6^\alpha + 7^\alpha + 8^\alpha + (n+4)^\alpha),$
8. $M_1^\alpha(CH_n) = n(3^{\alpha+1} + 4^{\alpha+1} + n^\alpha).$

Definition 6. *The flower Fl_n is the graph obtained from a helm H_n by joining each pendant vertex to the central vertex c of the helm.*

Theorem 3.11. *Let Fl_n be a flower. Then*

$$M(Fl_n; x, y) = nx^2y^4 + nx^2y^{2n} + nx^4y^4 + nx^4y^{2n}.$$

Proof. Let flower Fl_n is a graph having $(2n+1)$ vertices and $4n$ edges. The edge partition of Fl_n is given by,

$$\begin{aligned} \left|E_{\{2,4\}}\right| &= |uv \in E(Fl_n): d_u = 2 \quad and \quad d_v = 4| \quad = n, \\ \left|E_{\{2,2n\}}\right| &= |uv \in E(Fl_n): d_u = 2 \quad and \quad d_v = 2n| = n, \\ \left|E_{\{4,4\}}\right| &= |uv \in E(Fl_n): d_u = 4 \quad and \quad d_v = 4| \quad = n, \\ \left|E_{\{4,2n\}}\right| &= |uv \in E(Fl_n): d_u = 4 \quad and \quad d_v = 2n| \\ &= |E(Fl_n)| - |E_{\{2,4\}}| - |E_{\{2,2n\}}| - |E_{\{4,4\}}| = n. \end{aligned}$$

□

Corollary 3.12. *If Fl_n is a flower graph, then*

1. $M_1(Fl_n) = 4n(n+5)$,
2. $M_2(Fl_n) = 12n(n+2)$,
3. $M_2^m(Fl_n) = \frac{3n+6}{16}$,
4. $S_D(Fl_n) = \frac{3n^2}{2} + \frac{5n}{2} + 3$,
5. $H(Fl_n) = \frac{n}{n+1} + \frac{n}{n+2} + \frac{7n}{8}$,
6. $I_n(Fl_n) = \frac{4n}{3} + \frac{2n^2}{n+1} + \frac{4n^2}{n+2} + 2n$,
7. $\chi_\alpha(Fl_n) = n(6^\alpha + 8^\alpha + (2n+2)^\alpha + (2n+4)^\alpha)$,
8. $M_1^\alpha(Fl_n) = n(2^{\alpha+1} + 4^{\alpha+1} + n^\alpha 2^{\alpha+1})$.

Definition 7. *The sunflower graph SF_n is a graph obtained from a wheel with central vertex c, n-cycle $v_0, v_1, \ldots, v_{n-1}$ and additional n vertices $w_0, w_1, \ldots, w_{n-1}$ where w_i is joined by edges to v_i, v_{i+1} for $i = 0, 1, \ldots, n-1$ where $i+1$ is taken modulo n.*

Theorem 3.13. *Let SF_n be a sunflower. Then $M(SF_n; x, y) = 2nx^2y^5 + nx^5y^5 + nx^5y^n$.*

Proof. The sunflower graph SF_n is a graph having $(2n+1)$ vertices and $4n$ edges. The edge partition of SF_n is given by,

$$\begin{aligned}
|E_{\{2,5\}}| &= |uv \in E(SF_n): d_u = 2 \quad and \quad d_v = 5| = 2n,\\
|E_{\{5,5\}}| &= |uv \in E(SF_n): d_u = 5 \quad and \quad d_v = 5| = n,\\
|E_{\{5,n\}}| &= |uv \in E(SF_n): d_u = 5 \quad and \quad d_v = n|\\
&= |E(SF_n)| - |E_{\{2,5\}}| - |E_{\{5,5\}}| = n.
\end{aligned}$$

□

Corollary 3.14. *If SF_n is a sunflower graph, then*

1. $M_1(SF_n) = n^2 + 29n$,
2. $M_2(SF_n) = 5n(n+9)$,
3. $M_2^m(SF_n) = \frac{n}{5} + \frac{n}{25} + \frac{1}{5}$,
4. $S_D(SF_n) = \frac{n^2+39n+25}{5}$,
5. $H(SF_n) = \frac{4n}{7} + \frac{n}{5} + \frac{2n}{n+5}$,
6. $I_n(SF_n) = \frac{5n^2}{n+5} + \frac{5n}{2} + \frac{20n}{7}$,
7. $\chi_\alpha(SF_n) = n(2 \cdot 7^\alpha + 10^\alpha + (n+5)^\alpha)$,
8. $M_1^\alpha(SF_n) = n(2^{\alpha+1} + 5^{\alpha+1} + n^\alpha)$.

Definition 8. *The friendship graph f_n is a collection of n-triangles with a common vertex. Friendship graph can also be obtained from a wheel W_{2n} with cycle C_{2n} by deleting alternate edges of the cycle. That is $f_n = K_1 + nK_2$.*

Theorem 3.15. *Let f_n be a friendship graph. Then $M(f_n; x, y) = nx^2y^2 + 2nx^2y^{2n}$.*

Proof. Let friendship graph f_n is a graph having $(2n + 1)$ vertices and $3n$ edges. The edge partition of f_n is given by,

$$\begin{aligned} |E_{\{2,2\}}| &= |uv \in E(f_n): d_u = 2 \quad and \quad d_v = 2| = n, \\ |E_{\{2,2n\}}| &= |uv \in E(f_n): d_u = 2 \quad and \quad d_v = 2n| \\ &= |E(f_n)| - |E_{\{2,2\}}| = 2n. \end{aligned}$$

□

Corollary 3.16. *If f_n is a flower graph, then*

1. $M_1(f_n) = 4n(n + 2)$,
2. $M_2(f_n) = 4n(2n + 1)$,
3. $M_2^m(f_n) = \frac{n+2}{4}$,
4. $S_D(f_n) = 2(n^2 + n + 1)$,
5. $H(f_n) = \frac{n}{2} + \frac{2n}{n+1}$,
6. $I_n(f_n) = n + \frac{4n^2}{n+1}$,
7. $\chi_\alpha(f_n) = n(4^\alpha + 2^{\alpha+1}(n + 1)^\alpha)$,
8. $M_1^\alpha(f_n) = n2^{\alpha+1}(n + 2)$.

Definition 9. *A web graph is the graph obtained by joining a pendant edge to each vertex on the outer cycle of the closed helm. $W(t, n)$ is the generalized web with t cycles each of order n.*

Theorem 3.17. *Let $W(t, n)$ be a generalized web. Then*

$$M(W(t, n); x, y) = nxy^4 + n(2t - 1)x^4y^4 + nx^4y^n.$$

Proof. Let generalized web $W(t, n)$ is a graph having $(tn + n + 1)$ vertices and $n(2t + 1)$ edges. The edge partition of $W(t, n)$ is given by,

$$\begin{aligned} |E_{\{1,4\}}| &= |uv \in E(W(t, n)): d_u = 1 \quad and \quad d_v = 4| = n, \\ |E_{\{4,4\}}| &= |uv \in E(W(t, n)): d_u = 4 \quad and \quad d_v = 4| = n(2t - 1), \\ |E_{\{4,n\}}| &= |uv \in E(W(t, n)): d_u = 4 \quad and \quad d_v = n| \\ &= |E(W(t, n))| - |E_{\{1,4\}}| - |E_{\{4,4\}}| = n. \end{aligned}$$

□

Corollary 3.18. *If $W(t, n)$ be a generalized web, then*

1. $M_1(W(t,n)) = n(n+8(2t-1)+9)$,
2. $M_2(W(t,n)) = 4n(n+4(2t-1)+1)$,
3. $M_2^m(W(t,n)) = \frac{n}{4} + \frac{n(2t-1)}{16} + \frac{1}{4}$,
4. $S_D(W(t,n)) = \frac{n^2}{2} + \frac{n}{4} + 2n(2t-1) + 4n + 4$,
5. $H(W(t,n)) = \frac{2n}{5} + \frac{n(2t-1)}{4} + \frac{2n}{n+4}$,
6. $I_n(W(t,n)) = \frac{4n}{5} + 2n(2t-1) + \frac{4n^2}{n+4}$,
7. $\chi_\alpha(W(t,n)) = n(5^\alpha + (2t-1)8^\alpha + (4+n)^\alpha$,
8. $M_1^\alpha(W(t,n)) = 2n \cdot 4^\alpha + 2n \cdot 4^\alpha(2t-1) + n^{\alpha+1} + n$.

Definition 10. *The crown (or sun) CW_n is a corona of form $C_n \circ K_1$ where $n \geq 3$. That is crown is a helm without central vertex.*

Theorem 3.19. *Let CW_n be a crown graph. Then*
$$M(CW_n; x, y) = nxy^3 + nx^3y^3.$$

Proof. Let CW_n is a crown graph having $2n$ vertices and $2n$ edges. The edge partition of CW_n is given by,

$$\begin{aligned} |E_{\{1,3\}}| &= |uv \in E(CW_n): d_u = 1 \quad and \quad d_v = 3| = n, \\ |E_{\{3,3\}}| &= |uv \in E(CW_n): d_u = 3 \quad and \quad d_v = 3| \\ &= |E(CW_n)| - |E_{\{1,3\}}| = n. \end{aligned}$$

□

Corollary 3.20. *If CW_n is a flower graph, then*
1. $M_1(CW_n) = 10n$,
2. $M_2(CW_n) = 12n$,
3. $M_2^m(CW_n) = \frac{4n}{9}$,
4. $S_D(CW_n) = \frac{10n}{3}$,
5. $H(CW_n) = \frac{n}{2} + \frac{n}{3}$,
6. $I_n(CW_n) = \frac{9n}{4}$,
7. $\chi_\alpha(CW_n) = n(4^\alpha + 6^\alpha)$,
8. $M_1^\alpha(CW_n) = n(3^{\alpha+1} + 1)$.

The *duplication of an edge* [42] $e = uv$ by a new vertex v' in a graph G produces a new graph G' by adding a new vertex v' such that $N(v') = \{u, v\}$.

Definition 11. *Consider a wheel $W_n = C_n + K_1$ with $v_1, v_2, \ldots, v_n$ as its rim vertices and c as its central vertex. Let $e_1, e_2, \ldots, e_n$ be the rim edges of W_n which are duplicated by new vertices $w_1, w_2, \ldots, w_n$, respectively and let $f_1, f_2, \ldots, f_n$ be the spoke edges of W_n which are duplicated by the vertices $u_1, u_2, \ldots, u_n$, respectively. The resultant graph is called duplication of the wheel denoted by DuW_n.*

Theorem 3.21. *Let DuW_n be the duplication of the wheel. Then*

$$M(DuW_n; x, y) = 3nx^2y^6 + nx^2y^{2n} + nx^6y^6 + nx^6y^{2n}.$$

Proof. Let duplication of the wheel DuW_n is a graph having $(3n+1)$ vertices and $6n$ edges. The edge partition of DuW_n is given by,

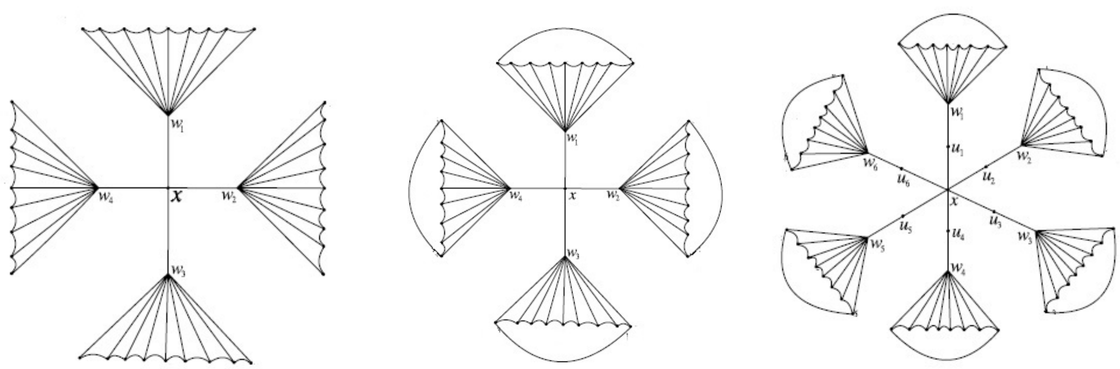
$$\begin{aligned}
|E_{\{2,6\}}| &= |uv \in E(DuW_n): d_u = 2 \quad and \quad d_v = 6| = 3n,\\
|E_{\{2,2n\}}| &= |uv \in E(DuW_n): d_u = 2 \quad and \quad d_v = 2n| = n,\\
|E_{\{6,6\}}| &= |uv \in E(DuW_n): d_u = 6 \quad and \quad d_v = 6| = n,\\
|E_{\{6,2n\}}| &= |uv \in E(DuW_n): d_u = 6 \quad and \quad d_v = 2n|\\
&= |E(DuW_n)| - |E_{\{2,6\}}| - |E_{\{2,2n\}}| - |E_{\{6,6\}}| = n.
\end{aligned}$$

□

Corollary 3.22. *If CW_n be the duplication of the wheel, then*

1. $M_1(DuW_n) = 4n(n+11)$,
2. $M_2(DuW_n) = 8n(2n+9)$, ,
3. $M_2^m(DuW_n) = \frac{5n+6}{18}$,
4. $S_D(DuW_n) = \frac{4n^2+17n+16}{4}$,
5. $H(DuW_n) = \frac{3n}{4} + \frac{n}{n+1} + \frac{n}{6} + \frac{n}{n+3}$,
6. $I_n(DuW_n) = \frac{9n}{2} + \frac{8n^2}{n+1} + 3n$,
7. $\chi_\alpha(DuW_n) = n(3 \cdot 8^\alpha + 12^\alpha + (2n+2)^\alpha + (2n+6)^\alpha)$,
8. $M_1^\alpha(DuW_n) = (4n \cdot 2^\alpha + 6n \cdot 6^\alpha + (2n)^{\alpha+1})$.

Definition 12. *A uniform n-fan split graph SF_n^r, contains a star S_{n-1} with hub at x such that the deletion of n edges of S_{n-1} partitions the graph into n independent fans $F_r^i = P_r^i + K_1$, $(1 \le i \le n)$ and a isolated vertex, Figure 2.*



SF_4^9 $\quad$ SW_4^9 $\quad$ KW(6, 9)

Figure 2. Self explanatory examples of SF_4^9, SW_4^9 and KW(6, 9) graphs.

Theorem 3.23. *Let SF_n^r be a uniform n-fan split graph. Then*

$$M(SF_n^r; x, y) = 2nx^2y^3 + 2nx^2y^{r+1} + n(r-3)x^3y^3 + n(r-2)x^3y^{r+1} + nx^ny^{r+1}.$$

Proof. The uniform n-fan split graph SF_n^r has $(nr + n + 1)$ vertices and $2nr$ edges. The edge set of SF_n^r can be partitioned as,

$$\begin{aligned}
|E_{\{2,3\}}| &= |uv \in E(SF_n^r): d_u = 2 \quad and \quad d_v = 3| = 2n,\\
|E_{\{2,r+1\}}| &= |uc \in E(SF_n^r): d_u = 2 \quad and \quad d_c = r+1| = 2n,\\
|E_{\{3,3\}}| &= |uc \in E(SF_n^r): d_u = 3 \quad and \quad d_c = 3| = n(r-3),\\
|E_{\{3,r+1\}}| &= |uc \in E(SF_n^r): d_u = 3 \quad and \quad d_c = r+1| = n(r-2),\\
|E_{\{n,r+1\}}| &= |uc \in E(SF_n^r): d_u = n \quad and \quad d_c = r+1|\\
&= |E(SF_n^r) - |E_{\{2,3\}}| - |E_{\{2,r+1\}}| - |E_{\{3,3\}}| - |E_{\{3,r+1\}}| = n.
\end{aligned}$$

□

Corollary 3.24. *If SF_n^r be a uniform n-fan split graph, then*

1. $M_1(SF_n^r) = n(r^2 + 11r + n - 9)$,
2. $M_2(SF_n^r) = n(3r^2 + nr + 10r + n - 17)$, ,
3. $M_2^m(SF_n^r) = \frac{9+n(3+4r+r^2)}{9(r+1)}$,
4. $S_D(SF_n^r) = \frac{3n^2+3(r+1)^2+n(r^3+9r^2+13r-10)}{3(r+1)}$,
5. $H(SF_n^r) = \frac{2n}{15}\left(\frac{15}{n+r+1} - \frac{90}{r+4} + \frac{30}{r+3} + 10r - 9\right)$,
6. $I_n(SF_n^r) = \frac{n^2(r+1)}{(n+r+1)} + \frac{n(45r^3+184r^2+83r-272)}{10(r+3)(r+4)}$,
7. $\chi_\alpha(SF_n^r) = 2n\,5^\alpha + 2n\,(r+3)^\alpha + n(r-3)6^\alpha + n(r-2)(r+4)^\alpha + n(n+r+1)^\alpha$
8. $M_1^\alpha(SF_n^r) = 4n \cdot 2^\alpha + 2n(r-3)3^\alpha + n(r-2)3^\alpha + 2n \cdot 3^\alpha + n^{\alpha+1} + n(r-2)(r+1)^\alpha + n(r+1)^\alpha$.

Definition 13. *The graph SW_n^r contains a star S_{n-1} with hub at x such that the deletion of the n edges of S_{n-1} partitions the graph into n independent wheels $W_r^i = C_r^i + K_1, (1 \leq i \leq n)$ and an isolated vertex, Figure 2.*

Theorem 3.25. *Let SW_n^r be the graph having $(nr + n + 1)$ vertices and $n(2r + 1)$ edges. Then*

$$M(SW_n^r; x, y) = nrx^3y^3 + nrx^3y^{r+1} + nx^ny^{r+1}.$$

Proof. Let SW_n^r is a graph having $(nr + n + 1)$ vertices and $n(2r + 1)$ edges. The edge partition of SW_n^r is given by,

$$\begin{aligned}
|E_{\{3,3\}}| &= |uv \in E(SW_n^r): d_u = 3 \quad and \quad d_v = 3| = nr, \\
|E_{\{3,r+1\}}| &= |uv \in E(SW_n^r): d_u = 3 \quad and \quad d_v = r + 1| = nr, \\
|E_{\{n,r+1\}}| &= |uv \in E(SW_n^r): d_u = n \quad and \quad d_v = r + 1| \\
&= |E(SW_n^r)| - |E_{\{3,r+1\}}| - |E_{\{3,3\}}| = n.
\end{aligned}$$

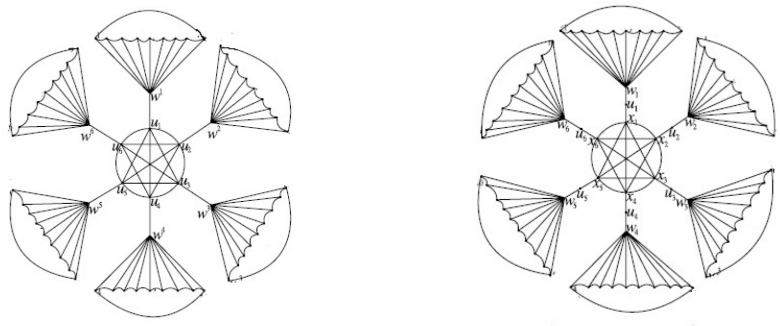
□

Corollary 3.26. *If SW_n^r graph, then*

1. $M_1(SW_n^r) = n^2 + n(r + 1) + nr(r + 10)$,
2. $M_2(SW_n^r) = n^2(r + 1) + 3nr(r + 4)$,
3. $M_2^m(SW_n^r) = \frac{nr^2+4nr+9}{9(r+1)}$,
4. $S_D(SW_n^r) = \frac{3n^2+3(r+1)^2+nr(r+4)^2}{3(r+1)}$,
5. $H(SW_n^r) = \frac{2n}{(n+r+1)} + nr\left(\frac{r+10}{3(r+4)}\right)$,
6. $I_n(SW_n^r) = \frac{9nr(r+2)}{2(r+4)} + \left(\frac{n^2(r+1)}{(n+r+4)}\right)$,
7. $\chi_\alpha(SW_n^r) = \mathrm{nr} \cdot 6^\alpha + \mathrm{nr}\,(\mathrm{r} + 4)^\alpha + \mathrm{n}(\mathrm{n} + \mathrm{r} + 1)^\alpha$,
8. $M_1^\alpha(SW_n^r) = 3\mathrm{nr} \cdot 3^\alpha + \mathrm{n}^{\alpha+1} + \mathrm{nr}(\mathrm{r} + 1)^\alpha + \mathrm{n}(\mathrm{r} + 1)^\alpha$.

Definition 14. *Let $u_i, (1 \leq i \leq n)$ be the vertices of the complete graph K_n. Let $W_r^i = C_r^i + K_1$ be the wheel with hubs $w^i, (1 \leq i \leq n)$, respectively. Let $u_iw^i, (1 \leq i \leq n)$ be an edge. The graph so constructed is called uniform n-wheel split graph $KW(n, r)$, Figure 2.*

Note: A uniform n-wheel split graph $KW(n, r)$ is a graph in which the deletion of n edges $u_iw^i, (1 \leq i \leq n)$ partitions the graph into a complete graph and n independent wheels W_r. This graph can be thought of as a generalization of the standard split graph in the sense that the elements of the independent sets are replaced by wheels here.



SW (6,9) KDW(6,9)

Figure 3. Graphs SW (6,9) and KDW(6,9).

Theorem 3.27. *Let $KW(n,r)$ be a uniform n-wheel split graph. Then*

$$M(KW(n,r);x,y) = nrx^3y^3 + nrx^3y^{r+1} + nx^ny^{r+1} + \binom{n}{2}x^ny^n.$$

Proof. Let $KW(n,r)$ uniform n-wheel split graph having $n(r+2)$ vertices and $\frac{n}{2}(4r+n+1)$ edges. The edge partition of $KW(n,r)$ is given by,

$$\begin{aligned}
|E_{\{3,3\}}| &= |uv \in E(KW(n,r)): d_u = 3 \quad and \quad d_v = 3| = nr,\\
|E_{\{3,r+1\}}| &= |uv \in E(KW(n,r)): d_u = 3 \quad and \quad d_v = r+1| = nr,\\
|E_{\{n,r+1\}}| &= |uv \in E(KW(n,r)): d_u = n \quad and \quad d_v = r+1| = n,\\
|E_{\{n,n\}}| &= |uv \in E(KW(n,r)): d_u = n \quad and \quad d_v = n|\\
&= |E(KW(n,r)) - |E_{\{3,3\}}| - |E_{\{3,r+1\}}| - |E_{\{n,r+1\}}| = \binom{n}{2}.
\end{aligned}$$

□

Corollary 3.28. *If $KW(n,r)$ be a uniform n-wheel split graph, then*

1. $M_1\big(KW(n,r)\big) = n^3 + n(r+1) + nr(r+10)$,
2. $M_2(KW(n,r)) = \frac{n^4-n^3+2n^2(r+1)+6nr(r+4)}{2}$,
3. $M_2^m(KW(n,r)) = \frac{1}{18}\left(\frac{9(r+3)+2nr(r+4)}{(r+1)} - \frac{9}{n}\right)$,
4. $S_D(KW(n,r)) = r - n + 1 + \frac{nr(r+4)^2}{3\,(r+1)} + n^2\left(\frac{r+2}{r+1}\right)$,
5. $H(KW(n,r)) = nr\left(\frac{r+10}{3(r+4)}\right) + n\left(\frac{n+r+3}{2(n+r+1)}\right) - \frac{1}{2}$,
6. $I_n(KW(n,r)) = \frac{1}{4}n^2(n+3) + \frac{9nr}{2} - \frac{9nr}{(r+4)} - \frac{n^3}{(n+r+1)}$,
7. $\chi_\alpha(KW(n,r)) = nr \cdot 6^\alpha + nr\,(r+4)^\alpha + n(n+r+1)^\alpha + \binom{n}{2}(2n)^\alpha$
8. $M_1^\alpha(KW(n,r)) = nr \cdot 3^{\alpha+1} + n^{\alpha+1} + n(n-1)n^\alpha + nr(r+1)^\alpha + n(r+1)^\alpha$.

Definition 15. *Let $u_i, (1 \leq i \leq n)$ be the vertices of a star S_{n-1} with a hub at x. Let $u_i w^i, (1 \leq i \leq n)$ be an edge. Let $W_r^i = C_r^i + K_1$ be wheels with hubs $w^i, (1 \leq i \leq n)$. The graph so obtained is denoted by $SW(n,r)$, Figure 3.*

Theorem 3.29. *Let $SW(n,r)$ be the graph having $n(r+2)+1$ vertices and $2n(r+1)$ edges. Then*

$$M(SW(n,r);x,y) = nx^2y^n + nx^2y^{r+1} + nrx^3y^3 + nrx^3y^{r+1}.$$

Proof. Let $SW(n,r)$ is a graph having $n(r+2)+1$ vertices and $2n(r+1)$ edges. The edge partition of $SW(n,r)$ is given by,

$$\begin{aligned}
|E_{\{2,n\}}| &= |uv \in E(SW(n,r)): d_u = 2 \quad and \quad d_v = n| = n,\\
|E_{\{2,r+1\}}| &= |uv \in E(SW(n,r)): d_u = 2 \quad and \quad d_v = r+1| = n,\\
|E_{\{3,3\}}| &= |uv \in E(SW(n,r)): d_u = 3 \quad and \quad d_v = 3| = nr,\\
|E_{\{3,r+1\}}| &= |uv \in E(SW(n,r)): d_u = 3 \quad and \quad d_v = r+1|\\
&= |E(SW(n,r)) - |E_{\{2,n\}}| - |E_{\{2,r+1\}}| - |E_{\{3,3\}}| = nr.
\end{aligned}$$

□

Corollary 3.30. *If $SW(n,r)$ be a graph, then*

1. $M_1(SW(n,r)) = n^2 + n(r+5) + nr(r+10)$,
2. $M_2(SW(n,r)) = 2n^2 + 2n(r+1) + 3nr(r+4)$,
3. $M_2^m(SW(n,r)) = \frac{2nr^2+8nr+9(n+r+1)}{18(r+1)}$,
4. $S_D(SW(n,r)) = \frac{3n^2(r+1)+3n(r^2+2r+5)+2(6(r+1)+nr(r+4)^2)}{6(r+1)}$,
5. $H(SW(n,r)) = \frac{nr(r+10)}{3\,(r+4)} + \frac{2n(n+r+5)}{(n+2)\,(r+3)}$,
6. $I_n(SW(n,r)) = \frac{2n^2}{n+2} + \frac{2n(r+1)}{r+3} + \frac{9nr(r+2)}{2(r+4)}$,
7. $\chi_\alpha(SW(n,r)) = n\,(n+2)^\alpha + n\,(r+3)^\alpha + nr \cdot 6^\alpha + nr\,(r+4)^\alpha$,
8. $M_1^\alpha(SW(n,r)) = n2^{\alpha+1} + nr \cdot 3^{\alpha+1} + n^{\alpha+1} + n(r+1)^\alpha + nr(r+1)^\alpha$.

Definition 16. *Let $x_i, (1 \leq i \leq n)$ be the vertices of the complete graph K_n. Let $W_r^i = C_r^i + K_1$ be wheel with hub $w^i, (1 \leq i \leq n)$. Let $x_i w^i, (1 \leq i \leq n)$ be an edge. Subdivide each edge $x_i w^i$ by $u_i, (1 \leq i \leq n)$. The graph so obtained is denoted by $KDW(n,r)$, Figure 3.*

Theorem 3.31. *Let $KDW(n,r)$ be the graph having $n(r+3)$ vertices and $\frac{n}{2}(4r+n+3)$. Then*

$$M(KDW(n,r);x,y) = nx^2y^n + nx^2y^{r+1} + nrx^3y^3 + nrx^3y^{r+1} + \binom{n}{2}x^ny^n.$$

Proof. Let $KDW(n,r)$ is a graph having $n(r+3)$ vertices and $\frac{n}{2}(4r+n+3)$ edges. The edge partition of $KDW(n,r)$ is given by,

$$
\begin{aligned}
|E_{\{2,n\}}| &= |uv \in E(KDW(n,r)) \colon d_u = 2 \quad and \quad d_v = n| = n,\\
|E_{\{2,r+1\}}| &= |uv \in E(KDW(n,r)) \colon d_u = 2 \quad and \quad d_v = r+1| = n,\\
|E_{\{3,3\}}| &= |uv \in E(KDW(n,r)) \colon d_u = 3 \quad and \quad d_v = 3| = nr,\\
|E_{\{3,r+1\}}| &= |uv \in E(KDW(n,r)) \colon d_u = 3 \quad and \quad d_v = r+1| = nr,\\
|E_{\{n,n\}}| &= |uv \in E(KDW(n,r)) \colon d_u = n \quad and \quad d_v = n|\\
&= |E(KDW(n,r))| - |E_{\{2,n\}}| - |E_{\{2,r+1\}}| - |E_{\{3,3\}}| - |E_{\{3,r+1\}}| = \binom{n}{2}.
\end{aligned}
$$

□

Corollary 3.32. *If* $KDW(n,r)$ *be a graph, then*

1. $M_1\big(KW(n,r)\big) = \quad n^3 + n(r+5) + nr(r+10)$,
2. $M_2(KW(n,r)) = \frac{n(n(n(n-1)+4)+4)+nr(3r+14)}{2}$,
3. $M_2^m(KW(n,r)) = \frac{9n^2-9(r+1)+2n(9(r+1)+nr(r+4))}{18n(r+1)}$,
4. $S_D(KW(n,r)) = \frac{3(r+1)(3n^2+4)+3n(r^2+3)+2nr(r+4)^2}{6\,(r+1)}$,
5. $H(KW(n,r)) = n\left(\frac{1}{2} + \frac{2}{n+2} + \frac{2}{r+3}\right) + nr\left(\frac{1}{3} + \frac{2}{r+4}\right) - \frac{1}{2}$,
6. $I_n(KW(n,r)) = \frac{n^3}{4} + n^2\left(\frac{2}{n+2} - \frac{1}{4}\right) + \frac{2n(r+1)}{r+3} + \frac{9nr(2+r)}{2(r+4)}$,
7. $\chi_\alpha(KW(n,r)) = n\,(n+2)^\alpha + n\,(r+3)^\alpha + nr \cdot 6^\alpha + nr\,(r+4)^\alpha + \binom{n}{2}(2n)^\alpha$,
8. $M_1^\alpha(KW(n,r)) = n \cdot 2^{\alpha+1} + nr \cdot 3^{\alpha+1} + n^{\alpha+1} + n(r+1)^\alpha + nr(r+1)^\alpha + (n-1)n^{\alpha+1}$

4. M−POLYNOMIAL OF SOME NANOSTRUCTURES

In science and technology, nanostructures play a vital role in small electronic devices to big satellites, pharmaceutical and medical treatments, communication and information, food science and so on. Among these, M-polynomial of dendrimers were studied in [33], V-phenylenic nanotubes and nanotori in [29] titania nanotubes in [34], Armchair polyhex nanotube and zig-zag polyhex nanotubes were encountered in [35]. In this paper, we consider $TUC_4C_8[p,q]$ nanotube, $TUC_4C_8[p,q]$ nanotorus, line graph of the subdivision graph of $TUC_4C_8[p,q]$ nanotube and $TUC_4C_8[p,q]$ nanotorus, V-tetracenic nanotube and V-tetracenic nanotorus and compute M-polynomial.

Let p and q denote the number of squares in a row and the number of rows of squares, respectively in nanotube and nanotorus of $TUC_4C_8[p,q]$. The nanotube and nanotorus of $TUC_4C_8[4,3]$ is shown in Figure 4 (a), (b) respectively. The line graph of subdivision graph of $TUC_4C_8[4,3]$ nanotube is given in Figure 5 (b). The line graph of subdivision graph of $TUC_4C_8[4,2]$ nanotorus is given in Figure 6 (b). The structures V-tetracenic nanotube and V-tetracenic nanotorus are given in Figures 7 and 8, respectively.

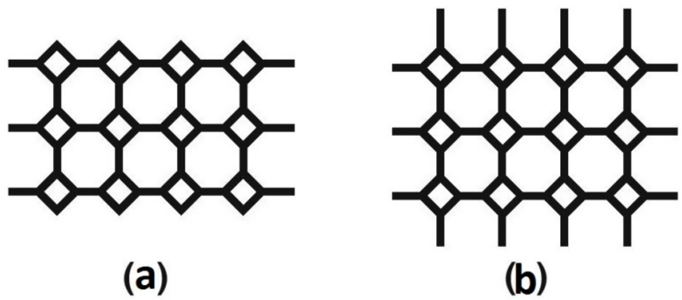


Figure 4. (a) $TUC_4C_8[4,3]$ nanotube; (b) $TUC_4C_8[4,3]$ nanotorus.

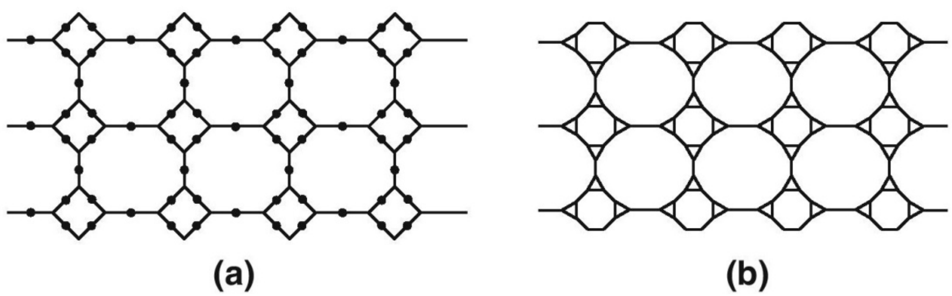


Figure 5. (a) Subdivision graph of $TUC_4C_8[4,3]$ of nanotube; (b) line graph of the subdivision graph of $TUC_4C_8[4,3]$ of nanotube.

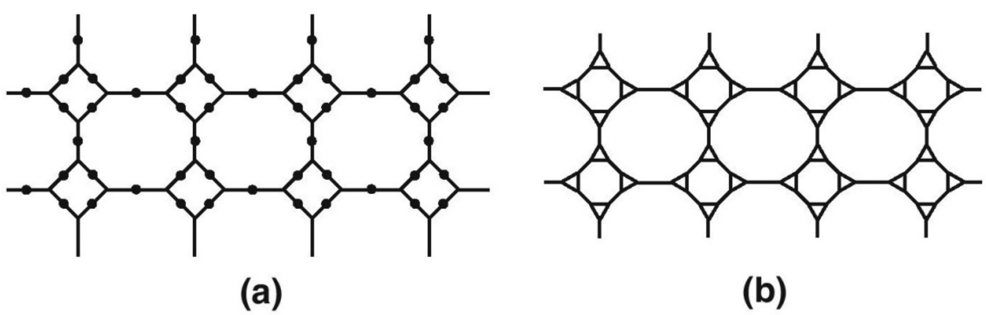


Figure 6. (a) Subdivision graph of $TUC_4C_8[4,2]$ of nanotorus; (b) line graph of the subdivision graph of $TUC_4C_8[4,2]$ of nanotorus.

We now obtain M-polynomial of these nanostructures as follows.

Theorem 4.1. *Let $A = TUC_4C_8[p,q]$ nanotube. Then*

$$M(A;x,y) = 4px^2y^3 + (6pq - 5p)x^3y^3.$$

Proof. The $TUC_4C_8[p,q]$ nanotube has $4pq$ vertices and $6pq - p$ edges. The edge set of $TUC_4C_8[p,q]$ nanotube can be partitioned as,

$$\begin{aligned} |E_{\{2,3\}}| &= |uv \in E(A): d_u = 2 \quad and \quad d_v = 3| = 4p, \\ |E_{\{3,3\}}| &= |uv \in E(A): d_u = 3 \quad and \quad d_v = 3| \\ &= |E(A) - |E_{\{2,3\}}| = 6pq - 5p. \end{aligned}$$

□

Theorem 4.2. *Let* $B = TUC_4C_8[p,q]$ *nanotorus. Then,* $M(B;x,y) = 6pqx^3y^3$.

Proof. The $TUC_4C_8[p,q]$ nanotorus is a 3-$regular$ graph with $6pq$ edges. Thus, from Lemma 2.1, M-polynomial of $TUC_4C_8[p,q]$ nanotorus is $M(B;x,y) = 6pqx^3y^3$.

Theorem 4.3. *Let* C *be the line graph of subdivision graph of* $TUC_4C_8[p,q]$ *nanotube. Then*

$$M(C;x,y) = 2px^2y^2 + 4px^2y^3 + p(18q-11)x^3y^3.$$

Proof. The line graph of subdivision graph of $TUC_4C_8[p,q]$ nanotube has $12pq - 2p$ vertices and $18pq - 5p$ edges. The edge partition of line graph of subdivision graph of $TUC_4C_8[p,q]$ nanotube is given by,

$$\begin{aligned} |E_{\{2,2\}}| &= |uv \in E(C): d_u = 2 \quad and \quad d_v = 2| = 2p, \\ |E_{\{2,3\}}| &= |uv \in E(C): d_u = 2 \quad and \quad d_v = 3| = 4p, \\ |E_{\{3,3\}}| &= |uv \in E(C): d_u = 3 \quad and \quad d_v = 3| \\ &= |E(C) - |E_{\{2,2\}}| - |E_{\{2,3\}}| = 18pq - 11p. \end{aligned}$$

□

Theorem 4.4. *Let* D *be the line graph of subdivision graph of* $TUC_4C_8[p,q]$ *nanotorus. Then* $M(D;x,y) = 18pqx^3y^3$.

Proof. The line graph of subdivision graph of $TUC_4C_8[p,q]$ nanotorus is a 3-$regular$ graph with $18pq$ edges. Thus, from Lemma 2.1 we have, $M(D;x,y) = 18pqx^3y^3$.

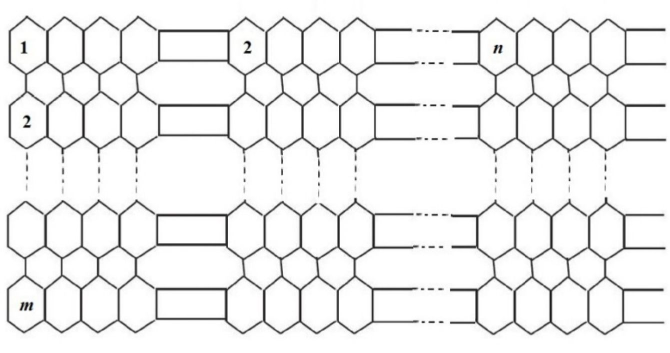


Figure 7. V-tetracenic nanotube $G[p,q]$.

Theorem 4.5. *Let* H *be the V-tetracenic nanotube. Then*

$$M(H;x,y) = 16px^2y^3 + (27q-20)px^3y^3.$$

Proof. The V-tetracenic nanotube has $18pq$ vertices and $27pq - 4p$ edges. The edge partition of V-tetracenic nanotube is obtained as,

$$\begin{aligned}|E_{\{2,3\}}| &= |uv \in E(H): d_u = 2 \ and \ d_v = 3| = 16p,\\ |E_{\{3,3\}}| &= |uv \in E(H): d_u = 3 \ and \ d_v = 3|\\ &= |E(H)| - |E_{\{2,3\}}| = 27pq - 20p.\end{aligned}$$

□

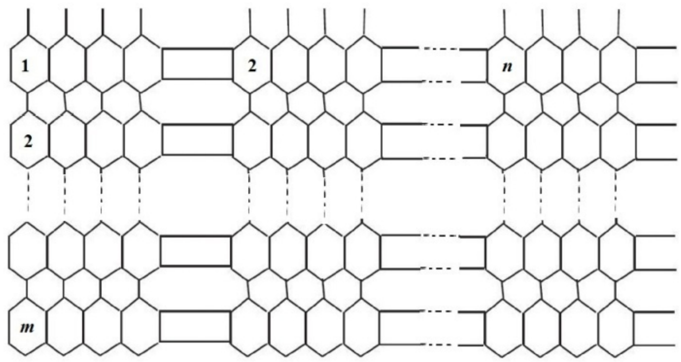


Figure 8. V-tetracenic nanotorus $G[p, q]$.

Theorem 4.6. *Let I be the V-tetracenic nanotorus. Then* $M(I; x, y) = 27pqx^3y^3$.

Proof. The proof follows from Lemma 2.1 as V-tetracenic nanotorus is 3-regular graph with $27pq$ edges.

□

We skip calculating topological indices of these nanostructures as it is routine work.

5. Concluding Remarks

In this paper, we have proposed new operators to derive general sum connectivity index and first general Zagreb index of a graph from the respective M-polynomial. Further, we have obtained M-polynomials of some graph operations and cycle related graphs. In addition, some degree based topological indices of these graphs are derived. The advantage of M-polynomial is that, from that one expression we can obtain several degree-based topological indices. It is very challenging to obtain new operators to derive all the degree-based topological indices from M-polynomial.

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