

Bounds for the Co-PI Index of a Graph

E. KAYA^A AND A. D. MADEN^B

^aDepartment of Mathematics, Faculty of Science and Literature, Iğdır University, Iğdır, Turkey

^bDepartment of Mathematics, Faculty of Science, Selçuk University, Campus, 42075, Konya, Turkey

Correspondence should be addressed to A. D. Maden; drdilekgungor@gmail.com

Received 10 December 2014; Accepted 2 January 2015

ACADEMIC EDITOR: ALI REZA ASHRAFI

ABSTRACT In this paper, we present some inequalities for the Co-PI index involving the some topological indices, the number of vertices and edges, and the maximum degree. After that, we give a result for trees. In addition, we give some inequalities for the largest eigenvalue of the Co-PI matrix of G .

KEYWORDS Co-PI index · Co-PI matrix · Co-PI spectral radius.

1. INTRODUCTION

Let $G = (V, E)$ be a finite connected simple graph with $n = |V|$ vertices and $m = |E|$ edges. For vertices $u, v \in V$, the distance $d(u, v)$ is defined as the length of the shortest path between u and v in G . The diameter $diam(G)$ is the greatest distance between two vertices of G . The degree $\deg_G(v)$ of a vertex v is the number of edges incident with it in G . Let $e = uv$ be an edge connecting vertices u and v in G . Define the sets:

$$N_u(e) = \{z \in V \mid d_G(z, u) < d_G(z, v)\}$$

$$N_v(e) = \{z \in V \mid d_G(z, v) < d_G(z, u)\}$$

which are sets consisting of vertices lying closer to u than to v and those lying closer to v than to u , respectively. The number of such vertices are denoted by

$$n_u = n_u(e) = |N_u(e)| \text{ and } n_v = n_v(e) = |N_v(e)|.$$

Other terminology and notations needed will be introduced as it naturally occurs in the following and we use [1,2,3] for those not defined here. A topological index is a number related to graph which is invariant under graph isomorphism.

In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacologic, toxicologic, biological and other properties of chemical compounds [2]. There exist several types of such indices, especially those based on vertex and edge distances. One of the most intensively studied topological indices is the Wiener index W , defined as the sum of distances between all pairs of vertices of the molecular graph [4,5]. The vertex PI index [8,9,10], Szeged index [5,6,7] and the first Zagreb index [11,12], defined as follows:

$$PI_v(G) = \sum_{e \in E(G)} n_u(e) + n_v(e),$$

$$Sz(G) = \sum_{e \in E(G)} n_u(e)n_v(e)$$

and

$$M_1(G) = \sum_{v \in V(G)} \deg^2(v),$$

respectively.

Recently, Hassani et al. introduced a new topological index similar to the vertex version of PI index [14]. This index is called the Co-PI index of G and defined as:

$$Co-PI_v(G) = \sum_{e=uv \in E(G)} |n_u(e) - n_v(e)|. \quad (1)$$

Here the summation goes over all edges of G . Fath-Tabar et al. proposed the Szeged matrix and Laplacian Szeged matrix in [13]. Then Su et al. introduced the Co-PI matrix of a graph [15]. The adjacent matrix $A(G) = [a_{ij}]_{n \times n}$ of G is the integer matrix with rows and columns indexed by its vertices, such that the ij -th entry is equal to the number of edges connecting i and j . Let the weight of the edge $e = uv$ be a non-negative integer $|n_u(e) - n_v(e)|$, we can define a weight function: $w: E \rightarrow R^+ \cup \{0\}$ on E , which is said to be the Co-PI weighting of G . The adjacency matrix of G weighted by the Co-PI weighting is said to be its Co-PI matrix and denoted by $M_{CPI}(G) = [c_{ij}]_{n \times n}$. That is,

$$c_{ij} = \begin{cases} |n_{v_i}(e) - n_{v_j}(e)|, & e = v_i v_j \\ 0 & , \text{ otherwise.} \end{cases}$$

Its eigenvalues are said to be the Co-PI eigenvalues of G and denoted by $\lambda_i^*(G)$ for $k = 1, 2, \dots, |V|$. Easy verification shows that the Co-PI index of G can be expressed as one half of the sum of all entries of $M_{CPI}(G)$, i.e.,

$$Co-PI_v(G) = \frac{1}{2} \sum_{i=1}^n M_{CPI_i}(G),$$

where M_{CPI_i} is the sum of i -th row of the matrix M_{CPI} .

In this paper, we establish some bounds for the Co-PI index, then we give a lower and upper bounds for trees. In addition, some inequalities for the largest eigenvalue of the Co-PI matrix are computed.

2. MAIN RESULTS

In this section, we give some bounds for the Co-PI index.

Theorem 2.1. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then,*

$$Co-PI_v(G) \leq m(n-2)$$

with equality if and only if G is isomorphic to S_n .

Proof. Let $e = uv$ be an arbitrary edge. Since $|n_u(e) - n_v(e)| \leq n-2$ by (1), we have

$$Co-PI_v(G) = \sum_{e=uv \in E(G)} |n_u(e) - n_v(e)| \leq \sum_{e=uv \in E(G)} (n-2) = m(n-2)$$

with equality if and only if $n_u(e) = n-1$ and $n_v(e) = 1$ if and only if G is isomorphic to the star graph S_n .

We present the our lemma which will be used for the following three main results (Theorem 2.2, 2.3 and 2.4).

Lemma 2.1. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then,*

$$\sum_{e=uv \in E(G)} [n_u^2(e) + n_v^2(e)] \leq m(n^2 - 2n + 2)$$

with equality if and only if G is isomorphic to S_n .

Proof. Let $e = uv$ be an edge of the graph G . As the maximum value of $n_u(e)$ is $n-1$ and the minimum value of $n_v(e)$ is 1, we get

$$n_u^2(e) + n_v^2(e) \leq (n-1)^2 + 1^2.$$

After summing over all edges of G , we have

$$\sum_{e=uv \in E(G)} [n_u^2(e) + n_v^2(e)] \leq \sum_{e=uv \in E(G)} [n^2 - 2n + 2] = m(n^2 - 2n + 2).$$

Equality holds if and only if $n_u(e) = n-1$ and $n_v(e) = 1$ if and only if G is isomorphic to the star graph S_n .

Theorem 2.2. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then,*

$$Co-PI_v(G) \leq \sqrt{m} \sqrt{m(n^2 - 2n + 2) - 2Sz(G)}$$

with equality if and only if G is isomorphic to S_n .

Proof. Let $e = uv$ be an arbitrary edge. From (1), the Cauchy-Schwarz inequality (shortly C-S inequality) and Lemma 2.1, we have

$$\begin{aligned} Co-PI_v(G) &= \sum_{e=uv \in E(G)} |n_u(e) - n_v(e)| \\ &\leq \sqrt{\sum_{e=uv \in E(G)} 1} \sqrt{\sum_{e=uv \in E(G)} (n_u(e) - n_v(e))^2} \\ &\leq \sqrt{m} \sqrt{m(n^2 - 2n + 2) - 2Sz(G)} \end{aligned}$$

with equality if and only if $n_u(e) = n-1$ and $n_v(e) = 1$ if and only if G is isomorphic to the star graph S_n .

Theorem 2.3. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then,*

$$Co-PI_v(G) \leq \sqrt{m(n^2 - 2n + 2) - 2Sz(G) + m(m-1)(n-2)^2}$$

with equality if and only if G is isomorphic to S_n .

Proof. Let $e = uv$ be an arbitrary edge. By (1), we have

$$\begin{aligned} (Co-PI_v(G))^2 &= \sum_{e=uv \in E(G)} \left[n_u^2(e) + n_v^2(e) - 2n_u(e)n_v(e) \right] \\ &\quad + 2 \sum_{\substack{e=uv \\ e'=ab \\ e \neq e'}} |n_u(e) - n_v(e)| |n_a(e')n_b(e')| \\ &\leq m(n^2 - 2n + 2) - 2Sz(G) + m(m-1)(n-2)^2 \end{aligned}$$

from Lemma 2.1. Thus,

$$Co-PI_v(G) \leq \sqrt{m(n^2 - 2n + 2) - 2Sz(G) + m(m-1)(n-2)^2}$$

with equality if and only if $n_u(e) = n-1$ and $n_v(e) = 1$ if and only if G is isomorphic to the star graph S_n .

Next result is a lower bound for the Co-PI index.

Theorem 2.4. *Let G be a connected graph with $n \geq 2$. Then,*

$$Co-PI_v(G) \geq \sqrt{\left| (n^2 - 2n + 2) - 2Sz(G) \right|}.$$

Equality holds if and only if G is isomorphic to the K_2 .

Proof. Let $e = uv$ be an arbitrary edge. Using (1)

$$\begin{aligned} (Co-PI_v(G))^2 &= \sum_{e=uv \in E(G)} \left[n_u^2(e) + n_v^2(e) - 2n_u(e)n_v(e) \right] \\ &\quad + 2 \sum_{\substack{e=uv \\ e'=ab \\ e \neq e'}} |n_u(e) - n_v(e)| |n_a(e')n_b(e')| \\ &\geq (n^2 - 2n + 2) - 2Sz(G) \end{aligned}$$

Hence the result. Equality holds if and only if $n_u(e) = n_v(e) = 1$ if and only if G is isomorphic to the K_2 .

We will establish relations for the Co-PI index using the following Ozeki inequality.

Theorem 2.5 (Ozeki Inequality [19]) *If (a_1, a_2, \dots, a_n) and (b_1, b_2, \dots, b_n) are $n-1$ tuples of real numbers satisfying $0 \leq m_1 \leq a_i \leq M_1$ and $0 \leq m_2 \leq b_i \leq M_2$ for $i = 1, 2, \dots, n$ then*

$$\left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{i=1}^n b_i^2 \right) - \left(\sum_{i=1}^n a_i b_i \right)^2 \leq \frac{1}{3} n^2 (M_1 M_2 - m_1 m_2)^2.$$

Theorem 2.6. *Let G be a connected graph with $n \geq 2$. Then,*

$$Co-PI_v(G) \leq \sqrt{\frac{m^2}{3} (n-2)^2 + PI_v^2(G) - 4mSz(G)}$$

equality holds if and only if G is isomorphic to the K_2 .

Proof. By setting in Theorem 2.5 the values $a_i = 1$ and $b_i = n_u(e) + n_v(e)$ for $i = 1, 2, \dots, m$, we have

$$\left(\sum_{i=1}^m 1 \right) \left(\sum_{e=uv \in E(G)} [n_u(e) + n_v(e)]^2 \right) \leq \frac{1}{3} m^2 (M_1 M_2 - m_1 m_2)^2 + \left(\sum_{e=uv \in E(G)} n_u(e) + n_v(e) \right)^2$$

Since $m_1 = M_1 = 1$, we need to estimate the upper and lower bounds for b_i , it is known that

$$M_2 = \max_{e=uv \in E(G)} \{n_u(e) + n_v(e)\} \leq n$$

and

$$m_2 = \min_{e=uv \in E(G)} \{n_u(e) + n_v(e)\} \geq 2.$$

Then we have,

$$m \sum_{e=uv \in E(G)} [n_u(e) + n_v(e)]^2 \leq \frac{1}{3} m^2 (n-2)^2 + \left(\sum_{e=uv \in E(G)} n_u(e) + n_v(e) \right)^2. \quad (2)$$

Using (1) and the C-S inequality, we obtain

$$\begin{aligned} \sum_{e=uv \in E(G)} |n_u(e) - n_v(e)| &\leq \sqrt{m \sum_{e=uv \in E(G)} (n_u(e) - n_v(e))^2} \\ &= \sqrt{m \sum_{e=uv \in E(G)} n_u^2(e) + n_v^2(e) - 2m \sum_{e=uv \in E(G)} n_u(e)n_v(e)}. \end{aligned} \quad (3)$$

Combining equalities (2) and (3), we complete the proof. The equality holds if and only if $n = 2$ if and only if G is isomorphic to the complete graph K_2 .

Theorem 2.7. *Let G be a connected graph with diameter 2. Then,*

$$Co-PI_v(G) \leq \sqrt{mM_1(\Delta - 2)}.$$

Proof. Let $e = v_i v_j$ be an arbitrary edge of G , such that it belongs to exactly $t(e)$ triangles. Since $diam(G) = 2$, then we get $n_{v_i}(e) = \deg(v_i) - t(e)$ and $n_{v_j}(e) = \deg(v_j) - t(e)$ equalities. Therefore, we obtain

$$|n_{v_i}(e) - n_{v_j}(e)| = |\deg(v_i) - \deg(v_j)|.$$

Hence, we have

$$\begin{aligned} \sum_{e=v_i v_j \in E(G)} |n_{v_i}(e) - n_{v_j}(e)| &= \sum_{e=v_i v_j \in E(G)} |\deg(v_i) - \deg(v_j)| \\ &\leq \sqrt{m \sum_{e=v_i v_j \in E(G)} (\deg(v_i) - \deg(v_j))^2} \quad (\text{from C-S inequality}) \\ &= \sqrt{m \sum_{e=v_i v_j \in E(G)} (\deg^2(v_i) - \deg^2(v_j)) - 2m \sum_{e=v_i v_j \in E(G)} \deg(v_i) \deg(v_j)}. \end{aligned} \quad (4)$$

Using above inequality, we have

$$\sum_{e=v_i v_j \in E(G)} \deg^2(v_i) + \deg^2(v_j) = \sum_{v_i \in V(G)} \deg(v_i) \deg^2(v_i) \leq \Delta M_1(G). \quad (5)$$

Combining inequalities (1), (4) and (5), we complete the proof.

Now we present a result for the Co-PI index of trees.

Corollary 2.1. *Let T be a tree with n vertices. Then,*

$$2(n-2) \leq Co-PI_v(T) \leq (n-1)(n-2).$$

Lower bound holds if and only if T is isomorphic to K_2 and upper bound holds if and only if T is isomorphic to S_n .

Proof. Lower Bound:

Since T is a tree, there are at least two pendant vertex. Let u be a pendant vertex and u is adjacent with u_1 . Since u is a pendant vertex, then $n_u(e) = 1$ and $n_{u_1}(e) = n - 1$. From these equalities, we have $|n_{u_1}(e) - n_u(e)| = n - 2$. Now let v be other pendant vertex and v is adjacent with v_1 . Since v is a pendant vertex, then $n_v(e) = 1$ and $n_{v_1}(e) = n - 1$. From this, we also have $|n_{v_1}(e) - n_v(e)| = n - 2$. From definition of the Co-PI index, we get

$$\sum_{e=uv \in E(T)} |n_u(e) - n_v(e)| \geq 2(n - 2).$$

The equality holds if and only if $n_u(e) = n_v(e) = n - 1$ if and only if T is isomorphic to the complete graph K_2 .

Upper Bound:

Since T is a tree with n vertices, then T has $n - 1$ edges. By (1), we have

$$\sum_{e=uv \in E(T)} |n_u(e) - n_v(e)| \leq \sum_{e=uv \in E(T)} (n - 2) = (n - 1)(n - 2).$$

The equality holds if and only if for all edges, $|n_u(e) - n_v(e)| = n - 2$ if and only if T is isomorphic to the star graph S_n .

Remark 2.1. *If we consider definition of the Co-PI index, we can give a different presentation of it as in the following*

$$\begin{aligned} \text{Co-PI}_v(G) &= \sum_{e=uv \in E(G)} |n_u(e) - n_v(e)| \\ &= \sum_{0 \leq i < n} \left| \{uv \in E(G) : n_u(e) \leq i \text{ and } n_v(e) > i\} \right|. \end{aligned}$$

Using this equality we obtained in above conjecture for the Co-PI index. We can not proof it, but we think that this result is one of the best results.

Conjecture 2.1. *Let G be a connected graph with n vertices and m edges. Then,*

$$\text{Co-PI}_v(G) \leq \sum_{e=uv \in E(G)} \min_{\substack{0 \leq i < m \\ n_u(e) \leq i \\ n_v(e) > i}} \{n_u(e), n_v(e)\}.$$

Now, we compare the results for two different graphs in the following example.

Example 2.1. Let $G_1 = (V_1, E_1)$ be a graph with vertex set $V_1 = \{v_1, v_2, v_3, v_4\}$ and edge set $E_1 = \{v_1v_2, v_2v_3, v_2v_4, v_3v_4\}$ and $G_2 = (V_2, E_2)$ be a graph with vertex set $V_2 = \{u_1, u_2, u_3, u_4, u_5\}$ and edge set $E_2 = \{v_1v_2, v_1v_4, v_2v_3, v_2v_5, v_4v_5\}$. Then,

Table 1. The Values of $Co-PI_v(G_1)$ and $Co-PI_v(G_2)$.

	Th 2.1	Th 2.2	Th 2.3	Th 2.6	Th 2.7	Conj 2.1
$Co-PI_v(G_1) = 4$	8	9,79	8,48	6,11	8,48	4
$Co-PI_v(G_2) = 7$	15	12,04	14,45	11,83	-	11

3. BOUNDS FOR THE LARGEST CO-PI EIGENVALUE OF G

The resistance distance is a metric function on a graph, proposed by Klein and Randić[21]. The resistance distance R_{ij} between the vertices v_i and v_j of a connected graph G is defined to be equal to the resistance between the respective two nodes of an electrical network, corresponding to G , in which the resistance between any two adjacent nodes is 1 Ohm. Maden et al. [16] proposed some results for the maximum eigenvalue of the resistance-distance matrix.

Now we present some inequalities for the largest Co-PI eigenvalue of G . The following lemma is one of the key point in our considerations.

Lemma 3.1 (B. Zhou [17,18]) Let $B = (B_{ij})$ be an $n \times n$ nonnegative, irreducible, symmetric matrix ($n \geq 2$) with row sums B_1, B_2, \dots, B_n . If $\lambda_1(B)$ is the maximum eigenvalue of B , then

$$\sqrt{\frac{\sum_{i=1}^n B_i^2}{n}} \leq \lambda_1(B) \leq \max_{1 \leq j \leq n} \sum_{i=1}^n B_{ij} \sqrt{\frac{B_j}{B_i}}$$

with equality holding if and only if $B_1 = B_2 = \dots = B_n$ or if there is a permutation matrix Q such that

$$Q' B Q = \begin{pmatrix} 0 & C \\ C^t & 0 \end{pmatrix}$$

where all the row sums of C are equal.

Theorem 3.1. *Let G be a connected graph with $n \geq 2$. Then,*

$$\sqrt{\frac{\sum_{i=1}^n M_{CPI_i}^2}{n}} \leq \lambda_1^*(G) \leq \max_{1 \leq j \leq n} \sum_{i=1}^n M_{CPI_{ij}} \sqrt{\frac{M_{CPI_j}}{M_{CPI_i}}} \quad (6)$$

where M_{CPI_i} is the sum of i -th row of the matrix M_{CPI} . Moreover equality holds if and only if $M_{CPI_1} = M_{CPI_2} = \dots = M_{CPI_n}$.

Proof. It is clear that the matrix M_{CPI} is irreducible for $n \geq 2$, and then, by Lemma 3.1, we obtain the inequality in (6). By definition, we know that $M_{CPI} \neq 0$ for $i \neq j$ and $M_{CPI} = 0$ otherwise. We note that for $n \geq 2$ there is no permutation matrix Q such that

$$Q^t M_{CPI} Q = \begin{pmatrix} 0 & C \\ C^t & 0 \end{pmatrix}$$

where all the row sums of C are equal. By Lemma 3.1, the equality in (6) holds if and only if $M_{CPI_1} = M_{CPI_2} = \dots = M_{CPI_n}$.

Corollary 3.1. *Let G be a connected graph with $n \geq 2$. Then,*

$$\lambda_1^*(G) \geq \frac{2Co - PI_v(G)}{n} \quad (7)$$

with equality holding if and only if $M_{CPI_1} = M_{CPI_2} = \dots = M_{CPI_n}$.

Proof. By the left part of the inequality given in (7) view of the C-S inequality, we obtain

$$\lambda_1^*(G) \geq \sqrt{\frac{\sum_{i=1}^n M_{CPI_i}^2}{n}} \geq \frac{\sum_{i=1}^n M_{CPI_i}}{n} = \frac{2Co - PI_v(G)}{n}$$

and equality holds if and only if $M_{CPI_1} = M_{CPI_2} = \dots = M_{CPI_n}$.

Note that $\text{trace}[M_{CPI}] = 0$ and denote by $N = N(G)$ the trace of M_{CPI}^2 . Therefore, for $i = 1, 2, \dots, n$, the eigenvalues $\lambda_i^*(G)$ of M_{CPI} satisfy the relations

$$\sum_{i=1}^n \lambda_i^*(G) = 0 \quad (8)$$

$$\sum_{i=1}^n \lambda_i^{*2}(G) = N(G) \quad (9)$$

Let Γ be the class of connected graphs whose Co-PI matrices have exactly one positive eigenvalue. In the following, we give upper and lower bounds for $\lambda_1^*(G)$ of graphs in the class Γ in terms of the number of vertices and $N(G)$.

Theorem 3.2. *Let $G \in \Gamma$ with $n \geq 2$ vertices. Then,*

$$\lambda_1^*(G) \leq \sqrt{\frac{n-1}{n}N(G)}.$$

Proof. By (8), we have $\lambda_1^*(G) = -\sum_{i=2}^n \lambda_i^*(G)$. Further, by the C-S inequality and using (9)

$$\begin{aligned} \lambda_1^{*2}(G) &= \left[\sum_{i=2}^n \lambda_i^*(G) \right]^2 \leq (n-1) \sum_{i=2}^n \lambda_i^{*2}(G) \\ &= (n-1) [N(G) - \lambda_1^{*2}(G)]. \end{aligned}$$

This implies the proof of Theorem 3.2.

Theorem 3.3. *Let $G \in \Gamma$ with $n \geq 2$ vertices. Then,*

$$\lambda_1^*(G) \geq \sqrt{\frac{N(G)}{2}}. \tag{10}$$

Proof. We first note that $\lambda_1^*(G) > 0$ and $\lambda_2^*(G) \leq 0$. Then by (9)

$$2\lambda_1^*(G) = \sum_{i=1}^n |\lambda_i^*(G)|.$$

From (8) and (9) we also have

$$\begin{aligned} \sum_{1 \leq i < j \leq n} |\lambda_i^*(G)\lambda_j^*(G)| &\geq \left| \sum_{1 \leq i < j \leq n} \lambda_i^*(G)\lambda_j^*(G) \right| \\ &= \frac{N(G)}{2} \end{aligned}$$

and so

$$\begin{aligned} 4\lambda_1^{*2}(G) &= \left[\sum_{i=1}^n |\lambda_i^*(G)| \right]^2 \\ &= \sum_{i=1}^n \lambda_i^{*2}(G) + 2 \sum_{i < j} |\lambda_i^*(G)\lambda_j^*(G)| \geq 2N(G) \end{aligned}$$

from which (10) follows.

4. NORDHAUS-GADDUM TYPE BOUNDS FOR THE LARGEST CO-PI EIGENVALUE OF G

In this section, we present Nordhaus-Gaddum type inequalities for the largest Co-PI eigenvalue of G . Before that, consider a connected graph G and its complement \bar{G} . Let G be a connected graph on $n > 2$ vertices, m edges. Further, assume that $G \in \Gamma$ has a connected complement \bar{G} with \bar{m} edges. As one can easily prove, the following equality:

$$2(m + \bar{m}) = n(n - 1).$$

Theorem 4.1 (Su et al. [15]) *Let G be connected graph with order $n \geq 3$, size m . Then,*

$$2m \leq \lambda_1^{*2}(G) + \lambda_2^{*2}(G) + \dots + \lambda_n^{*2}(G) \leq 2m(n - 2)^2.$$

In view of the Theorem 4.1, Theorem 3.2 and Theorem 3.3, we arrive at the following results, which will be presented without proof.

Theorem 4.2. *Let $G \in \Gamma$ with $n > 2$ vertices, and let \bar{G} be connected. Then,*

$$\lambda_1^*(G) + \lambda_1^*(\bar{G}) \leq \sqrt{\frac{n-1}{n}} \left[\sqrt{2m(n-2)^2} + \sqrt{(n(n-1)-2m)(n-2)^2} \right].$$

Theorem 4.3. *Let $G \in \Gamma$ with $n > 2$ vertices, and let \bar{G} be connected. Then,*

$$\lambda_1^*(G) + \lambda_1^*(\bar{G}) \geq \sqrt{m} + \sqrt{\frac{(n(n-1)-2m)}{2}}.$$

ACKNOWLEDGMENT. This work is from the first author's MsC Thesis (Ezgi KAYA (MsC Supervisor: A. Dilek MADEN), Some parametres in graphs and relations between these parametres, Selçuk University, 2013).

REFERENCES

1. N. Biggs, Algebraic Graph Theory, second ed., Cambridge University Press, Cambridge, 1994.
2. I. Gutman, O.E. Polansky, Mathematical Concepts in Organic Chemistry, Springer-Verlag, Berlin, 1986.
3. R. Todeschini, V. Consonni, Handbook of Molecular Descriptors, Wiley VCH, Weinheim, 2000.

4. A. A. Dobrynin, R. Entringer, I. Gutman, Wiener index of trees: theory and applications, *Acta Appl. Math.* 66, 2001, 211–249.
5. S. Klavžar, A. Rajapake, I. Gutman, The Szeged and the Wiener index of graphs, *Appl. Math. Lett.* 9, 1996, 45–49.
6. M. H. Khalifeh, H. Yousefi-Azari, A. R. Ashrafi, S. G. Wagner, Some new results on distance-based graph invariants, *European J. Combin.* 30, 2009, 1149–1163.
7. I. Gutman, A. Dobrynin, The Szeged index a success story, *Graph Theory Notes NY* 34, 1998, 37–44.
8. M. Mogharrab, H.R. Maimani, A.R. Ashrafi, A note on the vertex PI index of graphs, *J. Adv. Math. Studies* 2, 2009, 53–56.
9. K.C. Das, I. Gutman, Estimating the vertex PI index, *Z. Naturforsch.* 65a, 2010, 240–244.
10. M.H. Khalifeh, H. Yousefi-Azari, A.R. Ashrafi, Vertex and edge PI indices of Cartesian product graphs, *Discrete Appl. Math.* 156, 2008, 1780–1789.
11. I. Gutman, K. C. Das, The first Zagreb index 30 years after, *MATCH Commun. Math. Comput. Chem.* 50, 2004, 83–92.
12. A. Ilić, D. Stevanović, On comparing Zagreb indices, *MATCH Commun. Math. Comput. Chem.* 62, 2009, 681–687.
13. G. H. Fath-Tabar, T. Došlić, A. R. Ashrafi, On the Szeged and the Laplacian Szeged spectrum of a graph, *Linear Algebra Appl.* 433, 2010, 662–671.
14. F. Hasani, O. Khormali, A. Iranmanesh, Computation of the first vertex of Co-PI index of $TUC_4C_8(S)$ nanotubes, *Optoelectron. Adv. Mat. - Rapid Commun.* 4(4), 2010, 544–547.
15. G. Su, L. Xiong, L. Xu, On the Co-PI and Laplacian Co-PI eigenvalues of a graph, *Discrete Appl. Math.* 161, 2013, 277–283.
16. A. D. Maden (Güngör), I. Gutman, A. S. Çevik, Bounds for resistance-distance spectral radius, *Hacet. J. Math. Stat.* 42 (1), 2013, 43–50.
17. B. Zhou, On the spectral radius of nonnegative matrices, *Australas. J. Comb.* 22, 2000, 301–306.
18. B. Zhou, B. Liu, On almost regular matrices, *Util. Math.* 54, 1998, 151–155.
19. S. Izumino, H. Mori, Y. Seo, On Ozeki's inequality, *J. Inequal. Appl.* 2, 1998, 235–253.
20. D. J. Klein, M. Randić, Resistance distance, *J. Math. Chem.* 12, 1993, 81–95.