On the Saturation Number of Graphs

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ARTICLE INFO	ABSTRACT
	Let $G = (V, E)$ be a simple connected graph. A matching M in a graph G is a collection of edges of G such that no two edges from M share a vertex. A matching M is maximal if it cannot be extended to a larger matching in G. The cardinality of any smallest maximal matching in G is the saturation number of G and is denoted by $s(G)$. In this paper we study the saturation number of the corona product of two specific graphs. We also consider some graphs with certain constructions that are of importance in chemistry and study their saturation number.

1   INTRODUCTION

Matching theory is a branch of graph theory concerned with study of structural and enumerative aspects of matchings, collections of edges of a graph that do not share a vertex. For a general background on matching theory we refer the reader to the classical monograph by Lovász and Plummer in [15] and for graph theory terms not defined here we also recommend [17].

The cardinality of matching M of a graph G is called the size of the matching. It is intuitively clear that matchings of small size are not interesting (each edge is a matching of size one, and the empty set is a matching of size 0). Hence, we will be interested in matchings that are as large as possible.

A matching M is a maximum matching if there is no matching in G with greater size. The cardinality of any maximum matching in G is denoted by $\alpha'(G)$ and is called the matching number of G. Since each vertex can be incident to at most one edge of a

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matching, it follows that the matching number of a graph on n vertices cannot exceed $\lfloor n/2 \rfloor$. If each vertex of G is incident with an edge of M, the matching M is called perfect. So the number of vertices of a graph G admitting a perfect matching is even, but the opposite is generally not true. Perfect matchings are obviously also maximum matchings. The study of perfect matchings, also known as Kekulé structures has a long history in both mathematical and chemical literature. The development of matching theory has been strongly influenced and stimulated by the study of perfect matchings in benzenoid graphs. Additional impetus came with discovery of fullerenes, again mostly dealing with perfect matchings ([5,6,13,18]), but including also some structural results ([2,7]). For more details on perfect matching, we refer the reader to see [15].

There is, however, another way to quantify the idea of large matchings. A matching M is maximal if it cannot be extended to a larger matching in G. Obviously, every maximum matching is also maximal, but the opposite is generally not true. While there is vast literature on perfect and maximum matchings (see, for example, monographs in [15] and [3]), the results about maximal matchings are few and scattered through the literature. We mention here two papers that treat, among other topics, maximal matchings in trees ([14,16]), one concerned with the structure of equimatchable graphs ([11]) and a recent paper about the saturation number of benzenoid graphs ([8]).

The cardinality of any smallest maximal matching in G is the saturation number of G and is denoted by $s(G)$ (the same term, saturation number, is also used in the literature with a different meaning; we refer the reader to [10] for more information). It is easy to see that the saturation number of a graph G is at least one half of the matching number of G, i.e., $s(G) \geq \alpha'(G)/2$ ([8]). We recall that a set of vertices I is independent if no two vertices from I are adjacent. Clearly, the set of vertices that is not covered by a maximal matching is independent ([9]). This observation provides an obvious lower bound on saturation number of the graph G, i.e. $s(G) \geq (n - |I|)/2$ where G is graph of order n ([2]).

Next section is concerned with the saturation number of some specific graphs. We also study the saturation number of the corona product of two certain graphs. In Section 3, we consider some graphs with specific construction that are of importance in chemistry and study their saturation number. As consequences, we obtain the saturation number of some cactus chains.

2   SATURATION NUMBER OF SOME GRAPHS

This section is organized as follows. In the Subsection 2.1, we consider the saturation number of some certain graphs. We also examine the effects on the saturation number of them when they are modified by operations on their edges. Subsection 2.2 moves on to the saturation number of the corona product of two specific graphs.

2.1. SATURATION NUMBER OF SOME SPECIFIC GRAPHS

We start this section with the saturation number of some certain graphs. Let P_n, C_n and W_n be a path, cycle and wheel of order n, respectively. We have

(i) $\quad s(P_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil, & \text{if } n \equiv 2 \ (mod\ 3) \\ \left\lfloor \frac{n}{3} \right\rfloor, & \text{otherwise.} \end{cases}$

(ii) $\quad s(C_n) = \left\lceil \frac{n}{3} \right\rceil.$

(iii) $\quad s(W_n) = 1 + s(P_{n-2}).$

The union $G_1 \cup G_2$ of graphs G_1 and G_2 with disjoint point sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph with $V(G_1) \cup V(G_2)$ and $E(G_1) \cup E(G_2)$ ([12]). In addition, a maximum matching in a disconnected graph consists of the union of maximum matchings in each of its components. So we can conclude the following useful lemma.

Lemma 2.1 *For any two simple graphs G_1 and G_2, $s(G_1 \cup G_2) = s(G_1) + s(G_2)$.*

Here we consider the effect on the saturation number of the path and the cycle graphs when one edge of them is deleted.

Proposition 2.2 (*i*) *For every $e \in E(G)$, we have $s(C_n - e) = s(P_n) \leq s(C_n)$.*
(*ii*) *Let $V(P_n) = \{v_1, v_2, \dots, v_n\}$ and $e_i = v_i v_{i+1} \in E(P_n)$ where $1 \leq i \leq n - 1$. Then we have*

$$s(P_n - e_i) = \begin{cases} s(P_n) - 1, & \text{if } n \equiv 2 \ (mod\ 3) \text{ and } i = 1 \text{ or } \frac{n}{2} \text{ or } n - 1 \\ s(P_n) + 1, & \text{if } n \equiv 1 \ (mod\ 3) \text{ and } i = 2 \text{ or } n - 2 \\ s(P_n), & \text{otherwise.} \end{cases}$$

Proof. (*i*) It is clear that $C_n - e = P_n$. Considering the saturation number of P_n and C_n, we have the result.
(*ii*) For every $e \in E(P_n)$, we have $P_n - e = P_m \cup P_t$ where $m, t \in \mathbb{N}$ and $m + t = n$. By Lemma 2.1, we can conclude that $s(P_n - e) = s(P_m) + s(P_t)$. Now we consider three cases:

  1 If $n = 3k$ for some $k \in \mathbb{Z}$, then $m = 3k_1$ and $t = 3k_2$ where $k_1, k_2 \in \mathbb{Z}$ and $k_1 + k_2 = k$. So $s(P_n - e) = s(P_{3k_1}) + s(P_{3k_2}) = k_1 + k_2 = k = s(P_n)$; or $m = 3k_1 + 1$ and $t = 3k_2 + 2$ where $k_1, k_2 \in \mathbb{Z}$ and $k_1 + k_2 = k - 1$. Then $s(P_n - e) = s(P_{3k_1+1}) + s(P_{3k_2+2}) = k_1 + (k_2 + 1) = k = s(P_n)$.

2 Suppose that $n = 3k + 1$ for some $k \in \mathbb{Z}$. So we have $m = 3k_1$ and $t = 3k_2 + 1$ where $k_1, k_2 \in \mathbb{Z}$ and $k_1 + k_2 = k$. Therefore, $s(P_n - e) = s(P_{3k_1}) + s(P_{3k_2+1}) = k_1 + k_2 = k = s(P_n)$; or $m = 3k_1 + 2$ and $t = 3k_2 + 2$ where $k_1, k_2 \in \mathbb{Z}$ and $k_1 + k_2 = k - 1$. Then $s(P_n - e) = s(P_{3k_1+2}) + s(P_{3k_2+2}) = (k_1 + 1) + (k_2 + 1) = k + 1 = s(P_n) + 1$. Note that this case happens when $e = e_2$ or $e = e_{n-2}$.

3 If $n = 3k + 2$ for some $k \in \mathbb{Z}$, then $m = 3k_1 + 1$ and $t = 3k_2 + 1$ where $k_1, k_2 \in \mathbb{Z}$ and $k_1 + k_2 = k$. So we have $s(P_n - e) = s(P_{3k_1+1}) + s(P_{3k_2+1}) = k_1 + k_2 = k = s(P_n) - 1$. Note that this case happens if $e = e_1, e = e_{n-1}$ or $e = n/2$ (when n is even); or $m = 3k_1$ and $t = 3k_2 + 2$ where $k_1, k_2 \in \mathbb{Z}$ and $k_1 + k_2 = k$. Then
$$s(P_n - e) = s(P_{3k_1}) + s(P_{3k_2+2}) = k_1 + (k_2 + 1) = k + 1 = s(P_n).$$

2.2. SATURATION NUMBER OF THE CORONA PRODUCT OF TWO SPECIFIC GRAPHS

In this subsection, we consider the corona product of two certain graphs and study their saturation number. We recall that the corona of two graphs G_1 and G_2, written as $G_1 \circ G_2$, is the graph obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2, and then joining the i-th vertex of G_1 to every vertex in the i-th copy of G_2.

Theorem 2.3 *Let G be a simple graph of order n. Then $s(G \circ \overline{K_m}) = \alpha'(G) + l$, where $\alpha'(G)$ is the size of maximum matching M of G and l is the number of M-unsaturated vertices of G. In addition, if G has a perfect matching, then $s(G \circ \overline{K_m}) = n/2$.*

Proof. Suppose that M is a maximum matching of G and S has the smallest cardinality over all maximal matchings of $G \circ \overline{K_m}$. Let the vertex u be a M-unsaturated vertex of G and $(\overline{K_m})_u$ be a copy of $\overline{K_m}$ with the vertex set $\{v_1, v_2, \ldots, v_m\}$ such that the vertex u is adjacent to all vertices of $(\overline{K_m})_u$. Thus there exists $v \in \{v_1, v_2, \ldots, v_m\}$ such that $uv \in S$. Now if for M-unsaturated vertices $u_1, u_2, \ldots, u_l$ of G, put $S = M \cup \{u_i v | 1 \leq i \leq l, v \in V((\overline{K_m})_{u_i})\}$, then S has the smallest cardinality over all maximal matchings of $G \circ \overline{K_m}$ and so we have the result. Now suppose that the graph G has a perfect matching $N = \{e_1, e_2, \ldots, e_{n/2}\}$. To obtain a maximal matching S with the smallest cardinality for $G \circ \overline{K_m}$, it is enough to put $S = N$, and so $s(G \circ \overline{K_m}) = |S| = |N| = n/2$.

Corollary 2.4 *For every $n, m \in \mathbb{N}, s(P_n \circ \overline{K_m}) = s(C_n \circ \overline{K_m}) = \lceil n/2 \rceil$.*

Proof. Clearly, for every even n, the path P_n and the cycle C_n have perfect matching and by Theorem 2.3 we have $s(P_n \circ \overline{K_m}) = s(C_n \circ \overline{K_m}) = n/2$. Now suppose that n is odd. Then

for every maximum matching M, the path P_n and the cycle C_n have only one M-unsaturated vertex. So by Theorem 2.3, $s(P_n \circ \overline{K_m}) = s(C_n \circ \overline{K_m}) = \lfloor n/2 \rfloor + 1 = \lceil n/2 \rceil$.

Theorem 2.5 *For any simple graph G of order n, we have*

$$s(G \circ P_m) = \begin{cases} ns(P_m) + \alpha'(G) + l, & if \ m \equiv 1 \ (mod \ 3) \\ ns(P_m), & otherwise \end{cases}$$

where $\alpha'(G)$ is the size of maximum matching M of G and l is the number of M-unsaturated vertices of G.

Proof. Let $u \in V(G)$ and $(P_m)_u$ be a copy of P_m with the vertex set $\{v_1, v_2, \ldots, v_m\}$ such that the vertex u is adjacent to all vertices of $(P_m)_u$. Suppose that S and S_1 have the smallest cardinality over all maximal matchings of graphs $G \circ P_m$ and P_m, respectively. So $s(G \circ P_m) = |S|$ and $s(P_m) = |S_1|$. We have three cases:

1. Suppose that $m = 3k$ for some $k \in \mathbb{Z}$. It can be easily verified that if $v \in \{v_2, v_{m-1}\}$, then the edge uv belongs to S. Hence every vertex of G is S-saturated and so there is no edge of G in S. Also the number of edges in the smallest maximal matching of the graph $(P_m)_u - v$ with the edge uv is equal to the number of edges in the smallest maximal matching of the graph P_m. Thus we can conclude that $|S| = n|S_1|$ and we have $s(G \circ P_m) = |S| = n|S_1| = ns(P_m)$.

2. Assume that $m = 3k + 1$ for some $k \in \mathbb{Z}$. If M is a perfect matching of G, then put
$$S = \underbrace{S_1 \cup S_1 \cup \ldots \cup S_1}_{n-times} \cup M.$$
So $s(G \circ P_m) = |S| = n|S_1| + |M| = ns(P_m) + \alpha'(G)$. Now suppose that u is an M-unsaturated vertex of G and $(P_m)_u$ is a copy of P_m with the vertex set $\{v_1, v_2, \ldots, v_m\}$ such that the vertex u is adjacent to all vertices of $(P_m)_u$. Then there exists $v \in \{v_1, v_2, \ldots, v_m\}$ such that $uv \in S$. Suppose that $u_1, u_2, \ldots, u_l$ are M-unsaturated vertices of G and put
$$S = \underbrace{S_1 \cup S_1 \cup \ldots \cup S_1}_{n-times} \cup M \cup \{u_i v \mid 1 \le i \le l, v \in V((P_m)_{u_i})\}.$$
Then we have $s(G \circ P_m) = |S| = n|S_1| + |M| + l = ns(P_m) + \alpha'(G) + l$.

3. If $m = 3k + 2$ for some $k \in \mathbb{Z}$ and v is one of the vertices in the set $\{v_1, v_2, v_4, v_5, v_7, \ldots, v_{m-3}, v_{m-1}, v_m\}$, then the edge uv belongs to S. Therefore, every vertex of G is S-saturated and so there is no edge of G in S. Similar to the proof of (1), since the number of edges in the smallest maximal matching of the graph $(P_m)_u - v$ with the edge uv is equal to the number of edges in the smallest maximal matching of the path P_m, so $|S| = n|S_1|$ and it implies that $s(G \circ P_m) = |S| = n|S_1| = ns(P_m)$.

By similar method, we can have the following theorem which gives the saturation number of $G \circ C_m$.

Theorem 2.6 *For any simple graph G of order n, we have*
$$s(G \circ C_m) = \begin{cases} ns(C_m) + \alpha'(G) + l, & m \equiv 0 \pmod 3 \\ ns(C_m), & otherwise \end{cases}$$
where $\alpha'(G)$ is the size of maximum matching M of the graph G and l is the number of M-unsaturated vertices of G.

Proposition 2.7 *For any simple graph G, (i) $s(G) \leq s(K_1 \circ G) \leq 1 + s(G)$ and (ii) $s(\overline{K_m} \circ G) = m(s(K_1 \circ G))$.*

Proof. (i) Suppose that $K_1 = \{u\}$ and S_2 has the smallest cardinality over all maximal matchings of graph $G - v$ where $v \in V(G)$. So $s(G - v) = |S_2|$. To obtain the maximal matching S with the smallest cardinality for $K_1 \circ G$, put $S = \{uv\} \cup S_2$. Then we have $s(K_1 \circ G) = |S| = 1 + |S_2| = 1 + s(G - v)$. Also it can be easily verified that $s(G - v) = s(G)$ or $s(G - v) = s(G) - 1$ and it implies the result. (ii)It is clear that
$$\overline{K_m} \circ G = \underbrace{(K_1 \circ G) \ldots \cup (K_1 \circ G)}_{m-times}.$$
Now the result follows from Lemma 2.1.

We end this section with the following conjecture.

Conjecture 2.8 *Let G_1 and G_2 be two simple graphs and $|V(G_1)| = n$. Then*
$$ns(G_2) \leq s(G_1 \circ G_2) \leq ns(G_2) + \alpha'(G_1) + l$$
where $\alpha'(G)$ is the size of maximum matching M of the graph G_1 and l is the number of M-unsaturated vertices of G_1.

3.   SATURATION NUMBER OF THE LINK AND THE CHAIN OF GRAPHS

In this section, we investigate the saturation number of some graphs with specific construction that are of importance in chemistry. First we consider the link of graphs. Let $G_1, G_2, \ldots, G_k$ be a finite sequence of pairwise disjoint connected graphs and let $x_i, y_i \in V(G_i)$. The link G of the graphs $\{G_i\}_{i=1}^k$ with respect to the vertices $\{x_i, y_i\}_{i=1}^k$ is obtained by joining by an edge the vertex y_i of G_i with the vertex x_{i+1} of G_{i+1} for all $i = 1, 2, \ldots, k - 1$ (see Figure 1 for $k = 4$) ([4]).
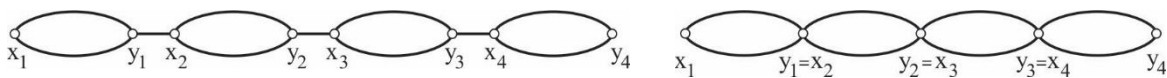
Figure 1. A link and a chain of four graphs, respectively.

In general, there is no a certain relation between $s(L(G_1, G_2, \ldots, G_k))$ and $s(G_i)$ where $1 \leq i \leq k$. For example,

$$s(L(C_4, P_5, P_4)) = s(C_4) + s(P_5) + s(P_4), s(L(C_4, P_5, C_4)) = 2s(C_4) + s(P_5) - 1,$$
$$s(L(C_4, P_5, C_4, P_5)) = 2s(C_4) + 2s(P_5) - 2.$$

But for some special graphs there are certain relations. In the following proposition, we consider the saturation number of the link of some cycles. As usual $d(x, y)$, denotes the distance between two vertices x and y.

Proposition 3.1 *Let C_m be a cycle of order m and $x_i, y_i \in V((C_m)_i)$ for every $1 \leq i \leq n$. If $d(x_i, y_i) = d_i$, then we have*

$$s\left(L\left(\underbrace{C_m, C_m, \ldots, C_m}_{n-times}\right)\right) = \begin{cases} ns(C_m), & m \equiv 0 \pmod 3, 1 \leq d_i \leq 5 \\ ns(C_m) - \left\lfloor \frac{n}{2} \right\rfloor, & m \equiv 1 \pmod 3, d_i \in \{1, 3, 4\} \\ ns(C_m) - (n-1), & m \equiv 1 \pmod 3, d_i \in \{2, 5\} \\ ns(C_m) - \left\lceil \frac{n-1}{3} \right\rceil, & m \equiv 2 \pmod 3, d_i \in \{1, 4\} \\ ns(C_m) - \left\lfloor \frac{n}{2} \right\rfloor, & m \equiv 2 \pmod 3, d_i \in \{2, 3, 5\} \end{cases}.$$
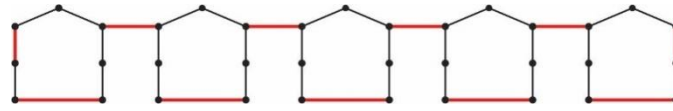
Proof. We consider all cases which has stated for the saturation number of the link of some cycles. The construction of maximal matchings with the smallest cardinality for examples of these cases, is shown in Figures 2, 3, 4, 5 and 6, respectively. So we have the result.
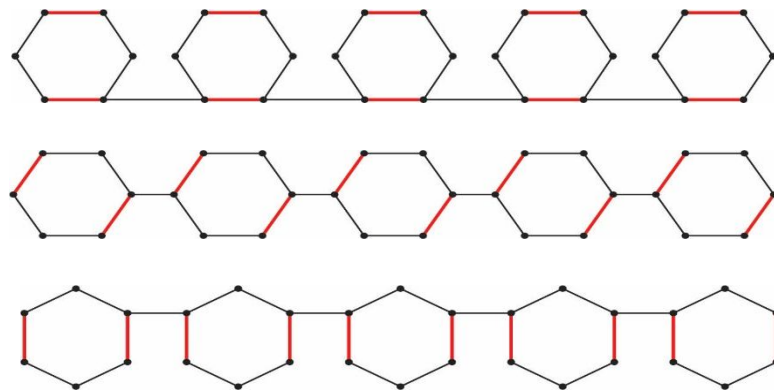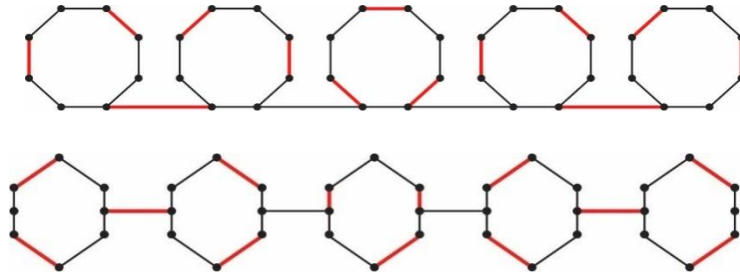

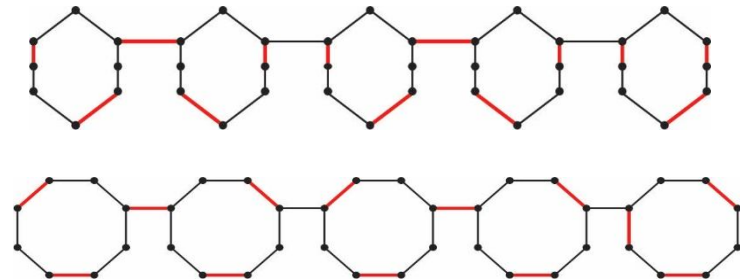**Figure 2.** $s(L(C_6, C_6, C_6, C_6, C_6)) = 10.$
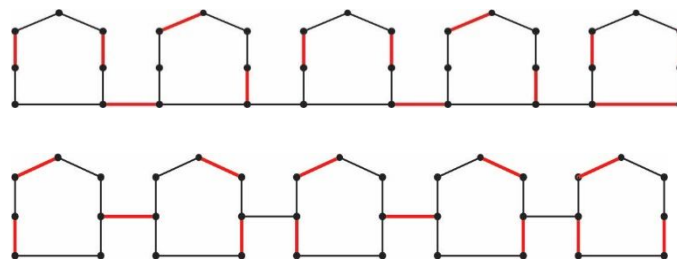
Figure 3. $s(L(C_7, C_7, C_7, C_7, C_7)) = 13.$

Figure 4. $s\big(L(C_7, C_7, C_7, C_7, C_7)\big) = 11$.

Figure 5. $s\big(L(C_8, C_8, C_8, C_8, C_8)\big) = 13$.

Figure 6. $s\big(L(C_8, C_8, C_8, C_8, C_8)\big) = 13$.

Here we study the saturation number of the chain of graphs. Let $G_1, G_2, \ldots, G_k$ be a finite sequence of pairwise disjoint connected graphs and let $x_i, y_i \in V(G_i)$. The chain G of the graphs $\{G_i\}_{i=1}^{k}$ with respect to the vertices $\{x_i, y_i\}_{i=1}^{k}$ is obtained by identifying the vertex y_i with the vertex x_{i+1} for $1 \le i \le k - 1$ (see Figure 1 for $k = 4$) ([4]). Similar to Proposition 3.1, the following observation presents the saturation number of the chain of paths and cycles, respectively.

Observation 3.2 (*i*) *Let P_m be a path of order m. Then*

$$s\big(C(\underbrace{P_m, P_m, \ldots, P_m}_{n-times})\big) = \begin{cases} ns(P_m) - \left\lfloor \dfrac{n}{3} \right\rfloor, & if\ m \equiv 0\ (mod\ 3) \\ ns(P_m), & if\ m \equiv 1\ (mod\ 3) \\ ns(P_m) + \left\lfloor \dfrac{-2(n-1)}{3} \right\rfloor, & if\ m \equiv 2\ (mod\ 3). \end{cases}$$

(*ii*) *Let C_m be a cycle of order m and $x_i, y_i \in V((C_m)_i)$ for every $1 \le i \le n$. If $d(x_i, y_i) = d_i$, then we have*

$$s(C(\underbrace{C_m, C_m, \dots, C_m}_{n-times})) = \begin{cases} ns(C_m) - \left\lfloor \frac{n-1}{2} \right\rfloor, & if\ m \equiv 0\ (mod\ 3),\ d_i \in \{1, 2, 4, 5\} \\ ns(C_m), & if\ m \equiv 0\ (mod\ 3),\ d_i = 3 \\ ns(C_m) - (n-1), & if\ m \equiv 1\ (mod\ 3), 1 \le d_i \le 5 \\ ns(C_m) - \left\lceil \frac{n}{2} \right\rceil, & if\ m \equiv 2\ (mod\ 3),\ d_i \in \{1, 4\} \\ ns(C_m) - (n-1), & if\ m \equiv 2\ (mod\ 3),\ d_i \in \{2, 3, 5\}. \end{cases}$$

Now we shall study the saturation number of cactus graphs that are of importance in chemistry. A cactus graph is a connected graph in which no edge lies in more than one cycle. Consequently, each block of a cactus graph is either an edge or a cycle. If all blocks of a cactus G are cycles of the same size k, the cactus is k-uniform. A triangular cactus is a graph whose blocks are triangles, i.e., a 3-uniform cactus. A vertex shared by two or more triangles is called a cut-vertex. If each triangle of a triangular cactus G has at most two cut-vertices and each cut-vertex is shared by exactly two triangles, we say that G is a chain triangular cactus. The number of triangles in G is called the length of the chain. Obviously, all chain triangular cactus of the same length are isomorphic. Hence we denote the chain triangular cactus of length n by T_n. An example of a chain triangular cactus is shown in Figure 7. Clearly, a chain triangular cactus of length n has $2n + 1$ vertices and $3n$ edges ([1]).

By replacing triangles in the definition of triangular cactus by cycles of length 4, we obtain a cactus whose every block is C_4. We call such a cactus, square cactus. Note that the internal squares may differ in the way they connect to their neighbors. If their cut-vertices are adjacent, we say that such a square is an ortho-square and if the cut-vertices are not adjacent, we call the square a para-square ([1]). An ortho-chain square cactus of length n, O_n, is shown in Figure 7. The following corollary gives the saturation number of the chain triangular cactus and the ortho-chain square cactus of length n.
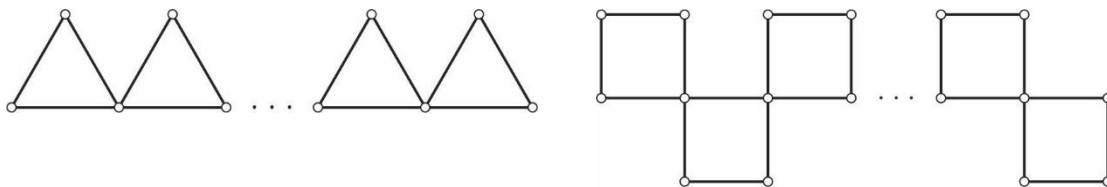
Figure 7. A chain triangular cactus T_n and an orth-chain square cactus O_n, respectively.

Corollary 3.3 Let T_n and O_n be a chain triangular cactus and an ortho-chain square cactus of length n, respectively. Then we have (i) for every $n \ge 1$, $s(T_n) = \left\lfloor \frac{n-2}{2} \right\rfloor + 2$ and (ii) for every $n \ge 1$, $s(O_n) = n + 1$.

Proof. (i) By the Observation 3.2 (ii),

$$s(T_n) = s(C(\underbrace{C_3, C_3, \ldots, C_3}_{n-times})) = n - \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Now the induction on n implies that $n - \left\lfloor \frac{n-1}{2} \right\rfloor = \left\lfloor \frac{n-2}{2} \right\rfloor + 2$ and so we have the result.

(ii) It follows from Observation 3.2 (ii).

REFERENCES

1. S. Alikhani, S. Jahari, M. Mehryar and R. Hasni, Counting the number of dominating sets of cactus chains, *Optoelectron. Adv. Mater. – Rapid Comm.* **8** (9–10) (2014) 955–960.

2. V. Andova, F. Kardoš and R. Škrekovski, Sandwiching saturation number of fullerene graphs, *arXiv:1405.2197* (2014).

3. S. J. Cyvin and I. Gutman, *Kekulé Structures in Benzenoid Hydrocarbons*, Vol. **46** Lecture Notes in Chemistry, Springer Science, Heidelberg, 1988.

4. E. Deutsch and S. Klavžar, Computing the Hosoya polynomial of graphs from primary subgraphs, *MATCH Commun. Math. Comput. Chem.* **70** (2013) 627–644.

5. T. Došlić, On lower bounds of number of perfect matchings in fullerene graphs, *J. Math. Chem.* **24** (1998) 359–364.

6. T. Došlić, Fullerene graphs with exponentially many perfect matchings, *J. Math. Chem.* **41** (2007) 183–192.

7. T. Došlić, Saturation number of fullerene graphs, *J. Math. Chem.* **43** (2008) 647–657.

8. T. Došlić and I. Zubac, Saturation number of benzenoid graphs, *MATCH Commun. Math. Comput. Chem.* **73** (2015) 491–500.

9. J. Edmonds, Paths, trees, and flowers, *Canad. J. Math.* **17** (1965) 449–467.

10. J. Faudree, R. J. Faudree, R. J. Gould and M. S. Jacobson, Saturation numbers for trees, *Electron. J. Combin.* **16** (2009).

11. A. Frendrup, B. Hartnell and P. D. Vestergaard, A note on equimatchable graphs, *Australas. J. Combin.* **46** (2010) 185–190.

12. F. Harary, *Graph Theory,* Addison-Wesley, Reading, MA, 1994.

13. F. Kardoš, D. Král, J. Miškuf and J. S. Sereni, Fullerene graphs have exponentially many perfect matchings, *J. Math. Chem.* **46** (2009) 443–447.

14. M. Klažar, Twelve countings with rooted plane trees, *European J. Combin.* **18** (1997) 195–210.

15. L. Lovász and M. D. Plummer, *Matching Theory*, Annals of Discrete Math. Vol. **29**, North-Holland, Amsterdam, 1986.

16. S. G. Wagner, On the number of matchings of a tree, *European J. Combin.* **28** (2007) 1322–1330.

17. D. B. West, *Introduction to Graph Theory*, Prentice Hall, Inc., Upper Saddle River, NJ, 1996.

18. H. Zhang and F. Zhang, New lower bound on the number of perfect matchings in fullerene graphs, *J. Math. Chem.* **30** (2001) 343–347.