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The Second Geometric–Arithmetic Index for Trees and Unicyclic Graphs

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ABSTRACT

Let G be a finite and simple graph with edge set $E(G)$. The second geometric-arithmetic Index is defined as

$$GA_2(G) = \sum_{uv\in E(G)} \frac{2\sqrt{n_u n_v}}{n_u+n_v},$$

where n_u denotes the number of vertices in G lying closer to u than to v. In this paper we find a sharp upper bound for $GA_2(T)$, where T is tree, in terms of the order and maximum degree of the tree. We also find a sharp upper bound for $GA_2(G)$, where G is a unicyclic graph, in terms of the order, maximum degree and girth of G. In addition, we characterize the trees and unicyclic graphs which achieve the upper bounds.

1 Introduction

Let G be a simple connected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The integers $n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$ are the order and the size of the graph G, respectively. We write $deg_G(v) = deg(v)$ for the degree of a vertex v and $\Delta = \Delta(G)$ for the maximum degree of G. Let $u, v \in V(G)$, then the distance $d_G(u,v)$ between u and v is defined as the length of a shortest path in G connecting u and v.

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In [5], a new class of topological descriptors, based on some properties of the vertices of a graph is presented. These descriptors are named as geometric-arithmetic indices, GA_{general}, and defined as:

$$GA_{\text{general}}(G) = \sum_{uv\in E(G)} \frac{2\sqrt{Q_u Q_v}}{Q_u+Q_v},$$

where Q_u is some quantity that in a unique manner can be associated with the vertex u of the graph G. The geometric-arithmetic index GA is defined in [6] as:

$$GA(G) = \sum_{uv\in E(G)} \frac{2\sqrt{deg(u)deg(v)}}{deg(u)+deg(v)}.$$

The geometric-arithmetic index is well studied in the literature, see for example [2, 4, 7]. Let uv be an edge of G. Define $N(u,G) = \{x \in V(G) \mid d_G(u,x) < d_G(u,x)\}$. In other words, $N(u,G)$ consists of vertices of G which are closer to u than to v. Note that the vertices equidistant to u and v are not included into either $N(u,G)$ or $N(v,G)$. Such vertices exist only if the edge uv belongs to an odd cycle. Hence, in trees, $n_u+n_v = n$ for all edges of the tree. It is also worth noting that $u \in N(u,G)$ and $v \in N(v,G)$, which implies that $n_u \geq 1$ and $n_v \geq 1$. The second geometric-arithmetic index GA_2 is defined in [5] as:

$$GA_2(G) = \sum_{uv\in E(G)} \frac{2\sqrt{n_u n_v}}{n_u+n_v},$$

where n_u=n_u (G) =$|N(u,G)|$. See [1, 3, 8] for more information on this index.

The following statements can be found in [5].

Theorem A. The path P_n is the n-vertex tree with maximum second geometric-arithmetic index.

Theorem B. Let S_n be a star of order n, then $GA_2(G) = \frac{2(n-1)\sqrt{n-1}}{n}$.

In this paper we first present some examples. Then we prove that for any tree T of order $n \geq 2$ with maximum degree Δ,

$$GA_2(T) \leq \frac{2}{n}\left((\Delta-1)\sqrt{n-1} + \sum_{i=1}^{n-\Delta}\sqrt{i(n-i)}\right).$$

Finally, we prove that for any unicyclic graph G of order $n \geq 3$ with maximum degree $\Delta \geq 3$ and girth k, if k is odd, then

$$\begin{aligned} GA_2(G) \leq & \frac{2}{n}\left((\Delta-2)\sqrt{n-1} + \sum_{i=1}^{n-k-\Delta+2}\sqrt{i(n-i)}\right) \\ & + \frac{2(k-1)}{n-1}\sqrt{\left(\frac{k-1}{2}+\Delta-2\right)\left(n-\frac{k-1}{2}-\Delta+1\right)} + \frac{2}{\Delta+k-3}\sqrt{\frac{k-1}{2}\left(\frac{k-1}{2}+\Delta-2\right)} \\ & + \frac{2}{n-\Delta+1}\sqrt{\frac{k-1}{2}\left(n-\frac{k-1}{2}-\Delta+1\right)}, \end{aligned}$$

and if k is even, then

$$GA_2(G) \leq \frac{2}{n}\left((\Delta-2)\sqrt{n-1} + \sum_{i=1}^{n-k-\Delta+2}\sqrt{i(n-i)} + k\sqrt{\left(\frac{k}{2}+\Delta-2\right)\left(n-\frac{k}{2}-\Delta+2\right)}\right).$$

We also characterize the trees and unicyclic graphs which achieve the upper bounds.

2 EXAMPLES

Dendrimers are nanostructures that can be precisely designed and manufactured for a wide variety of applications, such as drug delivery, gene delivery and diagnostic tests. In this section we calculate the second geometric-arithmetic index for Dendrimers of types A and B and for Tecto Dendrimers. See Figure 1.

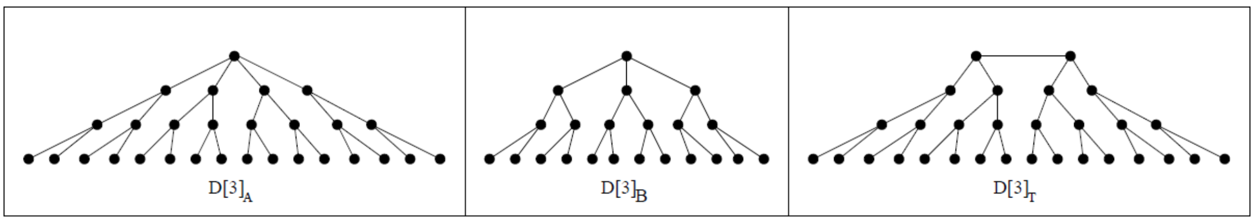

Figure 1: Dendrimers of types A and B and Tecto Dendrimers.

Example 1. In Dendrimers $D[n]$ type A, denoted $D[n]_A$, there are $4(2^n-1)+1$ vertices and $4(2^n-1)$ edges. Let e be an edge between the ith and the $(i+1)$th layers. Then

$$f_i(e) = \sqrt{(2^{n-i}-1)(2^{n+2}-2^{n-i}-2)} \text{ for } i = 1,2,\dots,n-1.$$

In addition, there are 2^{i+2} edges between the ith and the $(i+1)$th layers. Therefore, for $n \geq 2$,

$$GA_2(D[n]_A) = \frac{2}{4(2^n-1)+1}\left(4\sqrt{(2^n-1)(3(2^n-1)+1)} + \sum_{i=1}^{n-1} 2^{i+2} f_i(e)\right)$$

$$= \frac{8}{4(2^n-1)+1}\left(\sqrt{(2^n-1)(3(2^n-1)+1)} + \sum_{i=1}^{n-1} 2^{i} f_i(e)\right).$$

For examples,

$$GA_2(D[2]_A) = \frac{8}{13}\left(\sqrt{30}+2\sqrt{12}\right) = 7.63 \quad and$$

$$GA_2(D[3]_A) = \frac{8}{29}\left(\sqrt{154}+2\sqrt{78}+4\sqrt{28}\right) = 14.13.$$

Example 2. In Dendrimers $D[n]$ type B, denoted $D[n]_B$, there are $3(2^n-1)+1$ vertices and $3(2^n-1)$ edges. Let e be an edge between the ith and the $(i+1)$th layers. Then

$$f_i(e) = \sqrt{(2^{n-i}-1)(3(2^n)-2^{n-i}-1)} \text{ for } i = 1,2,\dots,n-1.$$

In addition, there are $3(2^i)$ edges between the ith and the $(i+1)$th layers. Therefore, for $n \geq 2$,

$$GA_2(D[n]_B) = \frac{8}{3(2^n-1)+1}\left(\sqrt{3(2^n-1)(2(2^n-1)+1)} + \sum_{i=1}^{n-1} 3(2^i)\, f_i(e)\right).$$

For example,

$GA_2(D[2]_B) = \frac{2}{10}\left(3\sqrt{21} + 18\right) = 6.35$ and

$GA_2(D[3]_B) = \frac{2}{22}\left(3\sqrt{105} + 6\sqrt{57} + 12\sqrt{21}\right) = 11.91.$

Example 3. In Tecto Dendrimers $D[n]_T$, there are $2^{n+2}-2$ vertices and $2^{n+2}-3$ edges. Let e be an edge between the ith and the $(i+1)$th layers. Then

$$f_i(e) = \sqrt{(2^{n-i}-1)(2^{n+2}-2^{n-i}-1)} \text{ for } i = 1,2,\dots,n-1.$$

In addition, there are 2^{i+2} edges between the ith and the $(i+1)$th layers. Therefore, for $n \geq 2$,

$$GA_2(D[n]_T) = \frac{8}{2^{n+2}-2}\left(4\sqrt{(2^n-1)(3(2^n)-1)} + \sum_{i=1}^{n-1} 2^{i+2} f_i(e) + 2^{n+1} - 1\right).$$

For example,

$GA_2(D[2]_T) = \frac{2}{14}\left(4\sqrt{33} + 8\sqrt{13} + 7\right) = 8.40$ and

$GA_2(D[3]_T) = \frac{2}{30}\left(4\sqrt{161} + 8\sqrt{81} + 16\sqrt{29} + 15\right) = 14.93.$

2 AN UPPER BOUND ON THE SECOND GEOMETRIC–ARITHMETIC OF TREES

In this section we present a sharp upper bound for the second geometric-arithmetic index of trees in terms of their order and maximum degree. We also characterize all trees whose the second geometric-arithmetic index achieves the upper bound. A *leaf* of a tree T is a vertex of degree 1, a *support vertex* is a vertex adjacent to a leaf and a *strong support vertex* is a support vertex adjacent to at least two leaves. An end-support vertex is a support vertex whose all neighbors with exception at most one are leaves. A *rooted tree* is a tree having a distinguished vertex v, called the *root*. Let $T_{n,\Delta}$ be the set of trees of order n and maximum degree Δ. Let T be a tree of order n and let $f : E(T) \to Z^+$ is a function defined by $f(xy) = \sqrt{n_x n_y}$. Hence $GA_2(T) = \frac{2}{n}\sum_{uv\in E(G)} f(uv)$. We start with an easy but useful observation.

Observation 4. Let $x \geq y \geq 1$ and $n \geq x + y + 2$ be positive integers. Then for every $1 \leq k \leq y$, $(x+k)(n-x-k) > (y-k+1)(n-y+k-1)$.

Proof. First note that $(x+k)(n-x-k)-(y-k+1)(n-y+k-1) = n(2k+x)-(x+k)^2$. Since $n \geq x+y+2$, it follows that $n(2k+x)-(x+k)^2 > 0$. So the result follows.

Lemma 5. Let T be a tree of order n with maximum degree Δ and v be a vertex of maximum degree. If T has a vertex of degree at least three different from v, then there is a tree $T' \in T_{n,\Delta}$ such that $GA_2(T) < GA_2(T')$.

Proof. Let T be the rooted tree at v. Let $u \neq v$ be a vertex of degree $deg(u) = k \geq 3$ such that $d(u,v)$ is as large as possible and let $N(u) = \{u_1, u_2, \dots, u_{k-2}, u_{k-1}, u_k\}$. Now we distinguish three cases.

Case 1. u is an end-support vertex.

We may assume that u_k is the parent of u. Let $S = \{uu_1, uu_2, \dots, uu_{k-2}, uu_{k-1}\}$ and let T' be the tree obtained by attaching the path $uu_1u_2 \dots u_{k-2}u_{k-1}$ to $T - \{u_1, u_2, \dots, u_{k-1}\}$. Suppose that $S' = \{uu_1, u_1u_2, \dots, u_{k-2}u_{k-1}\}$. Clearly, $T' \in T_{n,\Delta}$ and

$$\sum_{uv \in E(T)-S} f(uv) = \sum_{uv \in E(T')-S'} f(uv).$$

By definition

$$\frac{n}{2}GA_2(T) = \sum_{uv \notin S} f(uv) + \sum_{uv \in S} f(uv) = \sum_{uv \in E(T)-S} f(uv) + (k-1)\sqrt{n-1}, \quad (1)$$

and

$$\frac{n}{2}GA_2(T') = \sum_{uv \notin S'} f(uv) + \sum_{uv \in S'} f(uv) = \sum_{uv \in E(T')-S'} f(uv) + \sum_{i=1}^{k-1} \sqrt{i(n-i)}. \quad (2)$$

Combining (1), (2) and the fact that $k \geq 3$, we obtain $GA_2(T) < GA_2(T')$, as desired.

Case 2. u is a support vertex.

By Case 1, we may assume that u is not an end-support vertex and $deg(u_1) = 1$. Suppose $deg(u_2) = 2$ and T_{u_2} is the component of $T - uu_2$ containing u_2. Since, by the choice of vertex u, $d(u,v)$ is as large as possible, we may assume that T_{u_2} is the path $u_2x_1x_2 \dots x_t, t \geq 1$. Let T' be the tree obtained from $T - uu_1$ by adding the pendant edge x_tu_1 to this graph. Let $S = \{uu_1, uu_2, u_2x_1, x_1x_2, \dots, x_{t-1}x_t\}$ and $S' = \{uu_2, u_2x_1, x_1x_2, \dots, x_{t-1}x_t, u_1x_t\}$. Clearly, $T' \in T_{n,\Delta}$ and

$$\sum_{uv \in E(T)-S} f(uv) = \sum_{uv \in E(T')-S'} f(uv).$$

By definition

$$\frac{n}{2}GA_2(T) = \sum_{uv \in E(T)-S} f(uv) + \sum_{i=1}^{t+1} \sqrt{i(n-i)} + \sqrt{n-1}, \quad (3)$$

and

$$\frac{n}{2}GA_2(T') = \sum_{uv \in E(T')-S'} f(uv) + \sum_{i=1}^{t+2} \sqrt{i(n-i)}. \quad (4)$$

By (3), (4) and the fact that $n \geq t+4$, we obtain $GA_2(T) < GA_2(T')$.

Case 3. u is not a support vertex.

Suppose T_{u_1} and T_{u_2} are the components of $T-\{uu_1, uu_2\}$ containing u_1 and u_2, respectively. By the choice of vertex u, we may assume that $T_{u_1} = u_1x_1x_2 \dots x_s$, $s \geq 1$ and $T_{u_2} = u_2y_1y_2 \dots y_t$, $t \geq 1$. Then $deg(x_i) = deg(y_j) = 2$, $1 \leq i \leq s-1$, $1 \leq j \leq t-1$, and$deg(x_s) = deg(y_t) = 1$. Let T' be the tree obtained from $T - T_{u_2}$ by adding the path $x_sy_ty_{t-1} \dots y_1u_2$ to this graph. Let

$$S = \{uu_1, u_1x_1, x_1x_2, \dots, x_{s-1}x_s\} \cup \{uu_2, u_2y_1, y_1y_2, \dots, y_{t-1}y_t\},$$

and

$$S' = \{uu_1, u_1x_1, x_1x_2, \dots, x_{s-1}x_s\} \cup \{x_sy_t, u_2y_1, y_1y_2, \dots, y_{t-1}y_t\}.$$

Clearly, $T' \in T_{n,\Delta}$ and

$$\sum_{uv\in E(T)-S} f(uv) = \sum_{uv\in E(T')-S'} f(uv).$$

By definition we have

$$\frac{n}{2}GA_2(T) = \sum_{uv\in E(T)-S} f(uv) + \sum_{i=1}^{s+1}\sqrt{i(n-i)} + \sum_{i=1}^{t+1}\sqrt{i(n-i)}, \tag{5}$$

and

$$\frac{n}{2}GA_2(T') = \sum_{uv\in E(T')-S'} f(uv) + \sum_{i=1}^{s+t+2}\sqrt{i(n-i)}. \tag{6}$$

Applying Observation 4 and inequalities (5) and (6), we conclude that $GA_2(T) < GA_2(T')$. This complete the proof.

A spider is a tree with at most one vertex of degree more than 2, called the center of the spider (if no vertex is of degree more than two, then any vertex can be the center). A leg of a spider is a path from the center to a vertex of degree 1. Thus, a star with *k* edges is a spider of *k* legs, each of length 1, and a path is a spider of 1 or 2 legs.

Lemma 6. Let *T* be a spider of order *n* with $k \geq 3$ legs. If *T* has two legs of length at least 2, then there is a spider T' of order *n* with *k* legs such that $GA_2(T) < GA_2(T')$.

Proof. Let *v* be the center of *T* and $N(v) = \{v_1, v_2, \dots, v_k\}$. Root *T* at *v*. Assume, without loss of generality, that $deg(v_1)= deg(v_2) = 2$ and let $v_1x_1x_2 \dots x_s$ and $v_2y_1y_2 \dots y_t$ be two legs of *T*. Let T' be the tree obtained from *T* be deleting the edges $x_1x_2, \dots, x_{s-1}x_s$ and adding the edges $x_1y_t, x_1x_2, \dots, x_{s-1}x_s$. Suppose

$$S = \{vv_1, v_1x_1, x_1x_2, \dots, x_{s-1}x_s\} \cup \{vv_2, v_2y_1, y_1y_2, \dots, y_{t-1}y_t\},$$

and

$$S' = \{vv_1, y_tx_1, x_1x_2, \dots, x_{s-1}x_s\} \cup \{vv_2, v_2y_1, y_1y_2, \dots, y_{t-1}y_t\}.$$

Clearly

$$\sum_{uv\in E(T)-S} f(uv) = \sum_{uv\in E(T')-S'} f(uv).$$

By definition we have

$$\frac{n}{2}GA_2(T) = \sum_{uv\in E(T)-S} f(uv) + \sum_{i=1}^{s+1}\sqrt{i(n-i)} + \sum_{i=1}^{t+1}\sqrt{i(n-i)}, \tag{7}$$

and

$$\frac{n}{2}GA_2(T') = \sum_{uv\in E(T')-S'} f(uv) + \sum_{i=1}^{s+t+1}\sqrt{i(n-i)} + \sqrt{n-1}. \tag{8}$$

By Observation 4, equalities (7) and (8) and the fact that $n \geq s + t + 4$ we obtain $GA_2(T) < GA_2(T')$.

We are now ready to prove the main theorem of this section.

Theorem 7. For any tree $T \in T_{n,\Delta}$ of order $n \geq 2$,

$$GA_2(T) \leq \frac{2}{n}\left((\Delta - 1)\sqrt{n-1} + \sum_{i=1}^{n-\Delta}\sqrt{i(n-i)}\right).$$

The equality holds if and only if T is a spider with at most one leg of length at least two.

Proof. Let T_1 be a tree of order $n \geq 2$ with maximum degree Δ such that

$$GA_2(T_1) = \max\{GA_2(T) \mid T \text{ is a tree of order } n \text{ with maximum degree } \Delta\}.$$

Let v be a vertex with maximum degree Δ. Root T_1 at v. If $\Delta = 2$, then T_1 is a path of order n and the result follows by Theorem A. Let $\Delta \geq 3$. By the choice of T_1, we deduce from Lemma 5 that T_1 is a spider with center v. It follows from Lemma 6 and the choice of T_1 that T_1 has at most one leg of length at least two. First let all legs of T_1 have length one. Then T_1 is a star of order n and the result follows by Theorem B. Now let T_1 have only one leg of length at least two. Then

$$GA_2(T) = \frac{2}{n}\left((\Delta - 1)\sqrt{n-1} + \sum_{i=1}^{n-\Delta}\sqrt{i(n-i)}\right).$$

This completes the proof.

3 UNICYCLIC GRAPHS

A connected graph with precisely one cycle is called a unicyclic graph. Let the set $\varphi_{n,\Delta,k}$consist of all unicycle graphs of order n, maximum degree $\Delta \geq 3$ and grith k, where $3 \leq k \leq n$. Note that if G is a cycle of order n, then $GA_2(G) = n$. Let $G \in \varphi_{n,\Delta,k}$. In this section we assume that the k-cycle of G is $C_k = (w_1, w_2, \dots, w_k)$. In addition for a vertex $u \in V(C_k)$ we let T_u be the connected component of $G \setminus E(C_k)$ containing u. Note that T_u is a tree and we assume u is the root of this tree. Without loss of generality, we also assume one of the vertices of T_{w_1}, say v, is of degree Δ.

Lemma 8. *Let $G \in \varphi_{n,\Delta,k}$ and v be a vertex of maximum degree Δ. Let C be the only cycle of G, $u \in V(C)$ and $u \neq v$. If T_u is a spider with at least two legs, then there is a graph $G' \in \varphi_{n,\Delta,k}$such that $GA_2(G) < GA_2(G')$.*

Proof. Assume T_u has ℓ legs with lengths $t_1, t_2, \dots, t_\ell$ and $\sum_{i=1}^{\ell} t_i = s$. Let the graph G' be obtained from $G \setminus E(T_u)$ by attaching a path P_s to vertex u. Obviously, $G' \in \varphi_{n,\Delta,k}$. A simple calculation shows that

$$GA_2(G') - GA_2(G) = \frac{2}{n}\left[\sum_{i=1}^{s}\sqrt{i(n-i)} - \sum_{j=1}^{\ell}\sum_{i=1}^{t_j}\sqrt{i(n-i)}\right].$$

Apply Observation 4 to obtain $GA_2(G') - GA_2(G) > 0$.

Lemma 9. *Let* $G \in \varphi_{n,\Delta,k}$ *and* $deg(u) \geq 3$, *where* $u \in T_{w_i}, u \neq w_i$, *for some* $2 \leq i \leq k$. *Then there is a graph* $G' \in \varphi_{n,\Delta,k}$ *such that* $GA_2(G) < GA_2(G')$.

Proof. Without loss of generality, we may assume u has the largest distance from w_i among all the vertices of T_{w_i} whose degree is at least 3. This implies that T_u is a spider with at least two legs. Let G' be the graph obtained from G by replacing T_u with a path with the same order as T_u. A calculation similar to that presented in Lemma 8 shows that $GA_2(G') - GA_2(G) > 0$.

Lemma 10. *Let* $G \in \varphi_{n,\Delta,k}$ *and* T_{w_i} *and* T_{w_j} *be paths of length at least 1 for some* $2 \leq i, j \leq k, i \neq j$. *Then there is a graph* $G' \in \varphi_{n,\Delta,k}$ *such that* $GA_2(G) < GA_2(G')$.

Proof. Let ℓ_1 and ℓ_2 be the length of the paths T_{w_i} and T_{w_j}, respectively. Let G' be the graph obtained from G by removing T_{w_i} and T_{w_j} and attaching a path of length $\ell_1 + \ell_2$ to the vertex u. Then as before one can see that $GA_2(G) < GA_2(G')$.

Lemma 11. *Let* $G \in \varphi_{n,\Delta,k}$ *and assume the vertices of the cycle* C_k *are all of degree two except* w_1 *and* w_i, $i \neq 1$. *If the distance of* w_i *from* w_1 *is not* $\lceil (k-1)/2 \rceil$, *then there is a graph* $G' \in \varphi_{n,\Delta,k}$ *such that* $GA_2(G) < GA_2(G')$.

Proof. Let G' be the graph obtained from G by removing T_{w_i} and attaching it to vertex w_j, where $j = \lceil (k-1)/2 \rceil$. Then one can see that $GA_2(G) < GA_2(G')$.

Now we consider the graph $G \in \varphi_{n,\Delta,k}$ with $deg(w_i) = 2$ for all $2 \leq i \leq k, i \neq \lceil (k-1)/2 \rceil$ and $deg(w_j) \geq 2$, where $j = \lceil (k-1)/2 \rceil$. By Lemma 9, in order to maximize $GA_2(G)$, T_v must be a spider and $deg_G(w_1) = 3$ if $w_1 \neq v$.

Lemma 12. Let $G \in \varphi_{n,\Delta,k}$ and $w_1 \neq v$. Then there is a graph $G' \in \varphi_{n,\Delta,k}$ *such that* $GA_2(G) < GA_2(G')$.

Proof. Let G' be the graph obtained from $G \backslash T_{w_1}$ by attaching a path of order $|V(T_{w_1})| - \Delta + 2$ to the end vertex of the path T_{w_j} which is different from w_j, $j = \lceil (k-1)/2 \rceil$ and adding $\Delta - 2$ pendant edges at vertex w_1. Obviously, $G' \in \varphi_{n,\Delta,k}$ and it is straightforward to verify that $GA_2(G) < GA_2(G')$.

By Lammas 8–12 we obtain the following result.

Corollary 13. *Let $H \in \varphi_{n,\Delta,k}$ be the graph which consists of a cycle $C_k = (w_1, w_2, \dots, w_k)$ with $\Delta - 2$ pendant edges at vertex w_1 and a path of order $n - k - \Delta + 2$ at vertex w_j, where $j = \lceil (k-1)/2 \rceil$. Then for every $G \in \varphi_{n,\Delta,k}$, $GA_2(G) \le GA_2(H)$.*

We are now ready to state the main theorem of this section.

Theorem 14. For any unicycle graph G of order *n*, girth *k* and maximum degree $\Delta \ge 3$, if *k* is odd, then

$$GA_2(G) \le \frac{2}{n}\left((\Delta-2)\sqrt{n-1} + \sum_{i=1}^{n-k-\Delta+2}\sqrt{i(n-i)}\right) + \frac{2(k-1)}{n-1}\sqrt{\left(\frac{k-1}{2}+\Delta-2\right)\left(n-\frac{k-1}{2}-\Delta+1\right)} + \frac{2}{\Delta+k-3}\sqrt{\frac{k-1}{2}\left(\frac{k-1}{2}+\Delta-2\right)} + \frac{2}{n-\Delta+1}\sqrt{\frac{k-1}{2}\left(n-\frac{k-1}{2}-\Delta+1\right)},$$

and if *k* is even, then

$$GA_2(G) \le \frac{2}{n}\left((\Delta-2)\sqrt{n-1} + \sum_{i=1}^{n-k-\Delta+2}\sqrt{i(n-i)} + k\sqrt{\left(\frac{k}{2}+\Delta-2\right)\left(n-\frac{k}{2}-\Delta+2\right)}\right).$$

The equality holds if and only if G is the graph H given in Corollary 13.

REFERENCES

1 K. Ch. Das, I. Gutman and B. Furtula, *On second geometric-arithmetic index of graphs*, Iranian J. Math. Chem. **1** (2010), 17–28.

2 K. Ch. Das, I. Gutman and B. Furtula, *On the first geometric-arithmetic index of graphs*, Discrete Appl. Math. **159** (2011), 2030–2037.

3 H. Hua, *Trees with given diameter and minimum second geometric-arithmetic index*, J. Math. Chem. **64** (2010), 631–638.

4 A. Madanshekaf and M. Moradi, *The first geometric-arithmetic index of some nanostar dendrimers*, Iranian J. Math. Chem. **5** (2014), 1–6.

5 G. H. Fath-Tabar, B. Furtula and I. Gutman, *A new geometric-arithmetic index*, J. Math. Chem. **47** (2010), 477–486.

6 D. Vukicevic and B. Furtula, *Topological index based on the ratios of geometrical and arithmetical means of end-vertex degrees of edges*, J. Math. Chem. **4** (2009), 1369–1376.

7 Y. Yuan, B. Zhou and N. Trinajstic, *On geometric-arithmetic index*, J. Math. Chem. **47** (2010), 833–841.