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The Center and Periphery of Composite Graphs

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ABSTRACT. The center (periphery) of a graph is the set of vertices with minimum (maximum) eccentricity. In this paper, the structure of centers and peripheries of some classes of composite graphs are determined. The relations between eccentricity, radius and diameter of such composite graphs are also investigated. As an application we determine the center and periphery of some chemical graphs such as nanotorus and nanotubes covered by C_4.

Keywords: Eccentricity, radius, diameter, center, periphery.

1. INTRODUCTION

Graph operations play an important role in the study of graph decompositions into isomorphic subgraphs. For more details about graph operations see [4] and [1]. Facility location problems deal with the task of choosing a site subject to some criterion. Centrality questions are now examined using graphs and distance concepts. Two concepts, center and periphery is related to eccentricity in a graph. The center of some classes of graphs were determined in [8], [5] and [11]. We are interested in finding the center and periphery of some classes of composite graphs. Meanwhile the eccentricity of a vertex of composite graphs which are interested for us, was computed in [3], for obtaining the eccentric

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connectivity index of these composite graphs. The paper is organized as follows. In the next section the eccentricity, radius, diameter, center and periphery of a connected graph is introduced and we recall some preliminaries about six classes of composite graphs. Section 3 contains the main results about the center and periphery of composite graphs in terms of the radius and diameter of their components, respectively. As an application the center of some chemical graphs are obtained. All graphs are considered in this paper will be finite, simple and connected. The notation we use is mostly standard and taken from standard graph theory textbooks, such as [9].

2. DEFINITIONS AND PRELIMINARIES

Let G be a graph with the vertex and edge sets $V(G)$ and $E(G)$ respectively. The distance between a and b of $V(G)$ is denoted by $d_G(a, b)$ and it is defined as the number of edges in a shortest path connecting the vertices a and b. Let G be a connected graph and a be a vertex of G. The eccentricity $\varepsilon_G(a)$ is the distance to a vertex farthest from a. Thus

$$\varepsilon_G(a) = \max\{d_G(a,b) \mid b \in V(G)\}.$$

The radius $r(G)$ is the minimum eccentricity of vertices, whereas the diameter $d(G)$ is the maximum eccentricity. Now a is a central vertex if $\varepsilon_G(a) = r(G)$. The center of G, $C(G)$ is defined as

$$C(G) = \{a \in V(G) \mid \varepsilon_G(a) = r(G)\}.$$

Thus, the center consists of all vertices having minimum eccentricity. The vertex a is a peripheral vertex if $\varepsilon_G(a) = d(G)$, and periphery is the set of all such vertices. The periphery of G is denoted by P(G). It is defined as:

$$P(G) = \{a \in V(G) \mid \varepsilon_G(a) = d(G)\}.$$

Now we introduce the graph operations that we consider in this paper.
The Cartesian product $G\times H$ of graphs G and H has the vertex set $V(G\times H) = V(G)\times V(H)$ and (a, b)(c, d) is an edge of $G\times H$ if either ($a = c$ and $bd \in E(H)$), or ($ac \in E(G)$ and $b = d$). If $G_1, G_2, \ldots, G_s$ are graphs then we denote $G_1\times G_2\times\ldots\times G_s$ by $\prod_{i=1}^{s} G_i$. If $G_1 = G_2 = \ldots = G_s = G$, we have the s-th Cartesian power of G and denote it by G^s. The symmetric difference $G \oplus H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, and (a, b)(c, d) is an edge of $G \oplus H$ whenever $ac \in E(G)$ or $bd \in E(H)$, but not both.

The disjunction $G \vee H$ of two graphs G and H is the graph with vertex set $V(G)\times V(H)$, and (a, b)(c, d) is an edge of $G \vee H$ whenever $ac \in E(G)$ or $bd \in E(H)$.

The join G + H of graphs G and H with disjoint vertex sets V (G) and V (H) and edge sets E(G) and E(H) is the graph union G $\cup$ H together with all the edges joining vertices of V (G) and V (H).

Let G and H be two graphs. Their corona product GoH is defined as the graph obtained by taking one copy of G and joining the i-th vertex of G to every vertex in i-th copy of H. The vertex set of a corona product of two graphs is not the Cartesian product of their vertex sets. However, each vertex b of a copy of H attached to a vertex a from G can be uniquely described by the ordered pair (a, b). Hence the vertices in all copies of H can be described as the elements of the Cartesian product V (G)×V (H). This description can be extended to all vertices of GoH by introducing a special symbol ϕ so that V (GoH) = V (G)×{V (H)× ϕ}, where ordered pairs (a, ϕ) denote the vertices of G.

The composition G[H] of two graphs G and H is the graph with vertex set V (G)×V (H), and (a, b)(c, d) is an edge of G[H] whenever ac $\in$ E(G) or a = c, bd $\in$ E(H).

The following lemma is crucial in our study about radius, diameter, center and periphery of graph operations.

Lemma 2.1. Let G and H be graphs. Then we have:

1. $\varepsilon_{G\times H}(a,b) = \varepsilon_G(a) + \varepsilon_H(b)$,
2. $\varepsilon_{G\oplus H}(a,b) = 2$,
3. $\varepsilon_{G\vee H}(a,b) = \begin{cases} 1 & \varepsilon_G(a) = 1 \ \ and \ \ \varepsilon_H(b) = 1 \\ 2 & \varepsilon_G(a) \geq 2 \ \ or \ \ \varepsilon_H(b) \geq 2 \end{cases}$,
4. $\varepsilon_{G+H}(a) = \begin{cases} 1 & \varepsilon_G(a) = 1 \ \ or \ \ \varepsilon_H(a) = 1 \\ 2 & \varepsilon_G(a) \geq 2 \ \ or \ \ \varepsilon_H(a) \geq 2 \end{cases}$,
5. $\varepsilon_{GoH}(a,b) = \begin{cases} \varepsilon_G(a) + 1 & b = \phi \\ \varepsilon_G(a) + 2 & b \in V(H) \end{cases}$,
6. $\varepsilon_{G[H]}(a,b) = \begin{cases} 1 & \varepsilon_G(a) = 1 \ \ and \ \ \varepsilon_H(b) = 1 \\ 2 & \varepsilon_G(a) = 1 \ \ and \ \ \varepsilon_H(b) \geq 1 \\ \varepsilon_G(a) & \varepsilon_G(a) \geq 2 \end{cases}$.

Proof. For proofs we refer the reader to [3]. □

3. MAIN RESULTS

In this section, at first obtain the radius six composite graphs and then by using obtained results, we can introduce the center of these composite graphs.

Lemma 3.1. Let G and H be two graphs. Then

1. $r(G\times H)=r(G)+r(H)$,
2. $r(G\oplus H)=2$,
3. $r(G\vee H)=\begin{cases}1 & \exists(a,b)\in V(G\vee H);\ \varepsilon_G(a)=\varepsilon_H(b)=1\\ 2 & o.w.\end{cases}$,
4. $r(G+H)=\begin{cases}1 & r(G)=1\ \ or\ \ r(H)=1\\ 2 & r(G)\ge 2\ \ and\ r(H)\ge 2\end{cases}$,
5. $r(GoH)=r(G)+1$,
6. $r(G[H])=\begin{cases}1 & r(G)=1\ \ and\ r(H)=1\\ 2 & r(G)=1\ \ and\ r(H)\ge 2\\ r(G) & r(G)\ge 2\end{cases}$.

Proof. 1. By definition of radius of graph and Lemma 2.1 (1), we have

$$\begin{aligned} r(G\times H) &= \min\{\varepsilon_{G\times H}(a,b)\,|\,(a,b)\in V(G\times H)\}\\ &= \min\{\varepsilon_G(a)+\varepsilon_H(b)\,|\,(a,b)\in V(G\times H)\}\\ &= \min\{\varepsilon_G(a)\,|\,a\in V(G)\}+\min\{\varepsilon_H(b)\,|\,b\in V(H)\}\\ &= r(G)+r(H). \end{aligned}$$

2. By definition and Lemma 2.1 (2), the proof is straightforward.

3. By using Lemma 2.1 (3), if there exists a vertex (a, b) $\in$V (G$\vee$H) such that $\varepsilon_G(a)=\varepsilon_H(b)=1$, then $1\in E=\{\varepsilon_{G\vee H}(a,b)|(a,b)\in V(G\vee H)\}\subseteq\{1,2\}$. Thus, the set E minimizes in 1. Thus in this case r(G$\vee$H) = 1. Otherwise $E=\{\varepsilon_{G\vee H}(a,b)|(a,b)\in V(G\vee H)\}=\{2\}$, and it is clear that the set E minimizes in 2, then r(G$\vee$H) = 2.

4. By definition of radius of graph, $r(G+H)=\min\{\varepsilon_{G+H}(a)|a\in V(G+H)\}$. Suppose there exists a$\in$V(G+H) such that $\varepsilon_G(a)=1$ or $\varepsilon_H(a)=1$, then r(G) = 1 or r(H) = 1 and then $1\in E=\{\varepsilon_{G+H}(a)|a\in V(G+H)\}\subseteq\{1,2\}$. Thus the set E minimizes in 1. It means that r(G + H) = 1. Otherwise r(G + H) = 2.

5. By definition of corona product of graph and Lemma 2.1 (5), we have:

$$
\begin{aligned}
r(GoH) &= \min\{\varepsilon_{GoH}(a,b) \mid (a,b) \in V(GoH)\} \\
&= \min\{\varepsilon_G(a) + 1 \mid a \in V(G)\} \\
&= r(G) + 1.
\end{aligned}
$$

6. By definition and Lemma 2.1 (6), The proof is obvious. □

Now by using Lemma 3.1 we can determine the center of the Cartesian, symmetric difference disjunction, join and corona product of graphs.

Theorem 3.2. Let G and H be graphs. Then

1. $C(G \times H) = C(G) \times C(H)$,
2. $C(G \oplus H) = V(G \oplus H)$,
3. $C(G \vee H) = \begin{cases} C(G) \times C(H) & r(G) = r(H) = 1 \\ V(G \vee H) & o.w. \end{cases}$,
4. $C(G + H) = \begin{cases} C(G) \cup C(H) & r(G) = 1 \text{ and } r(H) = 1 \\ C(G) & r(G) = 1 \text{ and } r(H) \geq 2 \\ C(H) & r(G) \geq 2 \text{ and } r(H) = 1 \\ V(G + H) & r(G) \geq 2 \text{ and } r(H) \geq 2 \end{cases}$,
5. $C(GoH) \cong C(G)$,
6. $C(G[H]) = \begin{cases} C(G) \times C(H) & r(G) = 1 \text{ and } r(H) = 1 \\ C(G) \times V(H) & o.w. \end{cases}$.

Proof. 1. The center contains all vertices with minimum eccentricity. Thus

$$
\begin{aligned}
C(G \times H) &= \{(a,b) \mid \varepsilon_{G \times H}(a,b) = r(G \times H)\} \\
&= \{(a,b) \mid \varepsilon_G(a) + \varepsilon_H(b) = r(G) + r(H)\} \\
&= \{(a,b) \mid \varepsilon_G(a) = r(G),\ \varepsilon_H(b) = r(H)\} \\
&= C(G) \times C(H).
\end{aligned}
$$

2. By Lemma 3.1 (2), r(G$\oplus$H)=2. Then for each (a, b)$\in$V(G$\oplus$H), $\varepsilon_{G \oplus H}(a,b) = r(G \oplus H) = 2$. Hence $C(G \oplus H) = V(G \oplus H)$ and it means that $G \oplus H$ is self centered.

3. By Lemma 3.1 (3), r(G$\vee$ H) = 1 or 2. If r(G$\vee$ H) = 1, then

$$
\begin{aligned}
C(G \vee H) &= \{(a,b) \mid \varepsilon_{G \vee H}(a,b) = r(G \vee H) = 1\} \\
&= \{(a,b) \mid \varepsilon_G(a) = r(G),\ \varepsilon_H(b) = r(H)\} \\
&= C(G) \times C(H).
\end{aligned}
$$

It remains the case, r(G $\vee$ H) = 2. In this case for each (a, b) $\in$ V (G $\vee$ H), $\varepsilon_G(a) \geq 2$ or $\varepsilon_H(b) \geq 2$. Hence C(G $\vee$ H) = V (G $\vee$ H). The same proof can be applied for determining the periphery of G $\vee$ H.

4. By Lemma 3.1 (4), if r(G+H) = 1 then r(G) = 1 or r(H) = 1. If r(G) = 1 and r(H) = 1, then for each a $\in$ C(G) $\cup$ C(H), $\varepsilon_{G+H}(a)$ = r(G + H) = 1. Obviously, for each a $\notin$ C(G) $\cup$ C(H), $\varepsilon_{G+H}(a)$ = 2. Thus C(G+H) = C(G) $\cup$ C(H). Now if r(G) = 1 and r(H) $\geq$ 2, then for each a $\in$ C(G), $\varepsilon_{G+H}(a)$ = r(G+H) = 1. Obviously, for each a $\notin$ C(G), $\varepsilon_{G+H}(a)$ = 2. Thus C(G+H) = C(G). By similar argument we can show that, if r(G) $\geq$ 2 and r(H) = 1, then C(G+H) = C(H). It remains to exclude the case when r(G) $\geq$ 2 and r(H) $\geq$ 2. In this case for each a $\in$ V(G+H), $\varepsilon_{G+H}(a)$ = r(G+H) = 2 and it concludes that C(G+H) = V(G+H).

5. By definition of corona product of graphs, one can see that

$$\begin{aligned} C(GoH) &= \{(a,b) \,|\, (a,b) \in V(GoH),\ \varepsilon_{GoH}(a,b) = r(GoH)\} \\ &= \{(a,\phi) \,|\, a \in C(G)\} \\ &\cong C(G). \end{aligned}$$

6. The proof is straightforward. □

Corollary 3.3. Let $G_1, G_2, ..., G_s$ be graphs. Then

$$C(\prod_{i=1}^{s} G_i) = \prod_{i=1}^{s} C(G_i).$$

Proof. Apply induction and Lemma 2.1(1), one can see that,

$$\varepsilon_{\prod_{i=1}^{s} G_i}(a_1, a_2, ..., a_s) = \sum_{i=1}^{s} \varepsilon_{G_i}(a_i).$$

for each $a_i \in V(G_i)$. It is show that $r(\prod_{i=1}^{s} G_i) = \sum_{i=1}^{s} r(G_i)$ and then $C(\prod_{i=1}^{s} G_i) = \prod_{i=1}^{s} C(G_i)$ as desired. □

We now can determine the center of some interesting chemical graphs.

Corollary 3.4.

$$C(P_m \times P_n) \cong \begin{cases} K_1 & m \text{ and } n \text{ are odd} \\ K_2 & \text{either } m \text{ or } n \text{ is odd}, \\ C_4 & m \text{ and } n \text{ are evan} \end{cases}$$

$$C(L_n) \cong \begin{cases} K_2 & n \text{ is odd} \\ C_4 & n \text{ is even} \end{cases},$$

$$C(P_m \times C_n) = \begin{cases} K_1 \times C_n & m \text{ is odd} \\ K_2 \times C_n & m \text{ is even} \end{cases}.$$

Proof. Since $C(P_m) = \begin{cases} K_1 & m \text{ is odd} \\ K_2 & m \text{ is even} \end{cases}$ and $C(C_n) = C_n$ and by using Theorem 3.2, the proof is straightforward. □

Corollary 3.5. Every nanotorus covered by C_4 is self-centered.

Proof. By Corollary 3.3, the Cartesian product of two self centered graphs, is self centered. Since each cycle is self centered and a nanotorus covered by C_4 is Cartesian product of two cycles, hence it's self centered. □

Let G be a Hamming graph. Then G $G \cong K_{n_1} \times K_{n_2} \times .... \times K_{n_t}$, for some positive integers n_1, ..., n_t.Since for each positive integer n, $C(K_n) = K_n$, it means that K_n is self centered and then r(G) = t and C(G) = G.

Corollary 3.6. Let G and H be graphs. Then $G \oplus H$ is self-centered.

Corollary 3.7. If G and H be two self-centered graphs, then $G \vee H$ is self-centered.

Example 3.8. Graphs, $C_n \vee C_m$, $C_n \vee K_m$ and $K_n \vee K_m$ are self-centered.

Corollary 3.9. Let G and H be two connected graphs. Then G + H is self-centered if and only if one of the statements is hold:

i. Graphs G and H are self-centered and r(G) = r(H) = 1.
ii. min{r(G), r(H)} ≥ 2.

Corollary 3.10. Let G and H be two connected graphs. The following statements are hold:

i. If G and H are self-centered, then G[H] is self-centered.
ii. If G is self-centered and max{r(G), r(H)} ≥ 2, then G[H] is self-centered.

Lemma 3.11. Let G and H be graphs. Then

1. $d(G\times H)=d(G)+d(H)$,
2. $d(G\oplus H)=2$,
3. $d(G\vee H)=\begin{cases}1 & \forall(a,b)\in V(G\vee H);\ \varepsilon_G(a)=\varepsilon_G(b)=1\\ 2 & o.w.\end{cases}$,
4. $d(G+H)=\begin{cases}1 & r(G)=1\ \ and\ \ r(H)=1\\ 2 & r(G)\geq 2\ \ or\ \ r(H)\geq 2\end{cases}$,
5. $d(GoH)=d(G)+2$,
6. $d(G[H])=\begin{cases}1 & d(G)=1\ \ and\ d(H)=1\\ 2 & d(G)=1\ \ and\ d(H)\geq 2\\ d(G) & d(G)\geq 2\end{cases}$.

Proof. The proof is similar to Lemma 3.1. □

Remark. For each connected graphs G and H, we have d(GoH) = d(G)+2 > d(G)+1 ≥ r(G) + 1 = r(GoH). Then GoH is not self–centered.

Theorem 3.12. Let G and H be graphs. Then

1. $P(G\times H)=P(G)\times P(H)$,
2. $P(G\oplus H)=V(G\oplus H)$,
3. $P(G\vee H)=\begin{cases}V(G\vee H) & d(G)=d(H)=1\\ A & o.w.\end{cases}$,
where $A=\{(x,y)\,|\,\varepsilon_G(a)\geq 2\ \ or\ \ \varepsilon_H(b)\geq 2\}$,
4. $P(G+H)=\begin{cases}V(G)\cup V(H) & d(G)=1\ and\ \ d(H)=1\\ A & o.w.\end{cases}$,
where $A=\{a\in V(G)\,|\,\varepsilon_G(a)\geq 2\}\cup\{a\in V(H)\,|\,\varepsilon_H(a)\geq 2\}$,
5. $P(GoH)=P(G)\times V(H)$,
6. $P(G[H])=\begin{cases}V(G)\times V(H) & d(G)=1\ and\ \ d(H)=1\\ V(G)\times A & d(G)=1\ and\ \ d(H)\geq 1\\ P(G)\times V(H) & d(G)\geq 2\end{cases}$,
where $A=\{b\in V(H)\,|\,\varepsilon_H(b)\geq 2\}$.

Proof. The proof is similar to Lemma 3.2. □

In this paper the center and periphery of six graph operations are computed. In graph theory literature, there are many other graph operations. One of the most important such graph operations is tensor product. The tensor product $G\otimes H$ of G and H is a graph such that the vertex set of $G\otimes H$ is V(G)×V(H) and any two vertices (a, b) and (c, d) are adjacent if and only if a is adjacent with c and b is adjacent with d. In [4], the authors proved that if one of G and H are not bipartite then the tensor product $G\otimes H$ is connected. In [7], the author obtained eccentricity of a vertex in the tensor product $G\otimes H$, see [6] for more details. The splice and link are two important graph operations such that they have some application in chemistry. Let G and H be two simple and connected graphs with disjoint vertex sets. For given vertices $a\in V(G)$ and $b\in V(H)$, a splice of G and H is defined as the graph (G.H)(a, b) obtained by identifying the vertices a and b. Similarly, a link of G and H is defined as the graph (G~H)(a, b) obtained by joining a and b by an edge, see [2]. Moreover, in [10], the authors study on the eccentric connectivity index of graphs with subdivided edges. We would like to present open questions related to the results of this paper.

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