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The Irregularity and Total Irregularity of Eulerian Graphs

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ABSTRACT

For an arbitrary graph G, the irregularity and total irregularity of G are defined as $irr(G) = \sum_{uv\in E(G)} |d_G(u) - d_G(v)|$ and $irr_t(G) = 1/2 \sum_{u,v\in V(G)} |d_G(u) - d_G(v)|$, respectively, where $d_G(u)$ is the degree of vertex u. In this paper, we characterize all connected Eulerian graphs with the second minimum irregularity, the second and third minimum total irregularity, respectively.

1. INTRODUCTION

Throughout this paper, G is a simple and connected graph with the vertex and edge sets $V(G)$ and $E(G)$, respectively. For a graph G, there is a novel notion named ***third Zagreb polynomial***, defined as $M_3(G,x) = \sum_{uv\in E(G)} x^{|d_G(u)-d_G(v)|}$. Astaneh-Asl et al. [7] studied $M_3(G,x)$ of Cartesian product of two graphs and a type of dendrimers. In special case, the value of derivative of this polynomial at point $x = 1$ is well known as the ***irregularity*** of G and denoted by $irr(G)$, which was already proposed by Albertson [6]. In the other words

$$irr(G) = \sum_{uv\in E(G)} |d_G(u) - d_G(v)| \qquad (1)$$

In [6], Albertson gave some upper bounds on irregularity for trees, bipartite, and triangle-free graphs. Recall that the ***first Zagreb index*** M_1 and the ***second Zagreb index*** M_2 of G are defined as $M_1(G) = \sum_{u\in V(G)} d_G^2(u)$ and $M_2(G) = \sum_{uv\in E(G)} d_G(u).d_G(v)$, respectively. These indices were introduced in [16] and reflect the extent of branching of the molecular carbon-atom skeleton and can be viewed as molecular structure-descriptors [8,25]. Moreover, the values of these indices are computed for a class of nanostar dendrimers in [26]. Fath-Tabar

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[14] named the sum in (1) the ***third Zagreb index,*** and established new bounds on the first and second Zagrab indices that depend on $\mathrm{irr}(G)$. Zhou and Luo obtained the relationship between irregularity and first Zagreb index of graphs, and also they determined the graphs with maximum irregularity among trees and unicyclic graphs with given matching number and number of pendent vertices [19, 29]. Hansen and Melot determined the maximum irregularity of graphs with n vertices and m edges [17]. Moreover, Abdo and Dimitrov considered the irregularity of graphs under several graph operations [5]. Previously, we characterized all graphs with the second minimum of the irregularity in [20]. Also, we studied in [15, 21], trees and unicyclic graphs whose irregularity is extremal. More works about this graph invariant have been reported in [2, 9, 18, 22–24].

Recently, Abdo et al. [1] introduced a new measure of irregularity of a graph, so-called the ***total irregularity***, as $\mathrm{irr}_t(G) = 1/2\sum_{u,v\in V(G)}|d_G(u) - d_G(v)|$. For a connected graph G, the irregularity indices irr and irr_t were compared in [12], where it was shown that $\mathrm{irr}_t(G) \leq n^2/4\mathrm{irr}(G)$. Furthermore, they proved that if G is a tree, then $\mathrm{irr}_t(G) \leq (n-2) \times \mathrm{irr}(G)$. Abdo and Dimitrov [4] gave the upper bounds on irr_t of graphs under several graph operations including lexicographic, Cartesian, strong, direct, and corona products, also join, disjunction and symmetric difference. In [1], graphs with maximal total irregularity were characterized and the upper bound on the total irregularity of graphs was obtained. In special classes of graphs, such as trees, unicyclic and bicyclic graphs, this invariant has been studied in [13, 27, 28].

An ***Eulerian circuit*** is a closed walk in a graph that visits every edge of the graph once and only once. A graph containing an Eulerian circuit is called an ***Eulerian graph***. The study of these graphs was initiated in 1736. Their study is a very fertile field of research for graph theorists. Although, in the graph theory, the term Eulerian graph has two common meanings, i.e. a graph with an Eulerian circuit, or a graph with every vertex of even degree. Note that in the case of connected graph, these definitions are equivalent [10].

The aim of this paper is to study the irregularity and total irregularity of connected Eulerian graphs. In Section 2, we show that the irregularity of an Eulerian graph is a multiple of 4, and by using it, we characterize all Eulerian graphs with the second minimum irregularity value. Finally in Section 3, we determine graphs with the second and third minimum of total irregularity value over the class of all connected Eulerian graphs.

2. THE SECOND MINIMUM IRREGULARITY OF EULERIAN GRAPHS

In this section,we first restate a proven result in [10], which is needful for proving that the irregularity of Eulerian graphs is divisible by 4. Afterwards, we would able to determine the Eulerian graphs with the second minimum irregularity value.

Lemma 2.1. [10] *A connected graph is Eulerian if and only if each of its vertices has even degree.*

Theorem 2.2. *Let G be an Eulerian graph with n vertices, then $irr(G) = 4k$, for some non-negative integer k.*

Proof. We prove the theorem by induction on n. Obviously, for $n = 1$, we have $irr(K_1) = 0$. Suppose that for any Eulerian graph H on less than n vertices, $irr(H) = 4k$, for some non-negative integer k. Now, we shall show that if G is an Eulerian graph on n vertices, then there exists a non-negative integer k' with $irr(G) = 4k'$. To show this, we shall use induction on the number of edges. For $m = 0$, it is obvious that $irr(\overline{K_n}) = 0$. By induction on m, suppose that for any n-vertex Eulerian graph H, which has less than m edges, we have $irr(H) = 4k$, for some non-negative integer k. Let G be an n-vertex Eulerian graph with m edges. Let $C_q = v_1v_2 \cdots v_qv_1$ be the smallest simple cycle in G, and $H = G - E(C_q)$. If $H = \overline{K_n}$, then $G = C_q$, and therefore $irr(G) = 0$. If $H \neq \overline{K_n}$, then either H is an n-vertex Eulerian graph with less than m edges, or each of connected components of H is an Eulerian graph on less than n vertices. Therefore, by inductions' hypotheses, there is some $k \geq 0$ such that $irr(H) = 4k$. For convenience, we use the following notations:

$$E' = \{xv \in E(H) : v \in V(C_q) \,\&\, x \in V(G) \setminus V(C_q)\},$$

$$d_l(v) = |\{xv \in E' : d_H(x) \leq d_H(v)\}|,$$

$$d_g(v) = |\{xv \in E' : d_H(x) > d_H(v)\}|,$$

$$sign(s) = \begin{cases} 1 & ; s = l \\ -1 & ; s = g \end{cases}.$$

Assume that $v_{q+1} = v_1$. With above notations, one can immediately see that for any vertex v_i of C_q, $d_H(v_i) = d_l(v_i) + d_g(v_i)$. Note that by the choice of C_q, there is no non-consecutive indices i and j such that $v_iv_j \in E(G)$. Moreover, for any edge $xv \in E'$, if $d_H(x) \leq d_H(v)$, then

$$|d_G(x) - d_G(v)| - |d_H(x) - d_H(v)| = 2 = 2\,sign(l).$$

Moreover, if $d_H(x) > d_H(v)$ then

$$|d_G(x) - d_G(v)| - |d_H(x) - d_H(v)| = -2 = 2\,sign(g).$$

Now, we have:

$$\begin{aligned} irr(G) - irr(H) &= \sum_{uv\in E(C_q)} |d_G(u) - d_G(v)| + \sum_{xv\in E'} (|d_G(x) - d_G(v)| - |d_H(x) - d_H(v)|) \\ &= \sum_{uv\in E(C_q)} |d_G(u) - d_G(v)| + 2\sum_{v\in V(C_q)} \big(d_l(v) - d_g(v)\big) \\ &= \sum_{uv\in E(C_q)} \big(|d_G(u) - d_G(v)| + d_l(u) - d_g(u) + d_l(v) - d_g(v)\big) = \sum_{i=1}^{q} r_i. \end{aligned}$$

such that for any $i = 1,2,\dots,q$,

$$r_i = |d_G(v_i) - d_G(v_{i+1})| + d_l(v_i) - d_g(v_i) + d_l(v_{i+1}) - d_g(v_{i+1}).$$

One can easily check that if $d_G(v_{i+1}) \le d_G(v_i)$, then

$$r_i = 2d_l(v_i) - 2\,d_g(v_{i+1}) = 2\,\mathrm{sign}(l)d_l(v_i) + 2\,\mathrm{sign}(g)d_g(v_{i+1}),$$

and if $d_G(v_{i+1}) > d_G(v_i)$, then

$$r_i = -2\,d_g(v_i) + 2d_l(v_{i+1}) = 2\,\mathrm{sign}(g)d_g(v_i) + 2\,\mathrm{sign}(l)d_l(v_{i+1}).$$

Hence, for some suitable $s_i, s_i' \in \{l, g\}$, where $1 \le i \le q$, we can write the following:

$$\begin{aligned}
\mathrm{irr}(G) - \mathrm{irr}(H) = \sum_{i=1}^{q} r_i &= \left(2\,\mathrm{sign}(s_1)d_{s_1}(v_1) + 2\,\mathrm{sign}(s_2')d_{s_2'}(v_2)\right)\\
&+ \left(2\,\mathrm{sign}(s_2)d_{s_2}(v_2) + 2\,\mathrm{sign}(s_3')d_{s_3'}(v_3)\right) + \cdots\\
&+ \left(2\,\mathrm{sign}(s_q)d_{s_q}(v_q) + 2\,\mathrm{sign}(s_1')d_{s_1'}(v_1)\right)\\
&= \sum_{i=1}^{q}\left(2\,\mathrm{sign}(s_i)d_{s_i}(v_i) + 2\,\mathrm{sign}(s_i')d_{s_i'}(v_i)\right)\\
&= 2\sum_{i=1}^{q}\left(\mathrm{sign}(s_i)d_{s_i}(v_i) + \mathrm{sign}(s_i')d_{s_i'}(v_i)\right)\\
&= 2\sum_{i=1}^{q} t_i.
\end{aligned}$$

For each $i = 1,2,\dots,q$, there exist three cases as follow:

1) If $s_i = s_i' = l$, then $t_i = 2\,d_l(v_i)$.
2) If $s_i = s_i' = g$, then $t_i = -2\,d_g(v_i)$.
3) If $s_i \ne s_i'$, then $t_i = d_l(v_i) - d_g(v_i)$.

Since $d_G(v_i) = d_l(v_i) + d_g(v_i)$ is even, $d_l(v_i) - d_g(v_i)$ is even, too. Therefore, in all of the above cases, t_i is even. Thus,

$$\mathrm{irr}(G) - \mathrm{irr}(H) = 2\sum_{i=1}^{q} t_i = 4\sum_{i=1}^{q}\left(\frac{1}{2}\right)t_i = 4k''$$

where k'' is an integer. Hence, the theorem is proved by induction. □

Obviously, for a connected graph G, $\mathrm{irr}(G) = 0$ if and only if it is a regular graph. Therefore, we have the following result:

Corollary 2.3. *For a non-regular connected Eulerian graph G of order n,* $\mathrm{irr}(G) \ge 4$.

We know that the minimal irregularity of graphs is zero. Obviously, the irregularity of a graph is zero if and only if all of its connected components are regular. Since for each positive integer $r \ge 1$, each connected $2r$-regular graph is an Eulerian graph, hence the first minimum irregularity of Eulerian graphs is zero; and by Theorem 2.2, we conclude that the second minimum of the irregularity of Eulerian graphs is 4. In the following theorem we characterize connected Eulerian graphs with the second minimum irregularity.

Theorem 2.4. *There are* 12 *types of connected Eulerian graphs with irregularity value 4, where* the *general forms and examples of them are shown in Figure 1 and Table 1, respectively.*

Proof. Let G be a connected Eulerian graph with $irr(G) = 4$. For each edge uv of G, set $irr(uv) = |d_G(u) - d_G(v)|$, so we can write $irr(G) = \sum_{uv\in E(G)} irr(uv)$. The proof continues in three separate cases as follows:

Case 1. Let xy be an edge of G such that $irr(xy) = 4$. Since G is a connected Eulerian graph, there is a cycle $xyv_1v_2 \cdots v_kx$ in G containing edge xy. Clearly, since $irr(G) = irr(xy) = 4$, then $irr(yv_1) = irr(v_1v_2) = \cdots = irr(v_kx) = 0$ and we deduce that $d_G(y) = d_G(v_1) = d_G(v_2) = \cdots = d_G(v_k) = d_G(x)$, which is a contradiction. Therefore, this case does not occur.

Case 2. There are two edges xy and xz such that $irr(xy) = irr(xz) = 2$. It is clear that yxz is a path from vertex y to vertex z. Suppose $U = \{u_1, u_2, \dots, u_s\}$ and $V = \{v_1, v_2, \dots, v_r\}$ are subsets of vertices of G such that $x, y, z \notin U, V$. Also assume that $yu_1u_2 \cdots u_sxz$ and $yxv_1v_2 \cdots v_rz$ are two paths in G from vertex y to vertex z containing vertex x. Since $irr(G) = irr(xy) + irr(xz)$, then

$$irr(yu_1) = irr(u_1u_2) = \cdots = irr(u_sx) = irr(xv_1) = irr(v_1v_2) = \cdots = irr(v_rz) = 0.$$

Consequently, $d_G(x) = d_G(y) = d_G(z)$, which is a contradiction . Thus, two subcases will be considered as:

(I) There are two paths from vertex y to vertex z such that vertex x belongs to only one of them. Assume that $yu_1u_2 \cdots u_sz$ is a path in G, so $d_G(y) = d_G(z)$. Therefore, G is constructed of two separated components G_1 and G_2 that are connected by edges xy and xz, which $x \in V(G_1)$ and $y, z \in V(G_2)$. Let $|V(G_1)| = k$ and $|V(G_2)| = n - k$. Thus, we may consider two different parts as follows:

(i) $d_G(x) = a, d_G(y) = d_G(z) = a - 2$;

(ii) $d_G(x) = a, d_G(y) = d_G(z) = a + 2$.

In part(i), for any u in $V(G_1)\backslash\{x\}$, $d_{G_1}(u) = a$, $d_{G_1}(x) = a - 2$, and for any vertex u in $V(G_2)\backslash\{y, z\}$, $d_{G_2}(u) = a - 2$, $d_{G_2}(y) = d_{G_2}(z) = a - 3$. Therefore, $2|E(G_1)| = ka - 2$, $2|E(G_2)| = n(a - 2) - ka + 2(k - 1)$, G_1 is a $(2t + 2)$-regular graph, and G_2 is a $2t$-regular graph, for some $t \geq 1$. Consequently, ka and $n(a - 2)$ are even. Since a is even, k and n can be odd or even. Thus, four types will occur (see Table 1, types 1-4).

In part(ii), we have $2|E(G_1)| = ka - 2$, $2|E(G_2)| = (n - k)(a + 2) - 2$. Consequently, k and n can be odd or even. Thus, we have four further types (see Table 1, types 5–8). Note that in these types, G_1 is a $2t$-regular graph and G_2 is a $(2t+2)$-regular graph, for some $t \geq 1$.

(II) There is only one path, say yxz, joining vertices y and z which contains vertex x. Suppose $xu_1u_2\cdots u_sy$ and $xv_1v_2\cdots v_rz$ are two paths in G, where $u_1 \neq z$ and $v_1 \neq y$. Since $irr(G) = irr(xy) + irr(xz)$, then by above assumptions, $d_G(x) = d_G(y) = d_G(z)$, which is a contradiction to $irr(xy) = irr(xz) = 2$. Therefore, G is composed of three separate components G_1, G_2 and G_3 where G_1 and G_2 are connected by edge xy, G_1 and G_3 are connected by edge xz, $x \in V(G_1), y \in V(G_2), z \in V(G_3), V(G) = V(G_1) \cup V(G_2) \cup V(G_3)$ and $E(G) = E(G_1) \cup E(G_2) \cup E(G_3) \cup \{xy, xz\}$. Obviously, $2|E(G_2)| + 1 = \sum_{u\in V(G_2)} d_G(u)$ but $d_G(u)$ is even, for any vertex u of G. Therefore, this subcase does not occur.

Case 3. There are two distinct edges xy and uv such that $irr(xy) = irr(uv) = 2$. As case 2, we may again check this case in two subcases as follows:

(I) vertices y and u belong to all paths from vertex x to vertex v;

(II) There are two paths from vertex x to vertex v such that vertices y and u belong to only one of them.

Similar to case 2, in subcase **(I)**, G is constructed of three separate components G_1, G_2 and G_3, where G_1 and G_2 are connected by edge xy, and G_2, G_3 are connected by edge uv, $x \in V(G_1), y, u \in V(G_2), v \in V(G_3), V(G) = V(G_1) \cup V(G_2) \cup V(G_3)$ and $E(G) = E(G_1) \cup E(G_2) \cup E(G_3) \cup \{xy, uv\}$. Obviously, $2|E(G_1)| + 1 = \sum_{w\in V(G_1)} d_G(w)$ but $d_G(w)$ is even, for any vertex w of G. Therefore, this case does not occur.

In subcase **(II)**, we can see that G is composed of two separate components G_1, G_2 where G_1 and G_2 are connected by edges xy and uv, also $x, v \in V(G_1)$ and $y, u \in V(G_2)$. Let $d_G(x) = a, |V(G_1)| = k$ and $|V(G_2)| = n - k$. Without loss of generality, in the case **(II)**, we can consider following two parts:

(i) $d_G(x) = d_G(v) = a, d_G(y) = d_G(u) = a + 2$;

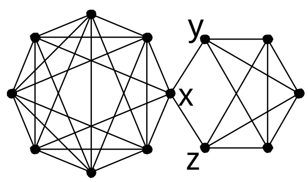
(ii) $d_G(x) = d_G(v) = a, d_G(y) = d_G(u) = a - 2$.

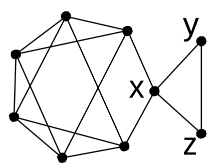
A similar argument as case 2, in part(i), k and n can be odd or even . Thus we have another four types (see Table.1, types 9–12). Note that, the graphs in parts(ii) and (i) are identical, where G_1 is $2t$-regular, and G_2 is $(2t+2)$-regular, for some $t \geq 1$. □

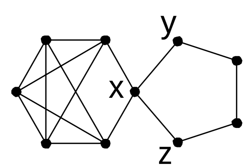
Note that, in generally, the irregularity of a graph is equal to the summation of its connected components' irregularities. Therefore, if G is an n-vertex (not necessary connected) Eulerian graph with $irr(G) = 4$, then Theorem 2.2 implies that $G \cong G' \cup \overline{K_s}$, where G' is a connected Eulerian graph on $n - s$ vertices with $irr(G') = 4$.

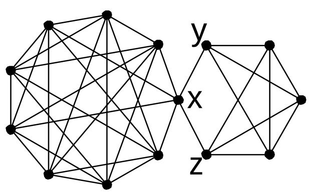
General form of types 1−8 General form of types 9−12

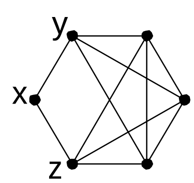
Figure 1. General forms of Eulerian graphs with the second minimum irregularity.

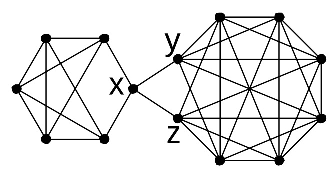
<table>
<tr><th>Type 1</th><th>Type 2</th><th>Type 3</th></tr>
<tr>
<td>

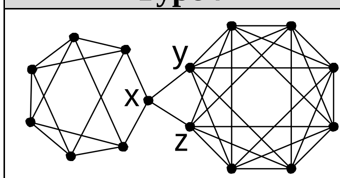
n=13,k=8,a=6,t=2
G_1: (2t + 2)-regular
G_2: 2t-regular</td>
<td>

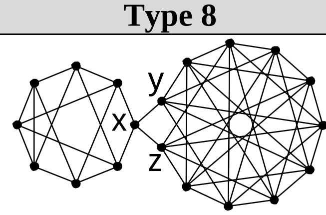
n=9,k=7,a=4,t=1
G_1: (2t + 2)-regular
G_2: 2t-regular</td>
<td>

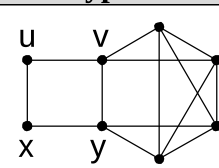
n=10,k=6,a=4,t=1
G_1: (2t + 2)-regular
G_2: 2t-regular</td>
</tr>
<tr><th>Type 4</th><th>Type 5</th><th>Type 6</th></tr>
<tr>
<td>

n=14,k=9,a=6,t=2
G_1: (2t + 2)-regular
G_2: 2t-regular</td>
<td>

n=6,k=1,a=2,t=1
G_1: 2t-regular
G_2: (2t + 2)-regular</td>
<td>

n=14,k=6,a=4,t=2
G_1: 2t-regular
G_2: (2t + 2)-regular</td>
</tr>
<tr><th>Type 7</th><th>Type 8</th><th>Type 9</th></tr>
<tr>
<td>

n=15,k=7,a=4,t=2
G_1: 2t-regular
G_2: (2t + 2)-regular</td>
<td>

n=19,k=8,a=4,t=2
G_1: 2t-regular
G_2: (2t + 2)-regular</td>
<td>

n=8,k=2,a=2,t=1
G_1: 2t-regular
G_2: (2t + 2)-regular</td>
</tr>
</table>

Table 1. Examples of Eulerian graphs with the second minimum irregularity.

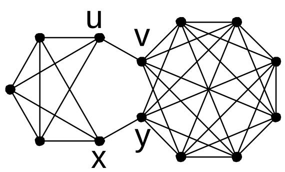
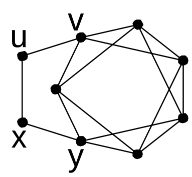
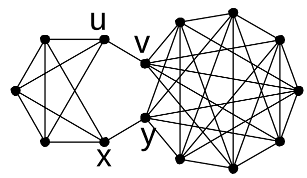
Type 10	**Type 11**	**Type 12**
 n=13,k=5,a=4,t=2 G_1: 2t-regular G_2: (2t + 2)-regular	 n=9,k=2,a=2,t=1 G_1: 2t-regular G_2: (2t + 2)-regular	 n=14,k=5,a=4,t=2 G_1: 2t-regular G_2: (2t + 2)-regular

Table 1. (Continued).

3. THE SECOND AND THIRD MINIMUM TOTAL IRREGULARITY FOR EULERIAN GRAPHS

In this section, first we express some initially basic definitions and a prominent proved result of [3], and then investigate the second and third minimum total irregularity of connected Eulerian graphs.

If $V(G) = \{v_1, v_2, \ldots, v_n\}$, then the sequence $(d_G(v_1), d_G(v_2), \ldots, d_G(v_n))$ is called a ***degree sequence*** of G [11]. Without loss of generality, we may assume that $d_G(v_1) \geq d_G(v_2) \geq \ldots \geq d_G(v_n)$. A ***bicyclic graph*** is a simple connected graph in which the number of edges equals to $n + 1$. A ***basic bicyclic*** ∞***-graph***, denoted by $\infty(p, q, l)$, is obtained from two vertex-disjoint cycles C_p and C_q by connecting one vertex of C_p and one of C_q with a path P_l of length $l - 1$ (in the case of $l = 1$, identifying the above two vertices) where $p, q \geq 3$ and $l \geq 1$.

Clearly, a graph G has total irregularity zero if and only if G is a regular graph. Note that the connected $2r$-regular graph, is an Eulerian graph with $irr_t = 0$. Hence,the first minimum total irregularity of Eulerian graphs is zero. Moreover the corresponding extremal Eulerian graphs with total irregularity 0 are exactly all $2r$-regular Eulerian graphs, where $r \geq 0$, and if $r > 0$ then the graph is connected. In [3], the authors characterized the non-regular graphs with the second and the third smallest total irregularity.

Lemma 3.1. [3] *Let G be a simple connected graph with n vertices. If G is a non-regular graph, then $irr_t(G) \geq 2n - 4$.*

In the following result, we show that the second minimum of the total irregularity of Eulerian graphs is 8 and determine the unique Eulerian graph with $irr_t = 8$.

Theorem 3.2. *Let G be a connected non-regular Eulerian graph of order$n$, then* $\mathrm{irr}_t(G) \geq 8$*, and the equality holds if and only if* $G \cong \infty(3,3,1)$*, where the bicyclic graph* $\infty(3,3,1)$ *is shown in Figure 2.*

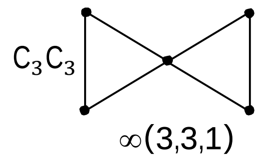


Figure 2. Unique Eulerian graph with the second minimum total irregularity.

Proof. By Lemma 3.1, if $n \geq 7$, then $\mathrm{irr}_t > 8$. If $n = 6$, then the degree sequence of G can be one of the following cases: $(4,4,4,4,4,2)$, $(4,4,4,4,2,2)$, $(4,4,4,2,2,2)$, $(4,4,2,2,2,2)$, and $(4,2,2,2,2,2)$. By a simple calculation, one can easily see that in these cases, $\mathrm{irr}_t(G) > 8$. If $n = 5$, then the degree sequence of G may be either $(4,4,2,2,2)$ or $(4,2,2,2,2)$. Note that the cases $(4,4,4,4,2)$ and $(4,4,4,2,2)$ do not occur. Also, the total irregularity of graph G with degree sequence $(4,4,2,2,2)$ is equal to 12 and with degree sequence $(4,2,2,2,2)$ is equal to 8. Additionally, the graph G with degree sequence $(4,2,2,2,2)$ is the bicyclic graph $\infty(3,3,1)$. Clearly, regular graphs C_3 and C_4 are the only Eulerian graphs with 3 and 4 vertices, which have total irregularity 0. □

Theorem 3.3. *Let* $G \ncong \infty(3,3,1)$ *be a connected non-regular Eulerian graph of order* n*, then* $irr_t(G) \geq 10$*, and the equality holds if and only if* $G \cong \infty(4,3,1)$ *or* H*, where graphs* $\infty(4,3,1)$ *and* H *are shown in Figure 3.*

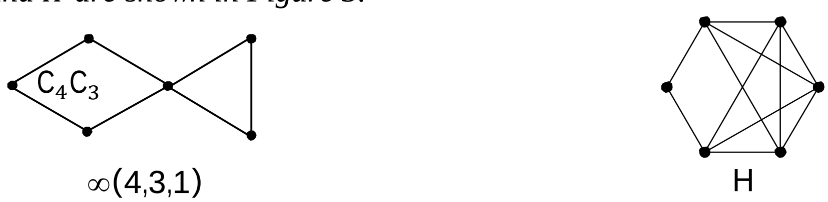


Figure 3. Eulerian graphs with the third minimum total irregularity.

Proof. By Lemma 3.1, if $n \geq 8$ then $\mathrm{irr}_t > 10$. If $n = 7$, then the degree sequence of G may be the following cases:

$(6,6,6,6,6,6,4)$, $(6,6,6,6,6,6,2)$, $(6,6,6,6,6,4,4)$, $(6,6,6,6,6,4,2)$, $(6,6,6,6,6,2,2)$,
$(6,6,6,6,4,4,4)$, $(6,6,6,6,4,4,2)$, $(6,6,6,6,4,2,2)$, $(6,6,6,6,2,2,2)$, $(6,6,6,4,4,4,4)$,
$(6,6,6,4,4,4,2)$, $(6,6,6,4,4,2,2)$, $(6,6,6,4,2,2,2)$, $(6,6,6,2,2,2,2)$, $(6,6,4,4,4,4,4)$,
$(6,6,4,4,4,4,2)$, $(6,6,4,4,4,2,2)$, $(6,6,4,4,2,2,2)$, $(6,6,4,2,2,2,2)$, $(6,6,2,2,2,2,2)$,
$(6,4,4,4,4,4,4)$, $(6,4,4,4,4,4,2)$, $(6,4,4,4,4,2,2)$, $(6,4,4,4,2,2,2)$, $(6,4,4,2,2,2,2)$,
$(6,4,2,2,2,2,2)$, $(6,2,2,2,2,2,2)$, $(4,4,4,4,4,4,2)$, $(4,4,4,4,4,2,2)$, $(4,4,4,4,2,2,2)$,
$(4,4,4,2,2,2,2)$, $(4,4,2,2,2,2,2)$, $(4,2,2,2,2,2,2)$.

By a simple calculation, one can easily see that in these cases, $irr_t(G) > 10$. If $n = 6$, then the degree sequence of G can be the following cases:

$(4,4,4,4,4,2)$, $(4,4,4,4,2,2)$, $(4,4,4,2,2,2)$, $(4,4,2,2,2,2)$, $(4,2,2,2,2,2)$.

The total irregularity of graph G with degree sequence $(4,4,4,4,4,2)$ or $(4,2,2,2,2,2)$ is equal to 10 and with the other degree sequences is more than 10. Note that if $(4,4,4,4,4,2)$ is degree sequence of graph G, then $G \cong H$, and if $(4,2,2,2,2,2)$ is degree sequence of graph G, then $G \cong \infty(4,3,1)$. Finally, if $n \le 5$, then by referring to the proof of Theorem 3.2, we see that the total irregularity value of G is not equal to 10. □

Corollary 3.4. *The second and third minimum value of the total irregularity of Eulerian graphs are* 8 *and* 10*, respectively.*

REFERENCES

1. H. Abdo, S. Brandt and D. Dimitrov, The total irregularity of a graph, *Discrete Math. Theor. Comput. Sci.* **16** (2014) 201−206.
2. H. Abdo, N. Cohen and D. Dimitrov, Graphs with maximal irregularity, *Filomat* **28** (7) (2014) 1315−1322.
3. H. Abdo and D. Dimitrov, Non-regular graphs with minimal total irregularity, arXiv:1407.1276v1.
4. H. Abdo and D. Dimitrov, The total irregularity of graphs under graph operations, *Miskolc Math. Notes* **15** (1) (2014) 3−17.
5. H. Abdo and D. Dimitrov, The irregularity of graphs under graph operations, *Discuss. Math. Graph Theory* **34** (2) (2014) 263−278.
6. M. O. Albertson, The irregularity of a graph, *Ars Comb.* **46** (1997) 219−225.
7. A. Astaneh-Asl and G. H. Fath-Tabar, Computing the first and third Zagreb polynomials of Cartesian product of graphs. *Iranian J. Math. Chem.* **2** (2) (2011) 73−78.
8. A.T. Balaban, I. Motoc, D. Bonchev and O. Mekenyan, Topological indices for structure-activity connection, *Topics Curr. Chem.* **114** (1983) 21−55.
9. F. K. Bell, A note on the irregularity of graphs, *Linear Algebra Appl.* **161** (1992) 45−54.
10. N. L. Biggs, E. K. Lloyd and R. J. Wilson, *Graph Theory*, Clarendon Press, Oxford, 1976.
11. J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, MacMillan, London, 1976.
12. D. Dimitrov and R. Skrekovski, Comparing the irregularity and the total irregularity of graphs, *Ars. Math. Contemp.* **9** (2015) 45−50.

13. M. Eliasi, The maximal total irregularity of some connected graphs. *Iranian J. Math. Chem*. **6** (2) (2015) 121−128.
14. G. H. Fath-Tabar, Old and new Zagreb index, *MATCH Commun. Math. Comput. Chem.* **65** (2011) 79−84.
15. G. H. Fath-Tabar, I. Gutman and R. Nasiri, Extremely irregular trees, *Bull. Acad. Serbe Sci. Arts* (*Cl. Sci. Math. Natur.*) **38** (2013) 1−8.
16. I. Gutman and N. Trinajstić, Graph theory and molecular orbitals total π-electron energy of alternant hydrocarbons, *Chem. Phys. Lett*. **17** (1972) 535−538.
17. P. Hansen and H. Melot, Variable neighborhood search for extremal graphs. 9. bounding the irregularity of a graph, *DIMACS Ser. Discrete Math. Theoret. Comput. Sci*. **69 (**2005) 253−264.
18. M. A. Henning and D. Rautenbach, On the irregularity of bipartite graphs, *Discrete Math.* **307 (**2007) 1467−1472.
19. W. Luo and B. Zhou, On the irregularity of trees and unicyclic graphs with given matching number, *Util. Math.* **83** (2010) 141−147.
20. R. Nasiri and G. H. Fath-Tabar, The second minimum of the irregularity of graphs, *Electronic. Notes Discrete Math*. **45** (2014) 133−140.
21. R. Nasiri, A. Golami, G. H. Fath-Tabar and H. R. Ellahi, Extremely irregular unicyclic graphs, *Kragujevac J. Math.* **43** (2) (2019) 281−292.
22. T. Reti and D. Dimitrov, On irregularities of bidegreed graphs, *Acta Polytechn. Hung*. **10** (7) (2013) 117−134.
23. M. Tavakoli, F. Rahbarnia and A. R. Ashrafi, Some new results on irregularity of graphs, *J. Appl. Math. Inform*. **32** (2014) 675−685.
24. M. Tavakoli, F. Rahbarnia, M. Mirzavaziri, A. R. Ashrafi and I. Gutman, Extremely irregular graphs, *Kragujevac J. Math*. **37** (1) (2013) 135−139.
25. R. Todeschini and V. Consonni, *Handbook of Molecular Descriptors*, Wiley-VCH, Weinheim, 2000.
26. Z. Yarahmadi and G. H. Fath-Tabar, The Wiener, Szeged, PI, vertex PI, the first and second Zagreb indices of N-branched phenylacetylenes dendrimers, *MATCH Commun. Math. Comput. Chem.* **65** (1) (2011) 201−208.
27. L. You, J. Yang and Z. You, The maximal total irregularity of unicyclic graphs, *Ars Comb.* in press.
28. L. You, J. Yang and Z. You, The maximal total irregularity of bicyclic graphs, *J. Appl. Math.* (2014) Article ID 785084, 9 pages.
29. B. Zhou and W. Luo, On irregularity of graphs, *Ars Combin*. **88** (2008) 55−64.