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On the Eccentric Connectivity Index of Unicyclic Graphs

YAŞAR NACAROĞLU[1,•] AND AYŞE DILEK MADEN[2]

[1]Department of Mathematics Faculty of Science and Arts, University of Kahramanmaras, Sutcu Imam, 46000, Kahramanmaras, Turkey

[2]Department of Mathematics, Faculty of Science, Selçuk University, Campus, 42075, Konya, Turkey

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ABSTRACT

In this paper, we obtain the upper and lower bounds on the eccentricity connectivity index of unicyclic graphs with perfect matchings. Also we give some lower bounds on the eccentric connectivity index of unicyclic graphs with given matching numbers.

1 INTRODUCTION

Throughout this paper, G will denote a simple connected graph with n vertices (labeled by $v_1, v_2, ..., v_n$). Moreover, for $1 \le i \le n$, the neighbor and the degree of each vertex v_i will be denoted by $N(v_i)$ and d_{v_i}, respectively. For two vertices u and v in $V(G)$, we denote by $d(u,v)$ the distance between u and v i.e. the length of the shortest $u-v$ path in G.

• Corresponding Author (Email address: yasarnacar@hotmail.com)
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A pendant vertex is a vertex of degree 1 and a pendant edge is an edge incident to a pendant vertex. Denote by PV the set of pendant vertices of a graph G.

The eccentricity of a vertex u in a graph G, denoted by e_u, is the maximum distance from u to any vertex. That is, $e_u = \max\{d(u,v) : v \in V\}$.

A matching M of a graph G is a subset of $E(G)$ such that no two edges in M share a common vertex. The matching number of G, denoted by m, is the number of edges of a maximum matching in G. If every vertex of G incident with an edge of M, then the matching M is perfect.

Denote by C_n the cycle on n vertices. A unicyclic graph is a connected graph with a unique cycle. Other undefined terminologies and notations of graphs may refer to [2].

Molecular descriptors have found a wide application in QSPR / QSAR studies [15]. Among them, topological indices have a prominent place.

Sharma, Goswami and Madan [14] introduced a distance-based topological index which named eccentric connectivity index $\xi^c(G)$ as follows:

$$\xi^c(G) = \sum_{u \in V(G)} d_u e_u .$$

The eccentric connectivity index has been employed successfully for the development of numerous mathematical models of biological activities of diverse nature [7,8,13,14].

Recently Ashrafi et. al. [1] obtained exact formulas for the eccentric connectivity index of $TUC_4C_8(s)$ nanotube and $TC_4C_8(s)$ nanotorus. Ilič and Gutman [6] examined the eccentric connectivity index of chemical trees. In [18], Zhou et al. gave mathematical properties of eccentric connectivity index. Also in [11], the authors investigated the eccentric connectivity index of trees. Moreover, in [17], Zhang investigated the eccentric connectivity index of unicyclic graphs. Recall that in mathematical chemistry a unicyclic graph with perfect matching is known as conjugated unicyclic graphs. We may refer [9,10] for more and some other details on conjugated unicyclic graphs.

In this paper, we present upper and lower bounds on the eccentric connectivity index of unicyclic graphs with perfect matchings. Also we give lower bounds on the eccentric connectivity index of unicyclic graphs with given matching numbers.

Throughout this paper, U_n denote unicyclic graphs with n vertices and $U(n,m)$ denote unicyclic graphs with n vertices and m matchings. Let $U(2m,m)$ be the set of unicyclic graphs with perfect matching and $2m$ vertices. We make use of the following results in this paper.

Lemma 1.1. [3] *Let $G \in U(2m,m)$, where $m \geq 3$, and let T be a branch of G with the root r. If $u \in V(T)$ is a pendant vertex furthest from the root r with $d_G(u,r) \geq 2$, then u is adjacent to a vertex v of degree two.*

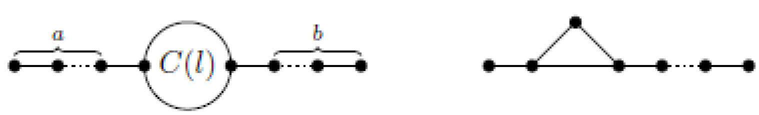
Lemma 1.2. [12] *Let* $G \in U(2m,m)$. *If* $PV \neq \emptyset$, *then for any vertex* $u \in V(G)$, $|N(u) \cap PV| \leq 1$.

Lemma 1.3. [16] *Let G be a graph in* $U(n,m)$ *and* G ® C_n, *where* $(n > 2m)$. *Then there is an* $m-$*matching M and a pendant vertex v such that M does not saturate v.*

Theorem 1.4. [6] *Let* w *be a vertex of a nontrivial connected graph G. For non-negative integers p and q, let G(p,q) denote the graph obtained from G by attaching to the vertex w pendent paths* $P = wv_1v_2...v_p$ *and* $Q = wu_1u_2...u_q$ *of lengths p and q, respectively. If* $p \geq q \geq 1$, *then*

$$\xi^c(G(p,q)) = \xi^c(G(p+1,q-1)).$$

Let P_n^3 denote the unicyclic graph obtained from C_3 by identifying one of its vertices with a pendant vertex of the path P_{n-2}. Let $C_{a,b}(l)$ denote the unicyclic graph obtained from $C(l)$ by attaching the path P_a at one vertex and the path P_b at another vertex. Let $C_{a,b}(l)$ and N_n^3 denote unicyclic graphs of the forms as depicted in Figure 1.



$C_{a,b}(l)$ where $a,b \geq 1$ $\qquad$ N_n^3

Figure 1

Lemma 1.5. [17] *Let G be a graph in* U_n *with* $n \geq 5$ *vertices. If* $G \in C_{a,b}(l)$ *and* G ® N_n^3, *then* $\xi^c(P_n^3) > \xi^c(N_n^3) > \xi^c(G)$.

Theorem 1.6. [17] *Let G be a graph in* U_n, $n \geq 5$. *Then*

$$\xi^c(G) \leq \xi^c(P_n^3) = \begin{cases} \frac{1}{2}(3n^2 - 4n - 6), & n \text{ is even} \\ \frac{1}{2}(3n^2 - 4n - 5), & n \text{ is odd}, \end{cases}$$

where the equality holds if and only if G ® P_n^3.

2 MAIN RESULTS

After all above material, we are ready to present our results on the bounds for the eccentric connectivity index of unicyclic graphs. Let H_1^m be the graph on $2m$ vertices obtained from C_3 by attaching a pendant edge together with $m-2$ paths of length 2 at one vertex. Let H_2^m be the graph on $2m$ vertices obtained from C_3 by attaching a pendant edge and $m-3$ paths of length 2 at one vertex, and single pendant edges at the other vertices. Let H_3^m be the graph on $2m$ vertices obtained from C_3 by attaching a pendant edge at one vertex and $m-2$ paths of length 2 at another vertex. Let H_4^m be the graph on $2m$ vertices obtained from C_4 by attaching $m-2$ paths of length 2 at one vertex.

Let H_5^m be the graph on $2m$ vertices obtained from C_4 by attaching a pendant edge atone vertex and $m-3$ paths of length 2 at another vertex. H_1^m, H_2^m, H_3^m, H_4^m and H_5^m are shown in Figure 2.

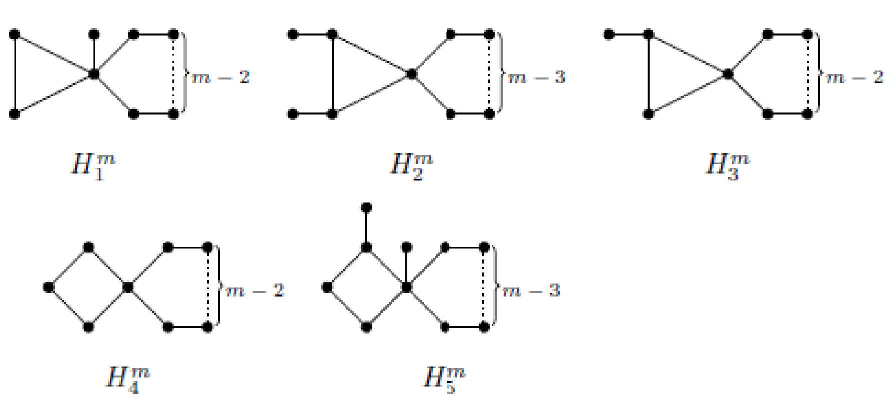


Figure 2

Theorem 2.1. *Let* $G \in U(2m,m) \setminus \{H_1^m, H_2^m, H_3^m\}$, *where* $m \geq 5$. *Then* $\xi^c(G) \geq 12m$. *The equality holds if and only if* $G \cong H_4^m$ *or* $G \cong H_5^m$.

Proof. We prove the result by induction on m. By direct calculation, we see that the result is provided for $m=5$. Let $m \geq 5$. We assume that the result holds for graphs in $U(2m-2, m-1)$.

Case 1. G has a pendant vertex u which is adjacent to a vertex v of degree two. Let w be adjacent of v different from u. In this case, $uv \in M$. Let $G' = G-u-v$. Then $G' \in U(2m-2, m-1)$. Obviously, if $e_u = e$, then $e_v = e-1$ and $e_w = e-2$ since $m \geq 5$. Also since $m \geq 5,\ e \geq 4$. We know that $e_t = e_t^{'}$ or $e_t = e_t^{'} + 1$ or $e_t = e_t^{'} + 2$ for $t \in V(G) - \{u, v, w\}$. Therefore, we write

$$\xi^c(G) - \xi^c(G^{'}) = e_u d_u + e_v d_v + e_w d_w - e_{w^{'}} d_{w^{'}} + \sum_{t \in V(G) - \{u,v,w\}} d_t \left[e_t - e_t^{'} \right]$$

$$= e.1 + 2(e-1) + d_w(e-2) - (d_w - 1)(e-2) + \sum_{t \in V(G) - \{u,v,w\}} d_t \left[e_t - e_t^{'} \right]$$

$$= 4e - 4 + \sum_{t \in V(G) - \{u,v,w\}} d_t \left[e_t - e_t^{'} \right]. \qquad (1)$$

Using the equality (1), we get the following inequality $\xi^c(G) \geq \xi^c(G^{'}) + 4e - 4$. Using the induction hypothesis and the fact $e \geq 4$, we have $\xi^c(G) \geq 12(m-1) + 4.4 - 4 = 12m$. This is the required result.

Case 2. G is a cycle C_k together with some pendant edges attached to some vertices on C_k. If $G \cong C_{2m}$ then $\xi(G) = 4m^2 \geq 12m$ for $m \geq 5$. Let $G ® C_{2m}$. Let p and q denote the number of pendant vertices and the number of vertices with degree two in G, respectively. Therefore, the number of vertices with degree three be p. Note that we have $2p + q = 2m$ and $p \leq m$. Let $V_1(G) = \{u \in V(G) :\ d_u = 1\}$, $V_2(G) = \{v \in V(G) : d_v = 2\}$ and $V_3(G) = \{t \in V(G) : d_t = 3\}$. If u is a pendant vertex, $e_u \geq \left\lfloor \frac{m+1}{2} \right\rfloor + 1$, if v is a vertex with degree two, $e_v \geq \left\lfloor \frac{m+1}{2} \right\rfloor$ and if t is a vertex with three degree, $e_t \geq \left\lfloor \frac{m+1}{2} \right\rfloor$. Thus

$$\xi^c(G) = \sum_{u \in V_1(G)} 1.e_u + \sum_{v \in V_2(G)} 2.e_v + \sum_{t \in V_3(G)} 3.e_t$$

$$\geq \sum_{u \in V_1(G)} 1.\left(\left\lfloor \frac{m+1}{2} \right\rfloor + 1 \right) + \sum_{v \in V_2(G)} 2.\left(\left\lfloor \frac{m+1}{2} \right\rfloor \right) + \sum_{t \in V_3(G)} 3.\left(\left\lfloor \frac{m+1}{2} \right\rfloor \right)$$

$$= p\left(\left\lfloor \frac{m+1}{2} \right\rfloor + 1 \right) + 2(2m - 2p)\left(\left\lfloor \frac{m+1}{2} \right\rfloor \right) + 3p\left(\left\lfloor \frac{m+1}{2} \right\rfloor \right)$$

$$= 4m\left\lfloor \frac{m+1}{2} \right\rfloor + p \geq 4m\left\lfloor \frac{m+1}{2} \right\rfloor + 1.$$

If m is odd, then since $m \geq 5$,

$$\xi^c(G) \geq 4m\left(\frac{m+1}{2} \right) + 1 = 2m^2 + 2m + 1 \geq 12m.$$

If m is even, then since $m \geq 5$,

$$\xi^c(G) \geq 4m\left(\frac{m}{2} \right) + 1 = 2m^2 + 1 \geq 12m.$$

The proof is now completed.

The next corollaries are the consequences of Theorem 2.1.

Corollary 2.2. *Let* $G \in U(2m,m) \setminus \{H_1^m\}$, *where* $m \geq 5$. *Then* $\xi^c(G) \geq 12m-1$. *The equality holds if and only if* $G \cong H_2^m$ *or* $G \cong H_3^m$.

Corollary 2.3. *Let* $G \in U(2m,m)$, *where* $m \geq 5$. *Then* $\xi^c(G) \geq 12m-3$. *The equality holds if and only if* $G \cong H_1^m$.

Let A_1^m denote the graph on n vertices obtained from C_3 by attaching $n-2m+1$ pendant edges and $m-2$ paths of length 2 together to one of three vertices of C_3. Let A_2^m denote the graph on n vertices obtained from C_3 by attaching $n-2m+1$ pendant edges and $m-3$ paths of length 2 together to one of three vertices, and two pendant edges to the other two vertices of C_3, respectively. Let A_3^m denote the graph on n vertices obtained from C_3 by attaching $n-2m$ pendant edges and $m-2$ paths of length 2 together to one of three vertices, and a pendant edge to another vertex of C_3, respectively. Let A_4^m denote the graph on n vertices obtained from C_4 by attaching $n-2m+1$ pendant edges and $m-3$ paths of length 2 together with one of the three vertices, and a pendant edge to another vertex of C_4, respectively.

Let A_5^m denote the graph on n vertices obtained from C_4 by attaching $n-2m$ pendant edges and $m-2$ paths of length 2 together to one of three vertices of C_3. A_1^m, A_2^m, A_3^m, A_4^m and A_5^m are shown in Figure 3.

Theorem 2.4. *Let* $G \in U(n,m) \setminus \{A_1^m, A_2^m, A_3^m\}$ $(n \geq 2m,\ m \geq 5)$. *Then* $\xi^c(G) \geq 5n+2m$. *The equality holds if and only if* $G \cong A_4^m$ *or* $G \cong A_5^m$.

Proof. We prove the result by induction on m. If $n = 2m$, then by Theorem 2.1, the result is clear. We assume that $n > 2m$. If $G \cong C_n$, then $n = 2m+1$, since G has an $m-$matching. So, since $m \geq 5$

$$\xi^c(G) = \sum_{u \in V(G)} 2\left\lfloor \frac{2m+1}{2} \right\rfloor = 4m^2 + 2m \geq 5n + 2m.$$

We assume that $G ® C_n$. By Lemma 1.3, G has an $m-$matching M and pendant vertex v such that M does not saturate v. Let $G' = G - v$. Then $G' \in U(n-1,m)$. Let $e_v = e$ and u be unique neighbor of v pendant vertex. Since $m \geq 4$, then $e \geq 3$. Thus, we have

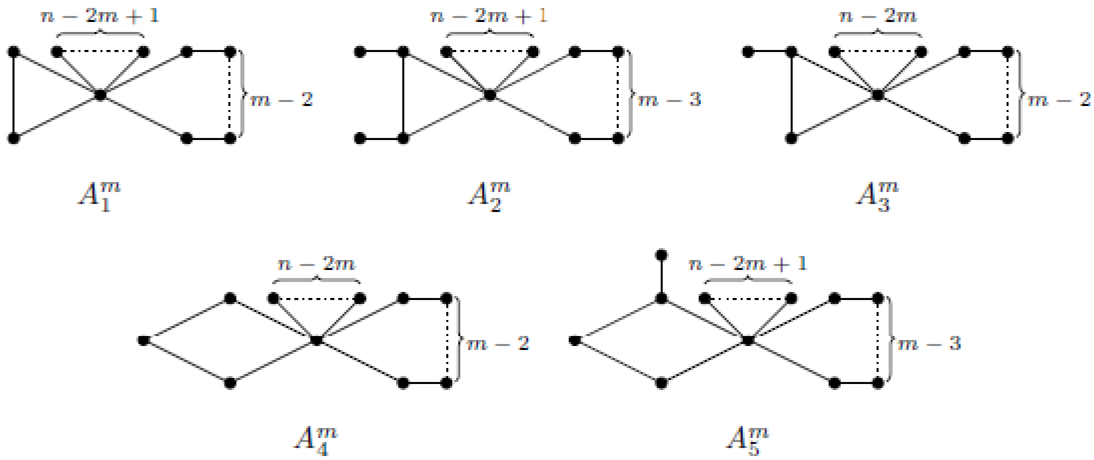


Figure 3

$$\begin{aligned}\xi^c(G)-\xi^c(G') &= 1e+d_u(e-1)-(d_u-1)(e-1)+\sum_{t\in V(G)-\{u,v\}} d_t\left[e_t-e_t^{'}\right] \\ &= 2e-1+\sum_{t\in V(G)-\{u,v\}} d_t\left[e_t-e_t^{'}\right]. \end{aligned} \quad (2)$$

Also, we have $0\le e_t-e_t^{'}\le 2.$ Using the equality (2), we get the following inequality

$$\xi^c(G)\ge \xi^c(G^{'})+2e-1.$$

Using the induction hypothesis and the fact $e\ge 3,$ we have

$$\begin{aligned}\xi^c(G) &\ge 5(n-1)+2m+2e-1 \\ &= 5n+2m+2e-6 \\ &\ge 5n+2m.\end{aligned}$$

This is the required result.

The next corollaries are the consequences of Theorem 2.4.

Corollary 2.5. *Let* $G\in U(n,m)\setminus\{A_1^m\}\left(n\ge 2m,\ m\ge 5\right)$. *Then* $\xi^c(G)\ge 5n+2m-1$. *The equality holds if and only if* $G\cong A_2^m$ *or* $G\cong A_3^m$.

Corollary 2.6. *Let* $G\in U(n,m)\setminus\{A_1^m\}\left(n\ge 2m,\ m\ge 5\right)$. *Then* $\xi^c(G)\ge 5n+2m-3$. *The equality holds if and only if* $G\cong A_1^m$.

Let U_1^m be the graph on $2m$ vertices obtained from C_3 by attaching a path of length $2m-3$ at one vertex. Let U_2^m be the graph on $2m$ vertices obtained from C_4 by attaching a path of length $2m-4$ one vertex. Let U_3^m be the graph on $2m$ vertices obtained from C_3 by attaching a pendant edge at one vertex and a path of length $2m-4$ at one vertex. Denote by U_1^m, U_2^m and U_3^m the graphs shown in Figure 2.

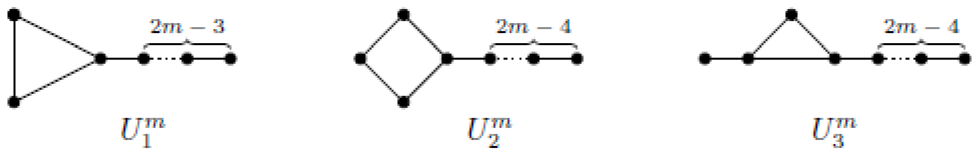


Figure 4

Theorem 2.7. *Let* $G \in U(2m,m) \setminus \{U_1^m, U_2^m\}$, *where* $m \geq 4$. *Then* $\xi^c(G) \leq 6m^2 - 4m - 7$ *with equality if and only if* $G \cong U_3^m$.

Proof. If $G \cong C_n$, then $\xi^c(G) = 4m^2 \leq 6m^2 - 4m - 7$ for $m \geq 4$. Let G ® C_n. Let G' be a graph obtained from G by using Theorem 1.4. In this case, we obtain $G' \in C_{a,b}(l)$. Then $\xi^c(G) \leq \xi^c(G')$. By applying Lemma 1.5, we get $\xi^c(G) \leq \xi^c(G') \leq \xi^c(U_1^m) = 6m^2 - 4m - 7$. The proof is completed.

The next corollaries are the consequences of Theorem 2.7.

Corollary 2.8. *Let* $G \in U(2m,m) \setminus \{U_1^m\}$, *where* $m \geq 4$. *Then* $\xi^c(G) \leq 6m^2 - 4m - 6$ *with equality if and only if* $G \cong U_2^m$.

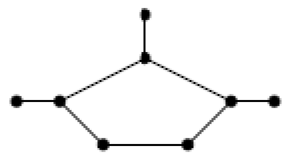
In Theorem 1.6, taking $n = 2m$, we have the following corollary.

Corollary 2.9. *Let* $G \in U(2m,m)$, *where* $m \geq 4$. *Then* $\xi^c(G) \leq 6m^2 - 4m - 3$ *with equality if and only if* $G \cong U_1^m$.

Remark 2.10. If $G \in U(2m+1,m)$ By Theorem 1.6, $\xi^c(G) \leq 6m^2 + 2m - 3$ with equality if and only if $G \cong P_n^3$ When $G \in U(n,m)(n \geq 2m+2)$ we do not know upper bounds on $\xi^c(G)$. The case maybe much more complicated.

Remark 2.11. We note that Theorem 2.7 and Corollary 2.9 hold in $U(6,3)$ from the table of connected graphs on six vertices in [4]. But Corollary 2.8 does not hold in case of equality, since $\xi^c(U_2^3) = \xi^c(C_6)$ are in $U(6,3)$.

From the table of unicyclic graphs on eight vertices in [5], we also see that Theorem 2.1 and Corollary 2.3 hold in $U(8,4)$. But Corollary 2.2 does not hold in case of equality, since $\xi^c(H_2^4) = \xi^c(H_3^4) = \xi^c(C_5^*)$ are in $U(8,4)$ where C_5^* is as in Figure 5.



C_5^*

Figure 5

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