Reciprocal Degree Distance of Grassmann Graphs

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ABSTRACT

Recently, Hua et al. defined a new topological index based on degrees and inverse of distances between all pairs of vertices. They named this new graph invariant as reciprocal degree distance as $RDD(G) = \sum_{\{u,v\}\subseteq V(G)}(d(u)+d(v))[d(u,v)]^{-1}$, where the $d(u,v)$ denotes the distance between vertices u and v. In this paper, we compute this topological index for Grassmann graphs.

Keywords: Grassmann graph, Harary index, vertex−transitive graphs.

1. INTRODUCTION

The Mathematical Chemistry is one of the main branches of theoretical chemistry. It can be used to prediction of chemical phenomena. The Chemical Graph Theory is a branch of mathematical chemistry for studying molecular structures [1−3]. This theory had an important effect on the development of the chemical sciences [4−5].

Throughout this paper, all graphs are simple and connected. We denote the vertex and edge sets of a graph G by $V(G)$ and $E(G)$, respectively. Suppose $\wp$ denotes the class of all graphs. A topological index is a real function $\Lambda : \wp \to \Re^+$ by this property that if $G \cong H$ then $\Lambda(G) = \Lambda(H)$. Obviously, the maps $\Lambda_1 = |V(G)|$ and $\Lambda_2 = |E(G)|$ are topological indices. If $x, y \in V(G)$ then the distance $d(x,y)$ between x and y is defined as the length of a minimum path connecting x and y. The Wiener index [6] is the first reported distance−based topological index and is defined as half sum of the distances between all the pairs of vertices in a molecular graph. In other words, the Wiener index is defined as follows:

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$$W(G) = \frac{1}{2}\sum_{x,y\in V(G)} d(x,y).$$

The reciprocal degree distance $RDD(G)$ of a graph G was proposed by Hua and Zhang [7] as

$$RDD(G) = \frac{1}{2}\sum_{x,y\in V(G)} \frac{d(x)+d(y)}{d(x,y)},$$

where $d(u)$ denotes the degree of the vertex u in G. The eccentricity of a vertex u is also defined as $\varepsilon(u) = \{d(x,y) \mid x\in V(G)\}$. The maximum eccentricity over all vertices of G is called the diameter of G and denoted by $d(G)$ and the minimum eccentricity among the vertices of G is called radius of G and denoted by $r(G)$.

The Harary index $H(G)$ of a graph G on n vertices is based on the concept of reciprocal distance and is defined the half-sum of the off–diagonal elements of the reciprocal distance matrix $RD(G)$:

$$H(G) = \frac{1}{2}\sum_{x,y\in V(G)} \frac{1}{d(x,y)},$$

where the ij-th entry of reciprocal distance matrix $RD(G)$ is

$$RD(G)_{ij} = \frac{1}{d(v_i,v_j)}.$$

In this paper, in the next section we introduce some properties of reciprocal distance index and then we compute this topological index for Grassmann graphs. Throughout this paper, our notation is standard and mainly taken from [3].

2. RESULTS AND DISCUSSION

The aim of this section is to study algebraic properties of reciprocal degree distance index and in the end of this section, we compute this topological index for Grassmann graphs.

Lemma 1. Let G be a regular graph of degree r, then

$$RDD(G)=2r \times H(G).$$

Proof. Since G is r-regular, then $d(u) = d(v) = r$ and so

$$RDD(G) = 2r\sum_{\{u,v\}\subseteq V(G)}\frac{1}{d(u,v)} = 2rH(G).$$

Let $d(u)$ denote the degree of vertex u in connected graph G. The first Zagreb index is defined by Gutman et al. as $M_1(G) = \sum_{uv\in E(G)}\frac{1}{d(u,v)}(d(u)+d(v))$. In the following proposition, we propose a bound for *RDD* via the first Zagreb index:

Proposition 2.

$$\frac{M_1(G)}{d(G)} \leq RDD(G) \leq M_1(G).$$

Proof. Note that for every pair of vertices x,y in $V(G)$, $1 \leq d(u,v) \leq d(G)$. This implies that

$$\frac{d(u)+d(v)}{d(G)} \leq \frac{d(u)+d(v)}{d(u,v)} \leq d(u)+d(v)$$

and so, the proof is completed.

A bijection σ on vertices set of graph G is named an automorphism of graph, if it preserves the edge set. In other word, σ is an automorphism if for every edge $e = uv$ of E then $\sigma(e) = \sigma(u)\sigma(v)$ is an edge of E. Let $Aut(G) = \{\alpha : V \to V, \alpha \text{ is bijection}\}$ then $Aut(G)$ under the composition of mappings forms a group. The automorphism group $Aut(G)$ acts transitively on V if for any vertices u and v in V there is $\alpha \in Aut(G)$ such that $\alpha(u) = v$. For given vertex u, let

$$D(u) = \sum_{v\in V(G)} d(u,v) \text{ and } T(u) = \sum_{v\in V(G)} d(u,v)^{-1}.$$

Then we have the following lemma:

Lemma 3. If G is a vertex–transitive graph, then for any pair (u,v) of vertices $D(u) = D(v)$ and $T(u) = T(v)$.

Proof. Since G is vertex–transitive, so there is an automorphism α in $Aut(G)$ such that $\alpha(u) = v$. Hence,

$$D(v) = \sum_{x\in V(G)} d(x,v) = \sum_{x\in V(G)} d(x, \alpha(u))$$
$$= \sum_{y\in V(G)} d(\alpha(y), \alpha(u)) = D(u).$$

The second claim can be obtained similarly.

Theorem 4. If G be a vertex–transitive graph on n vertices, then for a given vertex u,

$$W(G) = nD(u)/2 \text{ and } RDD(G) = nrT(u).$$

Proof. Similar to the proof of Lemma 3, it is sufficient to compute the $RDD(G)$ and so a similar result can be obtain for the Wiener index. Since G is vertex–transitive graph, G is r–regular for some integer r. So, the reciprocal degree distance is:

$$\begin{aligned} RDD(G) &= \sum_{\{u,v\}\subseteq V(G)} \frac{d(u)+d(v)}{d(u,v)} \\ &= 2r \times \sum_{\{u,v\}\subseteq V(G)} \frac{1}{d(u,v)} \\ &= r \times \sum_{u\in V(G)} T(u) = nrT(u). \end{aligned}$$

Consider a vector space V of dimension n over the finite field with $q = p^n$ elements. We denote a finite field of order $q = p^n$ by F_q where p is a prime number. A Grassmann graph $Gr(n,k)$ is a graph whose vertices are the k–subspaces of V and two vertices A and B of subspace of dimension k are adjacent if and only if $\dim(A\cap B) = k-1$ for $k \geq 2$, see [8–10]. The Grassmann graph is connected and the distance $d(A,B)$ between two vertices A,B in $Gr(n,k)$ is defined as the minimal number i such that there is a path of length i connecting A and B. We have the following distance formula:

$$d(A,B) = dim(A+B) - k = k - \dim(A\cap B).$$

In particular, the diameter of Grassmann graph $Gr(n,k)$ is $d = min\{k,n-k\}$. One can prove that this graph is distance–regular with intersection array

$$\{b_0, b_1, \ldots, b_{d-1}, c_1, c_2, \ldots, c_d\},$$

where

$$b_i = q^{2i+1} \frac{(q^{k-i}-1)(q^{n-k-i}-1)}{(q-1)^2} \text{ and } c_i = \frac{(q^i-1)^2}{(q-1)^2}.$$

Let $A, B \in V(Gr(n,k))$ where $\dim(A\cap B) = l$. Here, we introduce a path between A and B. To do this, suppose also $S = \{\alpha_1, \ldots, \alpha_l\}$ is a basis for $A\cap B$. By extending S to

$R=\{\alpha_1,...,\alpha_l,\beta_1,...,\beta_{k-l}\}$ and $T=\{\alpha_1,...,\alpha_l,\gamma_1,...,\gamma_{k-l}\}$, we achieve a basis for A and B, respectively. Let

$$\begin{aligned}&A_0=A,\\&A_1=\{\alpha_1,...,\alpha_l,\beta_{1,}...,\beta_{k-l},\gamma_1\},\\&\vdots\\&A_{k-l}=\{\alpha_1,...,\alpha_l,\beta_{1,}...,\beta_{k-l},\gamma_1,...,\gamma_{k-l}\}=B.\end{aligned}$$

One can easily that by this way, $d(A,B) = k - l$, see Figure 1. This implies that $D(A)=\sum_{B\in V(Gr(n,k))}d(A,B)=\sum_{\dim(A\cap B)=0}d(A,B)+...+\sum_{\dim(A\cap B)=k}d(A,B)$

$$=k\prod_{j=0}^{k-1}\frac{q^{n-j}-1}{q^{k-j}-1}+(k-1)\prod_{j=0}^{k-2}\frac{q^{n-j}-1}{q^{k-j}-1}+...+0. \tag{1}$$

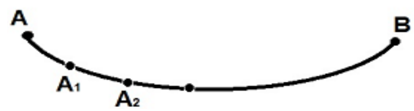

Figure 1.

Let,

$$\begin{bmatrix}n\\k\end{bmatrix}_q=\frac{q^n-q^i}{q^k-q^i}.$$

Let V be a vector space of dimension k over F_q and W be a fixed subspace of dimension m. Then the number of subspace of dimension i of V which intersect W in a space of dimension j is given by

$$q^{(i-j)(m-j)}\begin{bmatrix}k-m\\i-j\end{bmatrix}_q\begin{bmatrix}m\\j\end{bmatrix}_q.$$

This enables us, for example, to determine that the number of vertices at distance i from any fixed vertex of the graph $Gr(n,k)$ is

$$q^{i^2}\begin{bmatrix}k-n\\i\end{bmatrix}_q\begin{bmatrix}n\\i\end{bmatrix}_q.$$

In particular, since $Gr(n,k)$ is vertex-transitive, we can verify the following theorem:

Theorem 5. The Wiener index of Grassmann graph is:

$$W(Gr(n,k)) = \frac{1}{2}\begin{bmatrix} n \\ k \end{bmatrix}\left[k\prod_{j=0}^{k-1}\frac{q^{n-j}-1}{q^{k-j}-1} + (k-1)\prod_{j=0}^{k-2}\frac{q^{n-j}-1}{q^{k-j}-1} + \ldots + 0 \right].$$

Proof. Use Theorem 4 and Equation (1).

Corollary 6. The Harary index and the reciprocal degree distance of Grassmann graph are

$$H(Gr(n,k)) = \frac{1}{2}\begin{bmatrix} n \\ k \end{bmatrix}\left[\frac{1}{k}\prod_{j=0}^{k-1}\frac{q^{n-j}-1}{q^{k-j}-1} + (k-1)\prod_{j=0}^{k-2}\frac{q^{n-j}-1}{q^{k-j}-1} + \ldots + \prod_{j=0}^{1}\frac{q^{n-j}-1}{q^{k-j}-1} \right],$$

$$RDD(Gr(n,k)) = 4r\begin{bmatrix} n \\ k \end{bmatrix}\left[k\prod_{j=0}^{k-1}\frac{q^{n-j}-1}{q^{k-j}-1} + (k-1)\prod_{j=0}^{k-2}\frac{q^{n-j}-1}{q^{k-j}-1} + \ldots + 0 \right],$$

respectively.

Proof. Note that a Grassmann graph is *r*–regular where

$$r = \left(\frac{q^{k+1}-1}{q-1} - 1\right)\left(\frac{q^{n-k}-1}{q-1}\right).$$

Hence, by using Theorem 5, the proof is completed.

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