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The Generalized Wiener Polarity Index of Some Graph Operations

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ABSTRACT

Let G be a simple connected graph. The generalized polarity Wiener index of G is defined as the number of unordered pairs of vertices of G whose distance is k. Some formulas are obtained for computing the generalized polarity Wiener index of the Cartesian product and the tensor product of graphs in this article.

Keywords: Wiener index, Cartesian product, tensor product.

1. INTRODUCTION

Let G be a simple connected graph, with n vertices and m edges. The set $V(G)=\{v_1,...,v_n\}$ denotes the vertex set of G while $E(G)=\{e_1,...,e_n\}$ denotes it's edge set. When we mention $v_i = v_j$ it means the two vertices v_i and v_j are identical. The distance $d_G(v_i,v_j)$ between two vertices v_i and v_j in G is the length of a shortest path between the two vertices. The diameter of G is the greatest distance between two vertices of G, denoted by $D(G)$. The Wiener index $W(G)$ and the Wiener polarity index $W_P(G)$ are, respectively, defined as [1]

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$$W(G)=\sum_{\substack{1\le i<j\le n\\ v_i,v_j\in V(G)}} d_G(v_i,v_j)$$

$$W_P(G)=|\{(u,v)|d_G(u,v)=3,u,v\in V(G)\}|$$

by H. Wiener in the mid−20th century. Later in 1988, H. Hosoya [2] introduced the Hosoya(or wiener) polynomial

$$H(G,x)=\sum_{k\ge 1} d(G,k)x^k = \sum_{k=1}^{D(G)} d(G,k)x^k$$

where $d(G,k)$ denotes the number of unordered pairs of vertices whose distance is equal to k. Actually, $H(G,x)$ provided a generalized way of studying the wiener index and the wiener polarity index for $d(G,3)=W_p(G)$ and

$$\frac{dH(G,x)}{dx}|_{x=1}=W(G).$$

Motivated by the definition of $W_P(G)$, Ilić [3] recently defined $d(G,k)$ to be the generalized Wiener polarity index $W_k(G)$ for its significance. As a matter of fact, Gutman [4] showed that $d(G,k)$ is closely related with the first and the second Zagreb index M_1 and M_2. For example [4],

$$d(G,2)=\frac{1}{2}M_1-m-3n_{\square}$$

and

$$d(G,3)=M_2-M_1+m-4n_{\square}$$

where m is the number of edges, while $3n_{\square}$ and $4n_{\square}$ denotes the number of triangles and the number of quadrangles respectively. However, the relation between $d(G,k)$ and Zagreb indices has not yet been studied. The Cartesian product $G\times H$ of two graphs G and H is defined as:

$$E(G\times H)=\{(u_1,v_1)(u_2,v_2)|u_1u_2\in E(G)\ \ and\ \ v_1=v_2]\ or\ [u_1=u_2\ \ and\ \ v_1v_2\in E(H)\}$$

and the tensor product $G\otimes H$ of graphs G and H is defined as

$$E(G\otimes H)=\{(u_1,v_1)(u_2,v_2)|u_1u_2\in E(G)\ \ and\ \ v_1v_2\in E(H)\}.$$

More details about graph operations and generalized wiener polarity index can be found in articles [7, 8, 9].

This paper is organized as follows. In Section 2, we obtain some formulas for computing the generalized wiener polarity index of the cartesian product $G\times H$ of two simple connected graph G and H and of the tensor product of them in Section 3. Also, some corollaries of these formulas are given as applications in each section.

2. THE GENERALIZED WIENER INDEX OF THE CARTESIAN PRODUCT $G\times H$

Before we turn to our main theorem of this section, there is a lemma to introduce

Lemma 2.1. [5] Let G_1 and G_2 be two simple connected graphs, then

$$H(G_1+G_2,x)=2H(G_1,x)H(G_2,x)+|G_1|H(G_2,x)+|G_2|H(G_1,x).$$

Since $W_k(G)=0$ when $k>D(G)$, we only consider $k\le D(G)$ in this paper. Now we are ready to present our main theorem of the section

Theorem 2.1. Let G_1 and G_2 be two simple connected graphs, then

$$W_k(G_1\times G_2)=W_k(G_1)|V(G_2)|+W_k(G_2)|V(G_1)|+2\sum_{i=1}^{k-1}W_i(G_1)W_{k-i}(G_2)$$

Proof. According to Lemma 2.1 and direct computation, we have

$$\begin{aligned}W_k(G\times H)&=d(G_1+G_2,k)\\&=2\sum_{i=1}^{k-1}d(G_1,i)d(G_2,k-i)+|G_1|d(G_2,k)+|G_2|d(G_1,k)\\&=W_k(G_1)|V(G_2)|+W_k(G_2)|V(G_1)|+2\sum_{i=1}^{k-1}W_i(G_1)W_{k-i}(G_2)\end{aligned}$$

Thus the Theorem holds. □

As a direct application to Theorem 2.1, it follows that

Corollary 2.1. Let G be a simple connected graph, then

$$W_k(G\times G)=2W_k(G)|V(G)|+2\sum_{i=1}^{k-1}W_i(G)W_{k-i}(G)$$

The graphs $U=P_n\times C_m$ and $O=C_n\times C_m$ are usually called C_4-nanotube and C_4-nanotorus, where P_n denotes a path whose length is $n-1$ and C_m a cycle with m vertices. C_4–nanotube and C_4–nanotorus are studied widely in materialogy and quantum chemistry for their extraordinary thermal conductivity and mechanical and electrical properties. By applying Theorem 2.1 we obtain:

Corollary 2.2. Let $U=P_n\times C_m$ and $O=C_n\times C_m$, then

$$W_k(U) = 2kmn - mk^2, k < \min\{n-1, \left\lfloor \frac{n}{2} \right\rfloor\}$$

$$W_k(O) = 2kmn, k < \min\{\left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{m}{2} \right\rfloor\}$$

Proof. By Theorem 2.1 we have

$$\begin{aligned} W_k(U) &= W_k(P_n)|V(C_m)| + W_k(C_m)|V(P_n)| + 2\sum_{i=1}^{k-1} W_i(P_n)W_{k-i}(C_m) \\ &= (n-k)m + mn + 2\sum_{i=1}^{k-1}(n-i)m \\ &= (n-k)m + mn + (k-1)(2n-k)m \\ &= 2kmn - mk^2 \end{aligned}$$

and

$$\begin{aligned} W_k(O) &= W_k(C_n)|V(C_m)| + W_k(C_m)|V(C_n)| + 2\sum_{i=1}^{k-1} W_i(C_n)W_{k-i}(C_m) \\ &= 2mn + 2\sum_{i=1}^{k-1} mn \\ &= 2mn + 2(k-1)mn \\ &= 2kmn \end{aligned}$$

The result follows.

Though $W_k\left(P_n \times C_m\right)\left(\min\{n-1, \lfloor \frac{n}{2} \rfloor\} \le k \le \max\{n-1, \lfloor \frac{n}{2} \rfloor\}\right)$ and $W_k\left(C_n \times C_m\right)$ $\left(\min\{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{m}{2} \rfloor\} \le k \le \max\{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{m}{2} \rfloor\}\right)$ are not considered in Corollary 2.2, we can discussed them in a similar way by applying Theorem 2.1.

3. THE GENERALIZED WIENER INDEX OF THE TENSOR PRODUCT $G \otimes H$

In this section, we compute the generalized Wiener polarity index of the tensor product of graphs. A prepare work shall be introduced at first.

Lemma 3.1.[6] Let G_1 and G_2 be two simple connected graphs, and $u = (u_1, v_1)$, $v = (u_2, v_2) \in V(G_1) \times V(G_2)$, then $d_{G_1 \otimes G_2}(u, v) = \max\{d_{G_1}(u_1, u_2), d_{G_2}(v_1, v_2)\}$.

Now we are ready for

Theorem 3.1. Let G_1 and G_2 be two simple connected graphs, then

$$W_k(G_1 \otimes G_2) = W_k(G_1)|V(G_2)| + W_k(G_2)|V(G_1)| + 2W_k(G_1)\sum_{i=1}^{k-1} W_i(G_2)$$
$$+2W_k(G_2)\sum_{i=1}^{k} W_i(G_1)$$

Proof. Let $u=(u_1,v_1)$, $v=(u_2,v_2)\in V(G_1)\times V(G_2)$. According to Lemma 3.1, we have

$$d(u,v)=k \text{ if and only if } \begin{cases} d(u_1,u_2)=k, d(v_1,v_2)=0 \\ d(u_1,u_2)=0, d(v_1,v_2)=k \\ d(u_1,u_2)=k, d(v_1,v_2)<k \\ d(v_1,v_2)=k, d(u_1,u_2)<k \\ d(v_1,v_2)=k, d(u_1,u_2)=k \end{cases}$$

By summing up the numbers of five types of pairs of vertices above, we have

$$W_k(G_1 \times G_2) = |\{(u,v) \mid d(u_1,u_2)=k, v_1=v_2 \in V(G_2)\}|$$
$$+|\{(u,v) \mid d(v_1,v_2)=k, u_1=u_2 \in V(G_1)\}|$$
$$+\sum_{i=1}^{k-1}\Big|\{(u,v) \mid d(u_1,u_2)=k, d(v_1,v_2)=i\}|$$
$$+\sum_{i=1}^{k}\Big|\{(u,v) \mid d(v_1,v_2)=k, d(u_1,u_2)=i\}|$$
$$= W_k(G_1)|V(G_2)| + W_k(G_2)|V(G_1)| + 2W_k(G_1)\sum_{i=1}^{k-1} W_i(G_2)$$
$$+2W_k(G_2)\sum_{i=1}^{k} W_i(G_1)$$

Therefore, the theorem is established. □

A direct deduction of Theorem 3.1 is

Corollary 3.1. Let G be a simple connected graph, then

$$W_k(G \otimes G) = 2W_k(G)|V(G)| + 4W_k(G)\sum_{i=1}^{k-1} W_i(G) + 2W_k(G)^2$$

And as an application of Theorem 3.1 we have:

Corollary 3.2. Let $U = P_n \otimes C_m$ and $O = C_n \otimes C_m$, then

$$W_k(U) = 4kmn - 3mk^2, k < \min\{n-1, \left\lfloor \frac{n}{2} \right\rfloor\}$$

$$W_k(O) = 4kmn, k < \min\{\left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{m}{2} \right\rfloor\}$$

Proof. By Theorem 3.1 we have

$$\begin{aligned} W_k(U) &= W_k(P_n)\,|V(C_m)| + W_k(C_m)\,|V(P_n)| + 2W_k(P_n)\sum_{i=1}^{k-1} W_i(C_m) \\ &\quad + 2W_k\left(C_m\right)\sum_{i=1}^{k} W_i\left(P_n\right) \\ &= (n-k)m + mn + 2\left(n-k\right)\sum_{i=1}^{k-1} m + 2m\sum_{i=1}^{k}\left(n-i\right) \\ &= (n-k)m + mn + 2m\left(n-k\right)(k-1) + mk(2n-k-1) \\ &= 4kmn - 3mk^2 \end{aligned}$$

and

$$\begin{aligned} W_k(O) &= W_k(C_n)\left|V(C_m)\right| + W_k(C_m)\left|V(C_n)\right| + 2W_k(C_n)\sum_{i=1}^{k-1} W_i(C_m) \\ &\quad + 2W_k(C_m)\sum_{i=1}^{k} W_i(C_n) \\ &= 2mn + 2n\sum_{i=1}^{k-1} m + 2m\sum_{i=1}^{k} n \\ &= 4kmn \end{aligned}$$

The result follows. □

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