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On the General Sum–Connectivity Co–Index of Graphs

GUIFU SU [1,•] AND LAN XU [2]

[1] *Department of Mathematics, Beijing Institute of Technology, Beijing, 100081, P. R. China*
[2] *Department of Mathematics, Changji University, Xinjiang, 831100, P. R. China*

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ABSTRACT

In this paper, a new molecular-structure descriptor, the general sum–connectivity co–index $\overline{\chi_\alpha}$ is considered, which generalizes the first Zagreb co–index and the general sum–connectivity index of graph theory. We mainly explore the lower and upper bounds in terms of the order and size for this new invariant. Additionally, the Nordhaus–Gaddum–type result is also represented.

Keywords: general sum–connectivity co–index; first Zagreb co–index; lower and upper bounds.

1. INTRODUCTION

Throughout this paper all graphs will be considered simple, unless otherwise specified. That is to say, there is at most one edge joining any pair of vertices. Other terminology and notations will be introduced as it naturally occurs in the following and we use those not defined here [1]. Let $G=(V,E)$ be a simple graph, with vertex set V and edge set E, on $n=|V|$ vertices and $m=|E|$ edges.

The ***degree*** of a vertex $u \in V$ is denoted by $\deg_G(u)$, or $\deg(u)$ when no confusion is possible. The ***minimum*** and ***maximum*** degrees of G is denoted by $\delta(G)$ and $\Delta(G)$, respectively. As usual, a graph in which every vertex has equal degree k is said to be k-***regular***. The ***complement*** of G, denoted by $\overline{G}$, is a simple graph with vertex set $\overline{V}=V$, and edge set $\overline{E}$ in which two vertices u and v are adjacent if and only if they are not

•Corresponding author (Email: dfnh1983@126.com; gfs1983@126.com. Tel: 008610-137-1796-1438).

adjacent in G. For simplicity, we let $m = |E|$ amd $\overline{m} = |\overline{E}|$, hence $m + \overline{m} = \binom{n}{2}$ and the degree of the same vertex u in $\overline{G}$ is then given by $\deg_{\overline{G}}(u) = n - 1 - \deg_G(u)$, respectively. The first Zagreb index [2] was recently originally defined as follows:

$$M_1(G) = \sum_{u \in V(G)} [\deg(u)]^2$$

It can be also expressed as the sum over all edges of

$$G: M_1(G) = \sum_{uv \in E(G)} [\deg(u) + \deg(v)]$$

The first Zagreb index can be viewed as the contribution of pairs of adjacent vertices to additively weighted versions of Wiener numbers and polynomials [3]. Curiously enough, it turns out that analogous contribution of non–adjacent pairs of vertices must be taken into account when calculating the weighted Wiener polynomials of certain composite graphs [4]. Such quantity is said to be the first Zagreb co–index since the sums involved run over the edges of graph $\overline{G}$. The first Zagreb co–index of a graph G is more formally defined as [5]:

$$G: \overline{M_1}(G) = \sum_{uv \notin E(G)} [\deg(u) + \deg(v)]$$

This invariant was introduced in the hope of improving our ability to quantify the contribution of pairs of non–adjacent vertices to properties of graphs. We encourage the interested reader to [7] for some recent results on the extreme values of Zagreb co–indices over several classes of graphs. More recently, Zhou and his co-workers proposed a graph invariant, called the general sun–connectivity index [8]:

$$\chi_\alpha(G) = \sum_{uv \in E(G)} [\deg(u) + \deg(v)]^\alpha$$

Motivated by this information, in this paper, we focus also our attention to contributions from the pairs of non-adjacent vertices of graph G and introduce a new invariant, ***the general sum-connectivity co-index***, which is defined as:

$$\overline{\chi_\alpha}(G) = \sum_{uv \notin E(G)} [\deg(u) + \deg(v)]^\alpha$$

The reader should note that the general sum–connectivity co-index of G is not the general sum–connectivity index of $\overline{G}$, the defining sums run over all edges of $\overline{G}$, but the degrees are with respect to G. Note that for $\alpha = 1$ we obtain the first Zagreb co–index $\overline{M_1}(G)$. Thus the general sum–connectivity co–index generalizes the first Zagreb co–index and the general sum–connectivity index of chemical graph theory.

In Section 4, we will explore the lower and upper bounds in terms of order and size of graphs for general sum–connectivity co–index. The Nordhaus–Gaddum–type results for the general sum–connectivity co–index is also considered and will be viewed in Section 5.

2. CLOSED FORMULAE FOR SEVERAL FAMILIES OF GRAPHS

In this section, the general sum–connectivity index and co–index considered for several families of graphs can be given by closed formulae in terms of the number of vertices. Let K_n, C_n and P_n be the complete graph, the cycle graph and the path graph with order n. Let $K_{s,t}$ be the complete bipartite graph with s and t vertices in its two partite sets, and let Q_k denote the hypercube graph for $k > 2$ as usual, respectively.

1.1. Complete graph

(1) $\chi_\alpha(K_n) = 2^{\alpha-1} n(n-1)^{\alpha+1}$; (2) $\overline{\chi_\alpha}(K_n) = 0$.

1.2. Cycle graph

(1) $\chi_\alpha(C_n) = 2^{2\alpha} n$; (2) $\overline{\chi_\alpha}(C_n) = 2^{2\alpha-1} n(n-4)$.

1.3. Path graph

(1) $\chi_\alpha(P_n) = 2\cdot 3^\alpha + (n-3)2^{2\alpha}$; (2) $\overline{\chi_\alpha}(P_n) = 2^{2\alpha-1}(n-1)(n-2) + 2(n-1)3^\alpha + 2^\alpha$.

1.4. Complete bipartite graph

(1) $\chi_\alpha(K_{s,t}) = st(s+t)^\alpha$; (2) $\overline{\chi_\alpha}(K_{s,t}) = 2^{\alpha-1} s(s-1)t^\alpha + 2^{\alpha-1} t(t-1)s^\alpha$.

1.5. Hypercube graph

(1) $\chi_\alpha(Q_k) = 2^{\alpha+k-1} k^{\alpha+1}$; (2) $\overline{\chi_\alpha}(Q_k) = 2^{\alpha+k-1} k^\alpha (2^k - k - 1)$.

3. PRELIMINARY LEMMAS

To obtain our main results, we first give some lemmas as necessary preliminaries.

Lemma 3. 1. Let x_1, x_2, ..., x_k be k non–negative integers . Each of the following holds:

(1) If $\alpha > 1$, then $(x_1 + x_2 + \cdots + x_k)^\alpha \geq x_1^\alpha + x_2^\alpha + \cdots + x_k^\alpha$, with equality if and only if at most one $x_i \neq 0$.

(2) If $0 < \alpha < 1$, then $(x_1 + x_2 + \cdots + x_k)^\alpha \leq x_1^\alpha + x_2^\alpha + \cdots + x_k^\alpha$, with equality if and only if at most one $x_i \neq 0$.

Proof. By induction on k. We show firstly the case of $\alpha > 1$. It is trivial if $k = 1$. If $k = 2$, then $(x_1 + x_2)^\alpha \geq x_1^\alpha + x_2^\alpha$, since

$$\frac{x_1^{\alpha}+x_2^{\alpha}}{(x_1+x_2)^{\alpha}}=\left(\frac{x_1}{x_1+x_2}\right)^{\alpha}+\left(\frac{x_2}{x_1+x_2}\right)^{\alpha}\leq\left(\frac{x_1}{x_1+x_2}\right)+\left(\frac{x_2}{x_1+x_2}\right)=1$$

Assume that $(x_1+x_2+\cdots+x_t)^{\alpha}\geq x_1^{\alpha}+x_2^{\alpha}+\cdots+x_t^{\alpha}$. By the induction hypothesis,

$$(x_1+x_2+\cdots+x_{t+1})^{\alpha}\geq(x_1+x_2+\cdots+x_t)^{\alpha}+x_{t+1}^{\alpha}\geq x_1^{\alpha}+x_2^{\alpha}+\cdots+x_t^{\alpha}+x_{t+1}^{\alpha}$$

This completes the proof of the first statement. If $0<\alpha<1$, then

$$(x_1^{\alpha}+x_2^{\alpha}+\cdots+x_t^{\alpha})^{\frac{1}{\alpha}}\geq(x_1^{\alpha})^{\frac{1}{\alpha}}+(x_2^{\alpha})^{\frac{1}{\alpha}}+\cdots+(x_t^{\alpha})^{\frac{1}{\alpha}}=x_1+x_2+\cdots+x_t$$

i.e., $(x_1+x_2+\cdots+x_k)^{\alpha}\leq x_1^{\alpha}+x_2^{\alpha}+\cdots+x_k^{\alpha}$. Moreover, the equality is obtained if and only if there at most one $x_i\neq 0$.

We recall that if $\Phi''(x)\geq 0$ (≤ 0), for the real value function $\Phi(x)$ defined on an interval, then $\Phi(x)$ is a convex (concave) function. The fundamental discrete Jensen's inequalities say that:

Lemma 3. 2. (G. H. Hardy et al. [9]) Let C be a convex subset of a real vector space X, let $x_i\in C$ and $\sigma_i\geq 0$ ($i=1,2,\cdots,n$) with $\sum_{i=1}^{n}\sigma_i=1$. Then

(1) $\Phi\left(\sum_{i=1}^{k}\sigma_i x_i\right)\leq\sum_{i=1}^{k}\sigma_i\Phi(x_i)$ if $\Phi(x):C\to R$ is a convex function.

(2) $\Phi\left(\sum_{i=1}^{k}\sigma_i x_i\right)\geq\sum_{i=1}^{k}\sigma_i\Phi(x_i)$ if $\Phi(x):C\to R$ is a concave function.

The following result was obtained by K. Ch. Das and I. Gutman in a paper about Zagreb indices [5].

Lemma 3. 3. (K. Ch. Das and I. Gutman [5]) Let G be a simple graph with order n and size m. Then $\overline{M_1}(G)=2m(n-1)-M_1(G)$.

Ashrafi, Došlić and Hamzeh also gave the result of Lemma 3.3 in [6].

4. BOUNDS FOR THE GENERAL SUM–CONNECTIVITY CO–INDEX

Now we define a class Q_n of graphs of order n, which consists of simple graphs, each of which satisfies $\deg(u)+\deg(v)=c$ for each edge $e=uv$ and some constant c. In particular, the complete graph K_n and the complete bipartite graph $K_{s,t}$ are belong to Q_n.

Lemma 4. 1. Let G be a simple graph with order n and size m. Each of the following holds:

(1) If $0<\alpha<1$, then $\overline{\chi_\alpha}(G) \le \overline{M_1}(G)^\alpha \cdot \overline{m}^{1-\alpha}$, the upper bound attains on $H_1 \in Q_n$.

(2) If $\alpha>1$ or $\alpha<0$, then $\overline{\chi_\alpha}(G) \ge \overline{M_1}(G)^\alpha \cdot \overline{m}^{1-\alpha}$, and the lower bound attains on $H_2 \in Q_n$

Proof. Let $\Phi(x)=x^\alpha$, where $x \ge 0$ and $\alpha \in R-\{0,1\}$. It is obvious that the second derivative $\Phi''(x)=\alpha(\alpha-1)x^{\alpha-2}$, then $\Phi''(x)>0$ if $\alpha>1$ or $\alpha<0$ and $\Phi''(x)<0$ if $0<\alpha<1$, i.e., $\Phi(x)$ is a convex function if $\alpha>1$ or $\alpha<0$ and is concave one otherwise. If $0<\alpha<1$, in view of Lemma 3.2, we obtain

$$\left[\frac{1}{m}\overline{M_1}(G)\right]^\alpha = \left[\frac{1}{m}\sum_{uv\notin E(G)}[\deg(u)+\deg(v)]\right]^\alpha \ge \frac{1}{m}\sum_{uv\notin E(G)}[\deg(u)+\deg(v)]^\alpha = \frac{1}{m}\overline{\chi_\alpha}(G)$$

This states that $\overline{\chi_\alpha}(G) \le \overline{M_1}(G)^\alpha \cdot \overline{m}^{1-\alpha}$, and the upper bound attains if and only if $\deg(u)+\deg(v)$ is a constant for each edge uv, which implies that $G=H_1$ is a graph in Q_n By analogous arguments we can complete the proof of the rest part.

In view of Lemma 3.3, we immediately have the following.

Corollary 4. 2. Let G be a simple graph with order n and size m. Each of the following holds:

(1) If $0<\alpha<1$, then $\overline{\chi_\alpha}(G) \le (2mn-2m-M_1(G))^\alpha \cdot \overline{m}^{1-\alpha}$, the upper bound attains on $H_1 \in Q_n$.

(2) If $\alpha>1$ or $\alpha<0$, then $\overline{\chi_\alpha}(G) \ge (2mn-2m-M_1(G))^\alpha \cdot \overline{m}^{1-\alpha}$, and the lower bound attains on $H_2 \in Q_n$.

The authors in [10, 11] presented some lower and upper bounds for the first Zagreb index, from which and Corollary 4.2, we may deduce some bounds for the general sum–connectivity co–index of graphs. We present some examples:

(I) Let G be a simple graph with order n and size $m \ge 1$. Then $M_1(G) \le m\left[\frac{2m}{n-1}+n-2\right]$, with equality if and only if $G=K_n$, S_n or $\overline{K_{1,n-1}} = K_1 \bigcup K_{n-1}$, see [10], and thus if $\alpha>1$, we have

$$\overline{\chi}_\alpha(G) \ge m^\alpha \cdot \overline{m}^{1-\alpha}\left[n-\frac{2m}{n-1}\right]^\alpha$$

with equality if and only if $G = K_n$ or $\overline{K_{1,n-1}} = K_1 \cup K_{n-1}$.

(II) Let G be a simple graph with order n and size $m \ge 1$. Then for any real number $0<\alpha<1$, we have that

$$\overline{\chi}_\alpha(G) \le 2^\alpha \cdot m^\alpha \cdot \overline{m}^{1-\alpha}\left[n-1-\frac{2m}{n}\right]^\alpha$$

since $M_1(G) = \sum_{u\in V(G)} [\deg(u)]^2 \ge \frac{1}{n}\left[\sum_{u\in V(G)} \deg(u)\right]^2 = \frac{4m^2}{n}$.

(III) Let G be a simple graph with order n, size $m \ge 1$, maximum degree Δ and minimum degree δ. Then, see [11] $M_1(G) \le \frac{4m^2 + 2m(n-1)(\Delta-\delta)}{n+\Delta-\delta}$, with equality if and only if G is a regular graph or $S_{\delta,n-\delta}$ (called a double star obtained from S_δ and $S_{n-\delta}$ by connecting the center of S_δ with that of $S_{n-\delta}$) or $\overline{K_{n-\delta-1}} \cup K_{\Delta+1}$. Thus for $\alpha > 1$ we have

$$\overline{\chi}_\alpha(G) \ge 2^\alpha \cdot m^\alpha \cdot \overline{m}^{1-\alpha}\left[\frac{n(n-1)-2m}{n+\Delta-\delta}\right]^\alpha$$

with equality if and only if G is a regular graph or $S_{\delta,n-\delta}$.

Note that the lower bound is best possible. Let G be a k–regular, and for $\alpha > 1$, we let $G(n,\Delta,\delta) = 2^\alpha \cdot m^\alpha \cdot \overline{m}^{1-\alpha}\left[\frac{n(n-1)-2m}{n+\Delta-\delta}\right]^\alpha$, then the value of the general sum–connectivity co–index of k-regular graphs can be written

$$\overline{\chi}_\alpha(G) = \sum_{uv\notin E(G)} [\deg(u)+\deg(v)]^\alpha = \left[\frac{n(n-1)}{2} - \frac{kn}{2}\right]\cdot (kn)^\alpha = 2^{\alpha-1} n(n-k-1) = G(n,k,k).$$

Theorem 4. 3. Let G be a simple graph with order $n \ge 2$. Each of the following holds:

(1) If $0<\alpha<1$, then $\overline{\chi}_\alpha(G) \ge \overline{M_1}(G)^\alpha$, the lower bound attains either on $G = \overline{K_2} \cup K_{n-2}$ or $G = K_n$.

(2) If $\alpha < 0$, then $\overline{\chi}_\alpha(G) \le 2^{\alpha-1} n(n-2)$, the upper bound attains uniquely on $\frac{n}{2}K_2$.

Proof. If $0<\alpha<1$, then by Lemma 3.1, we obtain

$$\overline{\chi_\alpha}(G) = \sum_{uv \notin E(G)} [\deg(u) + \deg(v)]^\alpha \geq \left[\sum_{uv \notin E(G)} \deg(u) + \deg(v) \right]^\alpha = \overline{M_1}(G)^\alpha$$

The lower bound attains if and only if $|\overline{E}| \leq 1$, this implies that $G = \overline{K_2} \cup K_{n-2}$ or $G = K_n$. If $\alpha < 0$, we obtain

$$\overline{\chi_\alpha}(G) = \sum_{uv \notin E(G)} [\deg(u) + \deg(v)]^\alpha \leq \sum_{uv \notin E(G)} (2\delta)^\alpha \leq \frac{n\Delta(\overline{G})}{2} (2\delta)^\alpha \leq 2^{\alpha-1} n(n-2)$$

The upper bound attains if and only if $\overline{G}$ is regular and $\Delta(\overline{G})\delta^\alpha = n-2$, this implies that $G = \frac{n}{2} K_2$.

Theorem 4.4. Let G be a triangle-free graph with order n and size $m \geq 1$. Each of the following holds:

(1) If $\alpha > 0$, then $\overline{\chi_\alpha}(G) \leq 2^{-1}(n-1)n^{\alpha+1} - mn^\alpha$, the upper bound attains if and only if $\overline{G}$ is a complete bipartite graph.

(2) If $\alpha < 0$, then $\overline{\chi_\alpha}(G) \geq 2^{-1}(n-1)n^{\alpha+1} - mn^\alpha$, the lower bound attains if and only if $\overline{G}$ is a complete bipartite graph.

Proof. Let $\Phi(x) = x^\alpha$ be a function defined as previous, the derivative $\Phi'(x) = \alpha x^{\alpha-1}$, then $\Phi'(x) > 0$ if $\alpha > 0$ and $\Phi'(x) < 0$ if $\alpha < 0$, i.e., $\Phi(x)$ is an increasing function if $\alpha > 0$ and is a decreasing one if $\alpha < 0$. It is seen that $\deg(u) + \deg(v) \leq n$ holds for each edge uv of graph $\overline{G}$, so we obtain if $\alpha > 0$

$$\overline{\chi_\alpha}(G) = \sum_{uv \notin E(\overline{G})} [\deg(u) + \deg(v)]^\alpha \leq \sum_{uv \notin E(G)} n^\alpha = \overline{m}n^\alpha = 2^{-1}(n-1)n^{\alpha+1} - mn^\alpha$$

the upper bound attains if and only if $\deg(u) + \deg(v) = n$ for each edge of graph $\overline{G}$, this implies that $\overline{G}$ is a complete bipartite graph. By analogous argument, we can complete the rest proof.

From Theorem 4.4, we have the following:

Corollary 4.5. Let T be a tree with order $n > 2$. Each of the following holds:

(1) If $\alpha > 0$, then $\overline{\chi_\alpha}(T) \leq 2^{-1}(n-1)(n-2)n^\alpha$, the upper bound attains uniquely on $K_{1,n-1}$.

(2) If $\alpha < 0$, then $\overline{\chi_\alpha}(T) \geq 2^{-1}(n-1)(n-2)n^\alpha$, the lower bound attains uniquely on $K_{1,n-1}$.

Corollary 4.5 implies the following result, which characterizes trees with extremal general sum–connectivity co–index.

Corollary 4.6. Among all trees with order $n>2$, the star $K_{1,n-1}$ is the unique extremal structure with maximum general sum–connectivity co-index for $\alpha>0$, and with minimal general sum–connectivity co-index for $\alpha<0$.

5. NORDHAUS–GADDUM–TYPE RELATION FOR $\overline{\chi_\alpha}(G)$

Let I be an invariant of G, we denote by $\overline{I}$ the same invariant but in $\overline{G}$. Nordhaus–Gaddum–type relations for the graph invariant I are the inequalities of the following form:

$$L_1(n)\le I+\overline{I}\le U_1(n) \text{ and } L_2(n)\le I\cdot\overline{I}\le U_2(n),$$

where $L_1(n)$ and $L_2(n)$ are the lower bounding functions of the order n, and $U_1(n)$ and $U_2(n)$ upper bounding functions of the order n. These types of relations are named after Nordhaus and Gaddum [12], who were the first authors to give such relations, namely:

$$2\sqrt{n}\le\chi+\overline{\chi}\le n+1 \text{ and } n\le\chi\cdot\overline{\chi}\le\left\lfloor\left(\frac{n+1}{2}\right)\right\rfloor^2,$$

where χ is the chromatic number of a graph. The extremal graphs for the inequalities were characterized by Finck may be found in [13]. Since then many graph theorists have been interested in finding such relations for various graph invariants.

In this section, we will give Nordhaus–Gaddum–type result for the general sum–connectivity co–index in terms of the number of vertices of the graph.

Theorem 5.1. (Zhang and Wu [14]) Let G be a simple graph with order n. The following hold:

(1) If $\alpha>1$, then $2^{1-\alpha}n(n-1)^\alpha\le M_\alpha(G)+M_\alpha(\overline{G})\le n(n-1)^\alpha$, the upper bound attains uniquely on $G=K_n$, and the lower bound attains uniquely on $\frac{n-1}{2}$–regular graphs.

(2) If $0<\alpha<1$, then $n(n-1)^\alpha\le M_\alpha(G)+M_\alpha(\overline{G})\le 2^{1-\alpha}n(n-1)^\alpha$, the upper bound attains uniquely on $\frac{n-1}{2}$-regular graphs, and the lower bound attains uniquely on $G=K_n$.

(3) If $\alpha<0$, then $2^{1-\alpha}n(n-1)^\alpha\le M_\alpha(G)+M_\alpha(\overline{G})\le n[1+(n-1)^\alpha]$, the upper bound attains on the graph H_n obtained from K_n by deleting a perfect matching (n is even), and the lower bound attains uniquely on $\frac{n-1}{2}$–regular graphs.

From Lemma 3.3 and Theorem 5.1, we obtain the following.

Corollary 5.2. Let G be a simple graph with order n. Then $0 \le \overline{M_1}(G) + \overline{M_1}(\overline{G}) \le 2^{-1}n(n-1)^2$, the lower bound attains on $G = K_n$, and the upper bound attains uniquely on $\frac{n-1}{2}$–regular graphs.

Theorem 5.3. Let G be a simple graph with order n. The following hold:

(1) If $\alpha > 1$, then $0 \le \overline{\chi_\alpha}(G) + \overline{\chi_\alpha}(\overline{G}) \le 2^{1-\alpha} n(n-1)^{1+\alpha}$, the lower bound attains either on $G = K_n$ or $G = \overline{K_n}$, and the upper bound attains uniquely on $\frac{n-1}{2}$-regular graphs.

(2) If $\alpha < 0$, then $0 \le \overline{\chi_\alpha}(G) + \overline{\chi_\alpha}(\overline{G}) \le 2^{\alpha} n(n-2)$, the lower bound attains uniquely on $\frac{n}{2}K_2$, and the upper bound attains either on $G = K_n$ or $G = \overline{K_n}$.

Proof. (1) If $\alpha > 1$, $\Phi(x) = x^\alpha$, defined as in Theorem 4.1, is a strictly convex function, then we have by Corollary 5.2

$$\begin{aligned}\overline{\chi_\alpha}(G) + \overline{\chi_\alpha}(\overline{G}) &= \sum_{uv \notin E(G)} [\deg_G(u) + \deg_G(v)]^\alpha + \sum_{uv \notin E(\overline{G})} [\deg_{\overline{G}}(u) + \deg_{\overline{G}}(v)]^\alpha \\ &\ge (\overline{m} + m)\left[\frac{\sum_{uv \notin E(G)} [\deg_G(u) + \deg_G(v)] + \sum_{uv \notin E(\overline{G})} [\deg_{\overline{G}}(u) + \deg_{\overline{G}}(v)]}{\overline{m} + m}\right]^\alpha \\ &= (\overline{m} + m)^{1-\alpha}\left[\overline{M_1}(G) + \overline{M_1}(\overline{G})\right]^\alpha \ge 0\end{aligned}$$

with equality if and only if either $\deg_G(u) + \deg_G(v) = n-1$ for each edge uv or there is no edge in graph G. This implies that $G = K_n$ or $G = \overline{K_n}$. On the other hand,

$$\begin{aligned}\overline{\chi_\alpha}(G) + \overline{\chi_\alpha}(\overline{G}) &= \sum_{uv \notin E(G)} [\deg_G(u) + \deg_G(v)]^\alpha + \sum_{uv \notin E(\overline{G})} [\deg_{\overline{G}}(u) + \deg_{\overline{G}}(v)]^\alpha \\ &\le \overline{m}\left[\frac{n-1}{2} + \frac{n-1}{2}\right]^\alpha + m\left[\frac{n-1}{2} + \frac{n-1}{2}\right]^\alpha = (\overline{m} + m)(n-1)^\alpha = 2^{-1}n(n-1)^{\alpha+1}\end{aligned}$$

Note that the upper bound is best possible. In fact, for any $n = 4k+1$, $k \ge 1$, there exists a graph G_n with G_n and $\overline{G_n}$ are $\frac{n-1}{2}$-regular. Then G_n is a graph whose $\overline{\chi_\alpha}(G) + \overline{\chi_\alpha}(\overline{G})$ attains the upper bound.

(2) If $\alpha < 0$, by Theorem 4.3, we obtain that

$$\overline{\chi_\alpha}(G) + \overline{\chi_\alpha}(\overline{G}) \le 2^{\alpha-1} n(n-2) + 2^{\alpha-1} n(n-2) = 2^\alpha n(n-2)$$

the upper bound attains uniquely on $\frac{n}{2}K_2$. By analogous arguments as (1), we obtain that $\overline{\chi_\alpha}(G)+\overline{\chi_\alpha}(\overline{G})\geq 0$ for $\alpha<0$, and the lower bound attains uniquely on $G=K_n$ or $G=\overline{K_n}$. This completes the proof.

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REFRENCES

1. J. A. Bondy and U. S. R. Murty, Graph Theory with Applications, North–Holland, New York, 1976.
2. S. Nikolić, G. Kovaćević, A. Milićević and N. Trinajstić, The Zagreb indices 30 years after, *Croat. Chem. Acta* **76** (2003) 113–124.
3. D. J. Klein, T. Došlić and D. Bonchev, Vertex–weightings for distance moments and thorny graphs, *Discrete Appl. Math.* **155** (2007) 2294–2302.
4. T. Došlić, Vertex–Weighted Wiener Polynomials for Composite Graphs, *Ars Math. Contemp*. **1** (2008) 66–80.
5. A. R. Ashrafi, T. Došlić and A. Hamzeh, The Zagreb co–indices of graph operations, *Discrete Appl. Math.*, **158** (2010) 1571- 1578.
6. K. Ch. Das and I. Gutman, Some properties of the second Zagreb index, *MATCH Commun. Math. Comput. Chem*. **52** (2004) 103–112.
7. A. R. Ashrafi, T. Došlić and A. Hamzeh, Extremal graphs with respect to the Zagreb coindices, *MATCH Commun. Math. Comput. Chem*. **65** (2011) 85–92.
8. B. Zhou and N. Trinajstić, On general sum–connectivity index, *J. Math. Chem* **47** (2010) 210–218.
9. G. H. Hardy, J. E. Littlewood and G. Polya, Inequalities, in: Cambridge Mathematical Library Series, Cambridge University Press,1967.
10. D. de Caen, An upper bound on the sum of squares of degrees in a graph, *Discrete Math.*, **185** (1998) 245–248.
11. J. Li and Y. Pan, de Caen's inequality and bounds on the largest Laplacian eigenvalue of a graph, *Linear Algebra Appl*. **328** (2001) 153–160.
12. E. A. Nordhaus, J. W. Gaddum, On complementary graphs, *Amer. Math. Monthly* **63** (1956) 175–177.
13. S. J. Xu, Some parameters of graph and its complement. Theory of Graphs (Proc. Coll. Tihany), (1968) 99–113.
14. L. Zhang and B. Wu, The Nordhaus–Gaddum–type inequalities for some chemical indices, *MATCH Commun. Math. Comput. Chem*. **54** (2005) 189–194.