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Sharp Bounds on the PI Spectral Radius

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ABSTRACT

In this paper some upper and lower bounds for the greatest eigenvalues of the PI and vertex PI matrices of a graph G are obtained. Those graphs for which these bounds are best possible are characterized.

Keywords: PI Matrix, PI Energy, PI Spectral Radius.

1. INTRODUCTION

Let G be a connected simple graph with vertex and edge sets V(G) and E(G), respectively. As usual, the distance between the vertices u and v of G is denoted by $d_G(u,v)$ (d(u,v) for short) and it is defined as the number of edges in a minimal path connecting u and v. The diameter of G is the length of a longest shortest path of G denoted by diam(G). Suppose e= uv $\in$ E(G) and w $\in$ V(G). Define d(e,w) = Min{d(u,w),d(v,w)}. If f = ab $\in$ E(G) then f is said to be parallel with e and we write f || e, if d(e,a) = d(e,b). It is easy to see that the parallelism is not symmetric.

The edge PI index (PI index as short) of a graph G is defined as $PI(G) = \Sigma_{e=uv}[m_u(e) + m_v(e)]$, where $m_u(e)$ is the number of edges lying closer to u than to v and $m_v(e)$ is defined analogously. This topological index was introduced by Padmakar Khadikar [10,11]. The mathematical properties of this new index can be found in recent papers, [1,4,5,7,8,19]. There is a vertex version of this new index, named vertex PI index proposed very recently in [12]. It is defined as $PI_v(G) = \Sigma_{e=uv}[n_u(e) + n_v(e)]$, where $n_u(e)$ is the vertex contribution of the edge e and defined as the number of vertices lying closer to the vertex u than the vertex v and $n_v(e)$ is defined analogously, see [2,6,12−17] for mathematical properties of this new topological index.

Suppose G is a graph with adjacency matrix A(G) and λ is an eigenvalue of A(G). It is convenient to name λ an eigenvalue of G. The maximum degree of G is denoted by $\Delta(G)$. If

$V(G) = \{v_1, v_2, \ldots, v_n\}$ then the vertex PI–matrix of G, VPIM(G), is defined so that its (i,j)–entry, a_{ij}, is equal to

$$a_{ij} = \begin{cases} n_{v_i}(e) + n_{v_j}(e) & e = v_i v_j \\ 0 & Otherwise \end{cases}.$$

The PI-matrix of G, PIM(G), is defined analogously. Since the vertex PI–matrix is symmetric, all its eigenvalues δ_i, i = 1, 2, . . . , n, are real and can be labeled so that $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$. The eigenvalues of VPIM(G) are said to be the vertex PI–eigenvalues of G and the VPIM–spectrum of G is denoted by VPI–Spec(G).

The aim of this paper is to extend some results of Indulal [9]. Our notations are standard and taken mainly from [3].

2. PRELIMINARY RESULTS

The considerations in the subsequent sections are based on the applications of the following definitions.

Definition 1. Let G be a graph, $V(G) = \{v_1, v_2, ..., v_n\}$ and P = VPIM(G). Then the vertex PI–degree of v_i, P_i, is defined as $P_i = \sum_{j=1}^{n} a_{ij}$. Moreover, if $\{P_1, P_2, ..., P_n\}$ is the vertex PI–degree sequence of G then the sequence $T_1, T_2, \ldots, T_n$, where $T_i = \sum_{j=1}^{n} a_{ij} P_j$ is called the second vertex PI–degree of G. G is said to be k-vertex PI regular if $P_i = k$, $1 \leq i \leq n$. G is called pseudo k–vertex PI–degree regular if $\frac{T_i}{P_i} = k$, for all i.

A lot of theorems in algebraic graph theory can be extended to PI and vertex PI matrices of graphs. Before going to the main results of this paper, we calculate the PI and vertex PI matrices of some well-known graphs.

Example 1. Consider the complete graph K_n. Then $VPIM(K_n) = 2A(K_n)$.

Example 2. Suppose C_n denotes the cycle graph of length n. If n is even then $VPIM(C_n) = nA(C_n)$ and if n is odd then $VPIM(C_n) = (n-1)A(C_n)$.

Example 3. A k-regular graph G is said to be strongly regular with parameters (v,k,r,s) if |v(G)| = v, any two adjacent vertices of G have exactly r common neighbors and any two non-adjacent vertices of G have exactly s common neighbors. In this case, we denote G by Srg(v,k,r,s). It is well-known that if G is strongly regular with parameters (v,k,r,s) then r > 0 and moreover, strongly regular graphs have diameter 2. In [6], the authors proved that in a strongly regular graph G= Srg(v,k,r,s), $n_u(e) = n_v(e) = k - r$, for every edge e = uv. Therefore, VPIM(Srg(v,k,r,s)) = 2(k–r)A(Srg(v,k,r,s)).

Theorem 1. Suppose G is a k–PI regular graph then

a) k is an eigenvalue of VPIM(G),

b) The multiplicity of k is one,

c) For each eigenvalue δ, $|\delta| \le k$.

Proof. The proof is similar to [3, Proposition 3.1] and so it is omitted. ■

Theorem 2. Suppose G is a graph, d = diam(G) and e = e(G) is the number of distinct eigenvalues of G. Then $e \ge d + 1$. In particular, e(G) = 2 if and only if G is a complete graph.

Proof. It is clear that $2 \le n_u(e) + n_v(e) \le n$. Define $VPIM(G)^k = [d_{ij}^{(k)}]$. Then by induction on k, one can see that $2^k a_{ij}^{(k)} \le d_{ij}^{(k)} \le n^k a_{ij}^{(k)}$, where $a_{ij}^{(k)}$ is the number of walks of length k connecting vertices v_i and v_j. Therefore, $d_{ij}^{(k)} = 0$ if and only if $a_{ij}^{(k)}=0$. Now a similar argument as [3,Theorem 3.13] completes the proof. ■

Lemma 1. Let G be n-vertex graph and $VPI - Spec(G) = \{\delta_1, \dots, \delta_n\}$. Then $8m \le \sum_{i=1}^n \delta_i^2 \le 2mn^2$ and $24t \le \sum_{i=1}^n \delta_i^3 \le 6tn^3$, where t is the number of triangles in G.

Proof. Suppose $VPIM(G)^k = [d_{ij}^{(k)}]$. So, $2^k a_{ij}^{(k)} \le d_{ij}^{(k)} \le n^k a_{ij}^{(k)}$. If $VPI - Spec(G) = \{\lambda_1, \dots, \lambda_n\}$ then $Spec\ (VPIM(G))^2 = \{\lambda_1^2, \dots, \lambda_n^2\}$. Thus, $Tr(VPIM(G))^2 = \delta_1^2 + \cdots + \delta_n^2$ and so, $8m = \sum_{i=1}^n 2^2 a_{ii}^{(2)} \le \sum_{i=1}^n \delta_i^2 = \sum_{i=1}^n d_{ii}^2 \le \sum_{i=1}^n n^2 a_{ii}^{(2)} = n^2 \sum_{i=1}^n \lambda_i^2 = 2n^2 m$. We now assume that t_i denotes the number of triangles with vertex at v_i. Since, $Tr(VPIM(G))^3 = \delta_1^3 + \cdots + \delta_n^3$, $8t_i = 2^3 a_{ii}^{(3)} \le d_{ii}^{(3)} \le n^2 a_{ii}^{(3)} = n^3 t_i$. This implies that $\sum_{i=1}^n 8t_i \le \sum_{i=1}^n d_{ii}^3 = \sum_{i=1}^n \delta_i^3 \le \sum_{i=1}^n n^3 t_i$. Therefore, $48t \le \sum_{i=1}^n \delta_i^3 \le 6n^3 t$, where t is number of triangle, as desired. ■

Theorem 3. Let G be a connected graph with PI_v Spectrum $\delta_1 \ge \cdots \ge \delta_n$. Then $\chi(G) \ge 1 - \frac{\delta_1}{\delta_n}$.

Proof. The proof is similar to [3, Theorem 3.18] and so it is omitted. ■

3. MAIN RESULTS

In the following discussions G is always a simple connected graph with P as a vertex PI_v matrix. *We* also used the *following lemma***,** which gives a bound on the eigenvalues of G.

Lemma 2. Let G be n-vertex graph. Then for every PI_v eigenvalue δ of G, $|\delta| \leq n\Delta(G)$.

Proof. Let $X = [x_1, x_2, \ldots, x_n]^t$ be eigenvector corresponding to δ. Thus $MX = \delta X$. If $x_m = \mathrm{Max}\{x_1, x_2, \ldots, x_n\}$ then $\sum_{j=1}^{n} p_{mj} x_j = \lambda$. So, $|\lambda|.|x_m| \leq \sum_{j=1}^{n} p_{mj}|x_j| \leq n\Delta(G)|x_m|$. This implies that $|\lambda| \leq n\Delta(G)$. ■

Lemma 3. $T_1 + T_2 + \cdots + T_n = P_1^2 + P_2^2 + \cdots + P_n^2$.

Proof. By definition $P_i = \sum_{i=1}^{n} a_{ij}$ and $T_i = \sum_{j=1}^{n} a_{ij} P_j$. Now

$$\begin{aligned} T_1 + T_2 + \cdots + T_n &= [1,1,\ldots,1](P[P_1, P_2, \ldots, P_n]^t) \\ &= ([1,1,\ldots,1]P)[P_1, P_2, \ldots, P_n]^t \\ &= P_1^2 + P_2^2 + \cdots + P_n^2. \end{aligned}$$

The last equality is follows from the associativity of matrix multiplication. ■

Theorem 4. Let G be a graph. Then $\delta_1 \geq \frac{2PI_v(G)}{n}$ and the equality holds if and only if G is vertex PI regular.

Proof. Let $x = \frac{1}{\sqrt{n}}(1,1,\ldots,1)$ be a unit P–vector. Then by Raleigh principle, applied to the vertex PI– matrix P of G, we get $\delta_1 \geq \frac{xPx^t}{xx^t} = \frac{\frac{1}{\sqrt{n}}[P_1,P_2,\ldots,P_n]\frac{1}{\sqrt{n}}[1,1,\ldots,1]^t}{1} = \frac{1}{n}\sum_{i=1}^{n} P_i = \frac{2PI_v}{n}$. Suppose G is vertex PI–regular. Then the sum of each row of P is a constant, say k and $2PI_v = nk$. By Perron–Frobenius theorem, a real square matrix with positive entries has a unique largest real eigenvalue and that the corresponding eigenvector has strictly positive components [3, Theorem 0.2]. Apply this theorem to prove that k is the simple and it is the greatest eigenvalue of P. Thus $\delta_1 = k = \frac{nk}{n} = \frac{2PI_v}{n}$ and hence equality holds. Conversely if equality holds, then x is the eigenvector corresponding to δ_1 and hence $Px = \delta_1 x$. This then gives $P_i = \delta_1$ for all i. Since P_i is an integer it follows that G is vertex PI regular. Hence the theorem is complete. ■

Theorem 5. Let G be a graph with vertex PI degree sequence $\{P_1, P_2, \ldots, P_n\}$. Then

$$\delta_1 \geq \sqrt{\frac{{P_1}^2 + {P_2}^2 + \ldots + {P_n}^2}{n}}$$

with equality if and only if G is vertex PI–regular.

Proof. Let P be the vertex PI matrix of G and $X = (x_1, x_2, \dots, x_n)$ be the unit positive eigenvector of P corresponding to δ_1. Take $C = \frac{1}{\sqrt{n}}(1,1,\dots,1)$. Then C is a unit positive vector. So we have $PX = \delta_1 X \Rightarrow P^2X = \delta_1 PX = {\delta_1}^2 X \Rightarrow X^tP^2X = {\delta_1}^2$. Therefore $\delta_1 = \sqrt{X^tP^2X} \geq \sqrt{C^tP^2C}$. Now $CP = \frac{1}{\sqrt{n}}(1,1,\dots,1)P = \frac{1}{\sqrt{p}}(P_1, P_2, \dots, P_n)$. Hence $X^tP^2X = PX(PX)^t = \frac{\sum_{i=1}^n P_i^2}{n}$ thus $\delta_1 \geq \sqrt{\frac{\sum_{i=1}^n P_i^2}{n}}$ and hence the inequality holds. Assume that G is vertex PI regular. Then $P_i = k$ for all i and hence by the Perron–Frobenius theorem, k is the simple and it is the greatest eigenvalue of P. But then

$$\delta_1 = k = \sqrt{\frac{nk^2}{n}} = \sqrt{\frac{\sum_{i=1}^n {P_i}^2}{n}}$$

and hence equality holds. Conversely if equality holds, then C is the eigenvector corresponding to δ_1. Then as in the proof of Theorem 1, G is vertex PI–regular. ■

Theorem 6. Let G be a graph with vertex PI–degree sequence $\{P_1, P_2, \dots, P_n\}$ and second vertex PI degree sequence $\{T_1, T_2, \dots, T_n\}$. Then $\delta_1 \geq \sqrt{\frac{{T_1}^2 + {T_2}^2 + \dots + {T_n}^2}{{P_1}^2 + {P_2}^2 + \dots + {P_n}^2}}$. Equality holds if and only if G is pseudo vertex PI regular.

Proof. Let P be the vertex PI–matrix of G and $X = (x_1, x_2, \dots, x_n)$ be the unit positive eigenvector of P corresponding to δ_1. Take $C = \frac{1}{\sqrt{\sum_{i=1}^n P_i^2}}(P_1, P_2, \dots, P_n)$. Then C is a unit positive vector. So we have $PX = \delta_1 X \Rightarrow P^2X = \delta_1 PX = {\delta_1}^2 X \Rightarrow X^tP^2X = {\delta_1}^2$. Thus $\delta_1 = \sqrt{X^tP^2X}$. Since $PC = \frac{1}{\sqrt{\sum_{i=1}^n P_i^2}}[a_{ij}][P_1, P_2, \dots, P_n]^t = \frac{1}{\sqrt{\sum_{i=1}^n P_i^2}}[T_1, T_2, \dots, T_n]$, $X^tP^2X = PX(PX)^t = \frac{{T_1}^2 + {T_2}^2 + \dots + {T_n}^2}{{P_1}^2 + {P_2}^2 + \dots + {P_n}^2}$. Therefore, $\delta_1 \geq \sqrt{\frac{{T_1}^2 + {T_2}^2 + \dots + {T_n}^2}{{P_1}^2 + {P_2}^2 + \dots + {P_n}^2}}$. We now assume that G is pseudo vertex PI–regular, so $\frac{T_i}{P_i} = k$ or $T_i = kP_i$, for all i. Then $PC = kC$, showing that C is an eigenvector corresponding to k and hence $\delta_1 = k$. Thus the equality holds. Conversely if equality holds then as in the proof Theorem 4, we get C is the eigenvector corresponding to δ_1 and $PC = \delta_1 C$. This implies that $\frac{T_i}{P_i} = \delta_1$. In other words G is pseudo vertex PI regular. ■

Theorem 7. The bound for δ_1 is improving from Theorems 4 to 6.

Proof. By Lemma 3, $\sum_{i=1}^{n} T_i = \sum_{i=1}^{n} {P_i}^2$. Also by Cauchy–Schwartz inequality $(\sum_{i=1}^{n} T_i)^2 \le n(\sum_{i=1}^{n} {T_i}^2)$ and $(\sum_{i=1}^{n} P_i)^2 \le n(\sum_{i=1}^{n} {P_i}^2)$. Now

$$\delta_1 \ge \sqrt{\frac{\sum_{i=1}^{n} {T_i}^2}{\sum_{i=1}^{n} {P_i}^2}} \ge \sqrt{\frac{(\sum_{i=1}^{n} T_i)^2}{n(\sum_{i=1}^{n} {P_i}^2)}} = \sqrt{\frac{(\sum_{i=1}^{n} {P_i}^2)^2}{n(\sum_{i=1}^{n} {P_i}^2)}} = \sqrt{\frac{\sum_{i=1}^{n} {P_i}^2}{n}} \ge \sqrt{\frac{(\sum_{i=1}^{n} P_i)^2}{n\times n}} = \frac{2PI_v}{n}.$$

This completes our proof. ■

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