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On the Forgotten Topological Index

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ABSTRACT

The forgotten topological index of a graph G is defined as $F(G) = \Sigma_{u\in V(G)}d(u)^3$. In this paper, we compute some properties of forgotten index and then we determine it for some classes of product graphs.

1. INTRODUCTION

All graphs considered in this paper are undirected and finite without loops and multiple edges. Denoted by $V(G)$ and $E(G)$, we mean the set of vertices and the set of edges of graph G, respectively and suppose $n = |V(G)|$, $m = |E(G)|$. Two vertices are adjacent if and only if they are connected by an edge.

The ***Wiener index*** [17] is the first reported distance based topological index defined as half sum of the distances between all the pairs of vertices in a molecular graph [10,16]. ***Topological indices*** are abundantly being used in the *QSPR* and *QSAR* researches. So far, many various types of topological indices have been described.

Furtula and Gutman, in [4] introduced a new topological index namely, forgotten topological index and it is clearly stated that the forgotten index is a special case of the earlier much studied **general first Zagreb index**. They also established a few basic properties of it, see for example [1]. In 2014 unexpected chemical application of the *F*–index was discovered and it is proved that the forgotten topological index can significantly enhance the physico–chemical applicability of the first Zagreb index.

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2. NOTATION AND DEFINITIONS

There are two Zagreb indices [10]: the first M_1 and the second M_2, can be defined as:

$$M_1 = M_1(G) = \sum_{u\in V(G)} d(u)^2 \tag{1}$$

and

$$M_2 = M_2(G) = \sum_{uv\in E(G)} d(u)d(v) \tag{2}$$

respectively. The first Zagreb index can be rewritten also as

$$M_1 = M_1(G) = \sum_{uv\in E(G)} [d(u)+d(v)]. \tag{3}$$

For more details on these topological indices we refer to [8,11,14,16,18]. With this notation, the F– index is defined as [4,5]

$$F = F(G) = \sum_{u\in V(G)} d(u)^3 = \sum_{uv\in E(G)} [d(u)^2 + d(v)^2]. \tag{4}$$

In [7] it is shown that some topological indices have one of the following three algebraic forms:

$$TI_1 = TI_1(G) = \sum_{v\in V(G)} F_1(v) \tag{5}$$

$$TI_1 = TI_1(G) = \sum_{uv\in E(G)} F_2(u,v) \tag{6}$$

$$TI_1 = TI_1(G) = \sum_{\{u,v\}\subseteq V(G), u\neq v} F_3(u,v) \tag{7}$$

where F_1, F_2 and F_3 are functions dependent of a vertex or on a pair of vertices of the molecular graph G and the forgotten index is of the form Eq. (5).

In 2006, bearing in mind Eqs. (1) and (2), Došlić [2] put forward the concept of the first and second Zagreb coindices, defined as

$$\overline{M}_1 = \overline{M}_1(G) = \sum_{uv\notin V(G)} [d(u)+d(v)] \tag{8}$$

and

$$\overline{M}_2 = \overline{M}_2(G) = \sum_{uv\notin E(G)} d(u)d(v) \tag{9}$$

respectively, see also [9]. In formulas (8) and (9) it is assumed that $x \neq y$. In full analogy with Eqs. (8), and (9), relying on Eq. (4), we can now define the F–coindex as

$$Co-F = Co-F(G) = \sum_{uv\notin E(G)} [d(u)^2 + d(v)^2]. \tag{10}$$

Let α is an arbitrary real number, the generalized version of the first Zagreb index is defined in [12,13] as follows:

$$M_\alpha = M_\alpha(G) = \sum_{u\in V(G)} d(u)^\alpha = \sum_{uv\in E(G)} [d(u)^{\alpha-1} + d(v)^{\alpha-1}]. \tag{11}$$

The generalized first Zagreb index was studied in several works such as [6,15] and the aim of this paper is to investigate the properties of $M_\alpha(G)$ where $\alpha = 3$.

The Zagreb and forgotten co–indices of a graph G and of its complement $\overline{G}$ can be represented in terms of the Zagreb indices of G and forgotten index, respectively. The respective formulas are given in [5,9].

3. RESULTS AND DISCUSSIONS

In this section, we propose several bounds for the F–index and then we compute the F–index of some composite graphs. Throughout this paper we use standard notations of graph theory. The path, star, wheel and complete graphs with n vertices are denoted by P_n, S_n, W_n and K_n, respectively.

An automorphism of the graph G is a bijection σ on which preserves the edge set E, *i.e.* if $e=uv$ is an edge of G, then $e^\sigma = u^\sigma v^\sigma$ is a member of E, where the image of vertex u is denoted by u^σ. We denote the set of all automorphisms of G by $Aut(G)$ and this set under the composition of mappings forms a group. This group acts on the set of vertices, if for any pair of vertices $u, v \in V$, there is an automorphism $\alpha \in Aut(G)$ such that $u^\sigma = v$. An isomorphism of graphs G and H is a bijection $\alpha : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $\alpha(u)\alpha(v) \in E(H)$. Two isomorphic graphs G and H are denoted by $G \cong H$.

Theorem 1. Let G be a graph with orbits $V_1, V_2, ..., V_r$ under action of Aut(G) on the set of vertices $V(G)$. Then for $u_i \in V_i$, we have

$$F(G) = \sum_{i=1}^{r} |V_i| \mathrm{d}(u_i)^3.$$

Proof. Let $V_1, V_2, ..., V_r$ be all orbits of Aut(G) on the set of vertices. It is a well–known fact that for two vertices $x, y \in V_i$, d(x) = d(y). Then one can verify that

$$F(G) = \sum_{\mathrm{i}=1}^{\mathrm{r}} \sum_{u \in V_i} \mathrm{d}(u)^3 = \sum_{i=1}^{r} |V_i| \mathrm{d}(u_i)^3.$$

As an application of Theorem 1, consider the dendrimer D with r layers as depicted in Figure 1. The vertex degrees of this graph are 1 and 3, thus, it is bi–regular. The vertices of every layer are in the same orbit under the action of automorphism graph on the set of vertices. Hence,

$$F(G) = \sum_{i=1}^{r} |V_i| \mathrm{d}(u_i)^3 = \sum_{i=1}^{r-1} |V_i| 3^3 + |V_r|.$$

This graph has $1 + 3 + 2.3 + 2^2.3 + \cdots + 2^r.3 = 1 + 3(2^{r+1} - 1)$ vertices in which the last layer has $2^r.3$ vertices. Hence,

$$F(G) = 27[1 + \sum_{i=0}^{r-1} |V_i|] + |V_r|.$$

This means that $F(G) = 3.2^r + 27[1 + 3(2^r - 1)] = 84.2^r - 54.$

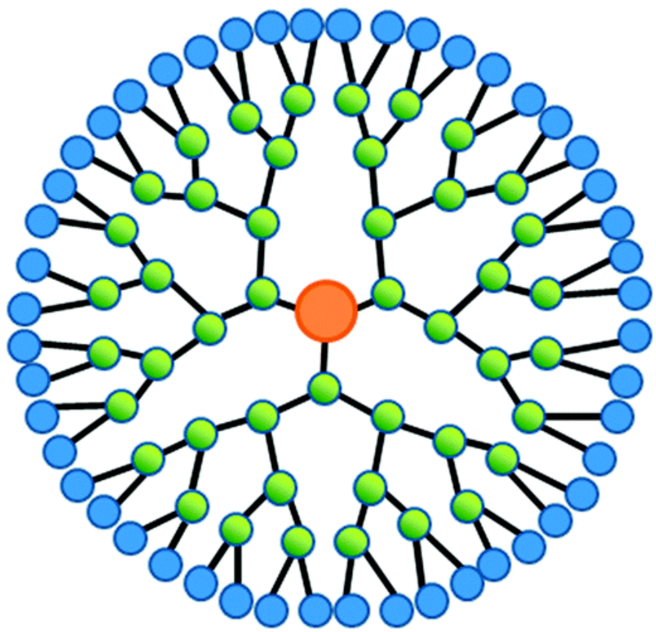

Figure 1. 2–*D* Graph of Dendrimer *D*.

Theorem 2. Let G be a graph on n vertices, then

$$F(G) \le M_1(G)^2 - 2M_2(G) \le M_1(G)^3.$$

Proof. We have

$$F(G) = \sum_{u\in V} \mathrm{d}(u)^3 \le [\sum_{u\in V} \mathrm{d}(u)]^3 = M_1(G)^3.$$

On the other hand,

$$F(G) = \sum_{uv\in E(G)}[d(u)^2 + d(v)^2]$$
$$= \sum_{uv\in E(G)}[(d(u)+d(v))^2 - 2d(u)d(v)]$$
$$\le M_1(G)^2 - 2M_2(G).$$

For two positive integers x and y, it is clear that x^3 is greater than $x^2 - 2y$ and the proof is completed.

Theorem 3. Let G be a graph on n vertices, then

$$\overline{F}(G) + F(G) \ge 2[M_2(G) + \overline{M}_2(G)].$$

Proof. For every pair of vertices $u,v \in V$, we have $(d(u)-d(v)) \geq 0$, hence $d(u)^2 + d(v)^2 \geq d(u)d(v)$ and then $F(G) \geq 2M_2(G)$. By a similar way, we can deduce that $\overline{F}(G) \geq 2\overline{M}_2(G)$. This confirms our claim.

Theorem 4. Let G be a graph on n vertices, m edges and maximum degree Δ. Then

$$\overline{F}(G) + F(G) \leq (n-1)M_1(G) + \Delta^2 m(n-3).$$

Proof. For each edge $uv \in E(G)$ and for a vertex $u \in V(G)$, the $n - 1 - d(u)$ vertices are non-adjacent with the vertex u. Let Δ be the maximum degree of G. For $uw \notin E(G)$, we have $d(u) + d(w) \leq [d(u) + \Delta][n-1-d(u)]$. So,

$$\begin{aligned}\overline{F}(G) &= \textstyle\sum_{uw \notin E(G)} d(u)^2 + d(w)^2 \\ &\leq \textstyle\sum_{uw \notin E(G)} [d(u)^2 + \Delta^2][n-1-d(u)] \\ &= (n-1)M_1(G) - F(G) + \Delta^2 (n-1)m - \Delta^2 2m.\end{aligned}$$

Hence,

$$\overline{F}(G) \leq (n-1)M_1(G) - F(G) + \Delta^2 m(n-3).$$

Theorem 5. Let u,v be two vertices of graph G. Let $G^* = G - \{vv_1, ..., vv_s\} + \{uv_1, ..., uv_s\}$. If $d(u) + s > d(v)$ then $F(G^*) > F(G)$.

Proof. Let $d_G(u) = d(u)$, for every vertex $x \in V \setminus \{u,v\}$, we have

$$d_{G^*}(u) = d_G(u) + s, d_{G^*}(v) = d_G(v) - s, d_{G^*}(x) = d_G(x).$$

Hence, by the definition of F–index, we have

$$\begin{aligned}F(G^*) - F(G) &= d_{G^*}(u)^3 + d_{G^*}(v)^3 - d_G(u)^3 - d_G(v)^3 \\ &= (d_G(u)+s)^3 + (d_G(v)-s)^3 - d_G(u)^3 - d_G(v)^3 \\ &= 3sd_G(u)(s + d_G(u)) + 3sd_G(v)(d_G(v) - s).\end{aligned}$$

Clearly $F(G^*) - F(G) > 0$ if and only if $d_G(u)(s + d_G(u)) + d_G(v)(d_G(v) - s) > 0$. On the other hand, $\{v_1, ..., v_s\} \subseteq N(v) \setminus N[u]$ implies that $d_G(v) - s > 0$ and so

$$d_G(u)(s + d_G(u)) + d_G(v)(d_G(v) - s) > 0.$$

The following bounds for the forgotten topological index were proposed in [5]:

$$F(G) \geq \frac{M_1(G)}{2m},$$

$$F(G) \geq \frac{M_1(G)^2}{m} - 2M_2(G),$$

$$F(G) \leq 2M_2(G) + m(n-1)^2.$$

Here, we establish some new bounds.

Theorem 6. Let G be a graph on n vertices and m edges. Then

$$F(G) \geq max\{6m-2n, 8m^3/n^2\}.$$

Proof. According to Bernoulli inequality, for every integer $\alpha \geq 1$, we have $(1+x)^\alpha \geq 1 + \alpha x$. Let $x = d(u_i) - 1$, then $d(u_i)^3 \geq 1 + 3(d(u_i) - 1) = 3d(u_i) - 2$. This means that $F(G) \geq 6m - 2n$. On the other hand, Let $x_1, ..., x_n$ be real numbers. Then, it is a well–known fact that

$$\frac{\sum_{i-1}^{n} x_i^k}{n} \geq \left(\frac{\sum_{i-1}^{n} x_i}{n}\right)^k.$$

By putting k=3 we have $F(G) \geq 8m^3/n^2$. This completes the proof.

Theorem 7. Let G be a graph on n vertices, m edges, minimum degree δ and maximum degree Δ. Then

$$F(G) \geq 6M_1(G) + 3n - 12m - \frac{2n}{\delta+1}\left(\frac{4}{(\delta+1)^2} + 3(\Delta-1)^2 + \frac{6(\Delta-1)}{\delta+1}\right).$$

Proof. For every real number a, we can prove that

$$\frac{a^2+1}{a+1} \geq \sqrt[3]{\frac{a^3+1}{2}}.$$

Thus

$$a^3 \leq 2\left(\frac{a^2+1}{a+1}\right)^3 - 1.$$

This implies that

$$F(G) = \sum_{i=1}^{n} d(u_i)^3 \leq 2\sum_{i=1}^{n}\left(\frac{d(u_i)^2+1}{d(u_i)+1}\right)^3 - n = 2\sum_{i=1}^{n}\left(\frac{d(u_i)^2-1+2}{d(u_i)+1}\right)^3 - n$$

$$= 2\left[\left(\sum_{i=1}^{n} d(u_i)^3 - 3\sum_{i=1}^{n} d(u_i)^2 + 3\sum_{i=1}^{n} d(u_i) - n\right)\right]$$

$$+\left[8\sum_{i=1}^{n}\frac{1}{\left(1+d(u_i)\right)^3} + 6\sum_{i=1}^{n}\frac{\left(d(u_i)-1\right)^2}{\left(1+d(u_i)\right)} + 12\sum_{i=1}^{n}\frac{\left(d(u_i)-1\right)}{\left(1+d(u_i)\right)^2}\right] - n.$$

But $d(u_i) \geq 1$, and so

$$F(G) \leq 2\left[(F(G) - 3M_1(G) + 6m - n) + \frac{8n}{(\delta+1)^3} + 6\sum_{i=1}^{n} \frac{(\Delta-1)^2}{\delta+1} + 12n\frac{(\Delta-1)}{(\delta+1)^2}\right] - n.$$

Thus, the proof is completed.

Let G be a connected graph with n vertices and A be its adjacency matrix, where $\lambda_1,\ldots,\lambda_n$ are its eigenvalues. The k–th spectral moment of G is defined as $\sum_{i-1}^{n} \lambda_i^k$ and it is equal to the number of all closed walks of length k in G. Similarly, if $\mu_1,\ldots,\mu_n$ are Laplacian eigenvalues, than the k–th Laplacian spectral moment is as follows:

$$S_k = \sum_{i-1}^{n} \lambda_i^k.$$

Theorem 8 (**[5]**) If the graph G is triangle–free, then

$$F(G) = \sum_{xy\in E(G)} [d(x) - d(y)]^2 - 2M_1(G) + 4m + \sum_{i=1}^{n}\sum_{j=1}^{n} (A^3)_{ij}$$

where **A** is the adjacency matrix of G.

Theorem 9. Let G be a connected graph, then

$$F(G) = S_3 - 3M_1(G) + 6t.$$

Proof. Let D be a diagonal matrix whose entries are the degree of vertices in G. We have

$$\sum_{i=1}^{n} \mu_i^3 = tr(D - A)^3 = tr(D^3 - A^3 + 3A^2D)$$
$$= \sum_{i=1}^{n} d(u_i)^3 + 3\sum_{i=1}^{n} d(u_i)^2 - 6t.$$

Thus for the k–th spectral moment we have

$$F(G) = S_3 - 3\sum_{i=1}^{n} d(u_i)^2 + 6t = S_3 - 3M_1(G) + 6t.$$

Corollary 10. Let G be a triangular–free graph, then

$$F(G) = S_3 - 3M_1(G).$$

4. COMPUTING THE F–INDEX OF SOME GRAPH PRODUCTS

In this section we present explicit formulas for the F–index of several classes of graphs that arise via binary graph operations known as graph products. We start from the most common operation, the Cartesian product. The disjunction and the symmetric difference share many properties with the Cartesian product: they have the same vertex sets, they are commutative and associative; hence they are considered next. The join of two or more graphs is also a commutative operation, but defined on the union instead on the Cartesian product of the vertex sets of the components.

4.1 CARTESIAN PRODUCT

The Cartesian product $G\times H$ of graphs G and H is a graph such that $V(G\times H)=V(G)\times V(H)$, and any two vertices (a,b) and (u,v) are adjacent in $G\times H$ if and only if either a = u and b is adjacent with v, or $b = v$ and a is adjacent with u. The degree of a vertex (u_1,u_2) of $G_1\times G_2$ is the sum of the degrees of its projections to the respective components,

$$d_{G_1\times G_2}(u_1,u_2)=d_{G_1}(u_1)+d_{G_2}(u_2).$$

Theorem 11. Let G_i (i = 1,2) be a graph on n_i vertices and m_i edges. Then

$$F(G_1\times G_2)=n_2F(G_1)+n_1F(G_2)+6m_2M_1(G_1)+6m_1M_1(G_2).$$

Proof. For $(u_1,u_2)\in G_1\times G_2$, we have $d_{G_1\times G_2}(u_1,u_2)=d_{G_1}(u_1)+d_{G_2}(u_2).$ This means that

$$\begin{aligned}F(\mathrm{G}_1\times\mathrm{G}_2)&=\sum_{(\mathrm{u}_1,\mathrm{u}_2)\in\mathrm{G}_1\times\mathrm{G}_2}d^3{}_{G_1\times G_2}(u_1,u_2)=\sum_{(\mathrm{u}_1,\mathrm{u}_2)\in\mathrm{G}_1\times\mathrm{G}_2}[d_{G_1}(u_1)+d_{G_2}(u_2)]^3\\&=\sum_{(\mathrm{u}_1,\mathrm{u}_2)\in\mathrm{G}_1\times\mathrm{G}_2}d^3{}_{G_1}(u_1)+d^3{}_{G_2}(u_2)+3d_{G_1}(u_1)d_{G_2}(u_2)[d_{G_1}(u_1)+d_{G_2}(u_2)]\\&=n_2F(\mathrm{G}_1)+n_1F(\mathrm{G}_2)+6m_2M_1(\mathrm{G}_1)+6m_1M_1(\mathrm{G}_2).\end{aligned}$$

4.2 SYMMETRIC DIFFERENCE AND DISJUNCTION

The **disjunction** $G\vee H$ of two graphs G and H is the graph with vertex set $V(G)\times V(H)$ in which (u_1,u_2) is adjacent with (v_1,v_2) whenever u_1 is adjacent with v_1 in G or u_2 is adjacent with v_2 in H. If $|V(G)| = n_1$, $|E(G)| = m_1$, $|V(H)| = n_2$, $|E(H)| = m_2$, the degree of a vertex (u_1,u_2) of $G\vee H$ is given by $\mathrm{d}_{G\vee H}((u_1,u_2)) = n_2\mathrm{d}_G(u_1) + n_1\mathrm{d}_H(u_2) - \mathrm{d}_G(u_1)\,\mathrm{d}_H(u_2)$.

Theorem 12. Let G_i (i = 1,2) be a graph on n_i vertices and m_i edges. Then

$$F(G_1\vee G_2)=n_2^2F(G_1)+n_1^2F(G_2)-4m_1m_2.$$

Proof. We have

$$\begin{aligned}F(G_1\vee G_2)&=\sum_{(u_1,u_2)\in G_1\vee G_2}d^3{}_{G_1\vee G_2}(u)=n_1\sum_{u\in G_1}d^3_{G_2}(u_2)-\sum_{u_1\in G_1}d_{G_1}(u_1)\sum_{u_2\in G_2}d_{G_2}(u_2)\\&=n_2^2F(G_1)+n_1^2F(G_2)-4m_1m_2.\end{aligned}$$

The **symmetric difference** $G\oplus H$ of two graphs G and H is the graph with vertex set $V(G)\times V(H)$ in which (u_1,u_2) is adjacent with (v_1,v_2) whenever u_1 is adjacent with v_1 in G

or u_2 is adjacent with v_2 in H, but not both. It follows from the definition that the degree of a vertex $(u_1,u_2)$of $G \oplus H$ is given by

$$d_{G\oplus H}((u_1,u_2)) = n_2 d_G(u_1) + n_1 d_H(u_2) - 2d_G(u_1)d_H(u_2).$$

Theorem 13. Let G_i (i = 1,2) be a graph on n_i vertices and m_i edges. Then

$$F(G_1 \oplus G_2) = n_1^2 F(G_2) + n_2^2 F(G_1) - 8m_1 m_2.$$

Proof. The proof is similar to the proof of Theorem 12.

4.3 JOIN

The join $G = G_1 + G_2$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2. Let n_1 and n_2 be number of vertices of G_1 and G_2, respectively. Then

$$d_{G_1+G_2}(u) = \begin{cases} d_{G_1}(u) + n_2 \\ d_{G_2}(u) + n_1 \end{cases}.$$

Theorem 14. Let G_i (i = 1,2) be a graph on n_i vertices and m_i edges. Then

$$F(G_1 + G_2) = F(G_1) + F(G_2) + n_1 n_2^3 + n_1^3 n_2 + 3n_2 M_1(G_1) + 3n_1 M_1(G_2) + 3n_1 m_2 + 3n_2 m_1.$$

Proof. We have

$$\begin{aligned} F(G_1 + G_2) &= \sum\nolimits_{u\in G_1+G_2} d^3{}_{G_1+G_2}(u) = \sum\nolimits_{u\in G_1} (d_{G_1}(u) + n_2)^3 + \sum\nolimits_{u\in G_2} (d_{G_2}(u) + n_1)^3 \\ &= F(G_1) + F(G_2) + n_1 n_2^3 + n_1^3 n_2 + 3n_2 M_1(G_1) + 3n_1 M_1(G_2) + 3n_1 m_2 + 3n_2 m_1. \end{aligned}$$

4.4 COMPOSITION

The composition $G = G_1[G_2]$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 such that $|V_1| = n_1$, $|V_2| = n_2$ and edge sets E_1 and E_2 such that $|E_1| = m_1$, $|E_2| = m_2$ is the graph with vertex set $V_1 \times V_2$ and $u = (u_1,u_2)$ is adjacent with $v = (v_1,v_2)$ whenever u_1 is adjacent with v_1 or $u_1 = v_1$ and u_2 is adjacent with v_2. It follows from the definition that the degree of a vertex (u_1,u_2) of $G_1[G_2]$ is given by

$$d_{G_1[G_2]}((u_1,u_2)) = n_2 d_{G_1}(u_1) + n_1 d_{G_2}(u_2).$$

Theorem 15. Let G_i (i = 1,2) be a graph on n_i vertices and m_i edges. Then

$$F(G_1[G_2]) = n_2^3 F(G_1) + F(G_2) + 6n_2^2 m_2 M_1(G_1) + 6n_2 m_1 M_1(G_2).$$

Proof. We have

$$F(G_1[G_2]) = \sum_{(u_1,u_2)\in G_1[G_2]} d^3{}_{G_1[G_2]}(u) = \sum_{u\in G_1} [n_2 d_{G_1}(u_1) + d_{G_2}(u_2)]^3$$
$$= n_2^3 F(G_1) + F(G_2) + 6n_2^2 m_2 M_1(G_1) + 6n_2 m_1 M_1(G_2).$$

4.5 CORONA PRODUCT

The corona $G_1 o G_2$ was defined by Frucht and Harary [3] as the graph G obtained by taking one copy of G_1 of order p_1 and p_1 copies of G_2, and then joining the i–th node of G_1 to every node in the i–th copy of G_2, see Figure 2. Suppose p_1, p_2, q_1 and q_2 are the number of vertices and the number of edges of graphs G_1 and G_2, respectively. It is easy to see that the number of vertices and the number of edges of $G_1 o G_2$ are $p_1(1 + p_2)$ and $q_1 + p_1 q_2 + p_1 p_2$, respectively.

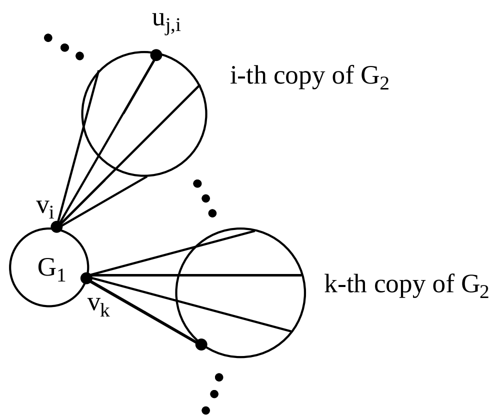

Figure 2. The Corona Product $G_1 \circ G_2$.

Example 1. For the graphs $G_1 = K_2$ and $G_2 = P_3$, the two different coronas $G_1 o G_2$ and $G_2 o G_1$ are shown in Figure 3.

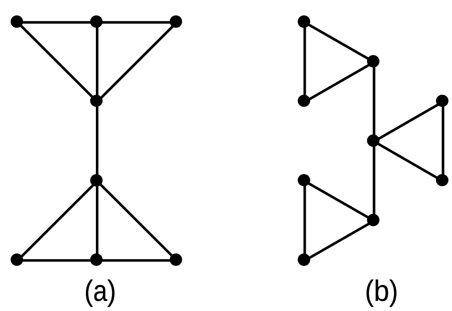

Figure 3. (a) The Corona Product $K_2 o P_3$ and (b) $P_3 o K_2$.

Theorem 16. Let G_i (i = 1,2) be a graph on n_i vertices and m_i edges. Then

$$F(G_1 o G_2) = F(G_1) + n_1 F(G_2) + n_1 n_2^3 + 3n_2 M_1(G_1) + 6n_2^2 m_1 + n_1 n_2 + 3n_1 M_1(G_2) + 6n_1 m_2.$$

Proof. It is not difficult to see that

$$d_{G_1oG_2}(a)=\begin{cases} d_{G_1}(a)+n_2 & a\in V(G_1)\\ d_{G_2}(u)+1 & a\in V(G_2)\end{cases}.$$

This means that

$$F(G_1oG_2)=\sum_{u\in G_1oG_2} d^3{}_{G_1oG_2}(u)=\sum_{u\in G_1} d^3_{G_1}(u)+\sum_{i=1}^{n_1}\sum_{u\in G_i} d^3_{G_i}(u)$$
$$=F(G_1)+n_1F(G_2)+n_1n_2^3+3n_2M_1(G_1)+6n_2^2m_1+n_1n_2+3n_1M_1(G_2)+6n_1m_2.$$

4.6 TENSOR PRODUCT

For given graphs G_1 and G_2 their tensor product $G_1\otimes G_2$ is defined as the graph on the vertex set $V(G_1)\times V(G_2)$ with vertices $u=(u_1, u_2)$ and $v=(v_1, v_2)$ connected by an edge if and only if either $u_1v_1\in E(G_1)$ and $u_2v_2\in E(G_2)$, see Figure 4. In other words, $G_1\otimes G_2$ has exactly n_1n_2 vertices and $2m_1+2m_2-12$ edges, where n_1, n_2 are the number of vertices and m_1, m_2 are the number of edges of G_1 and G_2, respectively.

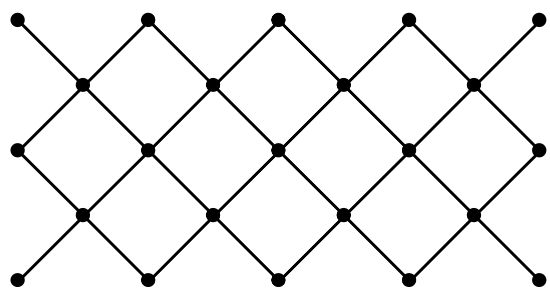

Figure 4. The Tensor Product $P_3\otimes P_5$.

Theorem 17. Let G_i (i = 1,2) be a graph on n_i vertices and m_i edges. Then

$$F(G_1\otimes G_2)=n_1F(G_2)+n_2F(G_1)+F(G_1)F(G_2)+3[2m_2(M_1(G_1)+F(G_1))+2m_1M_1(G_2)$$
$$+F(G_1)M_1(G_2)+F(G_2)M_1(G_1)]+6M_1(G_1)M_1(G_2).$$

Proof. Notice that the degree of every vertex of the tensor product can be computed as

$$d_{G_1\otimes G_2}(u_1, u_2) = d_{G_1}(u_1)+d_{G_2}(u_2)+d_{G_1}(u_1)d_{G_2}(u_2).$$

Similar to the proof of Theorem 11, the proof is straightforward.

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