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Graphs with Smallest Forgotten Index

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ABSTRACT

The forgotten topological index of a molecular graph G is defined as $F(G) = \sum_{v\in V(G)} d(v)^3$, where $d(v)$ denotes the degree of vertex v in G. The first through the sixth smallest forgotten indices among all trees, the first through the third smallest forgotten indices among all connected graph with cyclomatic number $\gamma = 1, 2$, the first through the fourth for $\gamma = 3$, and the first and the second for $\gamma = 4, 5$ are determined. These results are compared with those obtained for the first Zagreb index.

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1. INTRODUCTION

All graphs considered are assumed to be simple and finite. The sets of vertices and edges of a graph G are denoted by $V(G)$ and $E(G)$, respectively. By n and m we denote the number of vertices and edges of G, i.e., $n = |V(G)|$ and $m = |E(G)|$. If G has p components, then $\gamma = \gamma(G) = m - n + p$ is called the *cyclomatic number* of G. In this work we shall be mainly concerned with connected graphs, for which $p = 1$. A connected graph with $\gamma = 0$ is said to be a tree. Graphs with $\gamma = 1, 2, 3, 4, 5$ are then called *unicyclic, bicyclic, tricyclic, tetracyclic* and *pentacyclic*, respectively.

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The set of all connected graphs with exactly n vertices and cyclomatic number γ is denoted by $C^{\gamma}(n)$. In particular, $C^{0}(n)$ is the set of all n-vertex trees.

The number of the first neighbors of a vertex $u \in V(G)$ is said to be its *degree*, and will be denoted by $d(u) = d_G(u)$. As well known,

$$\sum_{u \in V(G)} d(u) = 2m.$$

Consequently, for all graphs belonging to a set $\boldsymbol{C}^{\gamma}(n)$, the sum of the vertex degrees is the same.

Let $V(G) = \{v_1, v_1, \ldots, v_n\}$, and let the vertices of G be labeled so that $d(v_1) \geq d(v_2) \geq \cdots \geq d(v_n)$. Then the *degree sequence* of G is $[d(v_1), d(v_2), \ldots, d(v_n)]$. As customary, we shall write this degree sequence in an abbreviated manner, as the below two self–explanatory examples show:

$$[4,3,3,3,3,2,2,2,2,2,2,2,1,1,1,1] \equiv [4,3^4,2^7,1^4]$$
$$[4,4,4,1,1,1,1,1,1,1,1] \equiv [4^3,1^8]\,.$$

The greatest vertex degree of the graph G will be denoted by $\Delta = \Delta(G)$. The number of vertices of degree i in G will be denoted by $n_i = n_i(G)$. If we assume that the graph G has no isolated vertices (= vertices of degree zero), which is a necessary condition for being connected, then $n_0 = 0$. For such graphs,

$$\sum_{i=1}^{\Delta(G)} n_i = n \quad and \quad \sum_{i=1}^{\Delta(G)} i n_i = 2m.$$

For a subset W of $V(G)$, let $G - W$ be the subgraph of G obtained by deleting the vertices of W and the edges incident with them. Similarly, for a subset E' of $E(G)$, $G - E'$ denotes the subgraph of G obtained by removing the edges of E'. If $W = \{v\}$ and $E' = \{xy\}$, then the subgraphs $G - W$ and $G - E'$ will be shorter written as $G - v$ and $G - xy$, respectively. Finally, if x and y are non-adjacent vertices of G, then $G + xy$ is the graph obtained from G by adding an edge xy. Our other notations are standard and can be taken from the most of textbooks on graph theory. The *first Zagreb index*, $M_1(G)$, of the graph G is defined as

$$M_1 = M_1(G) = \sum_{u \in V(G)} d(u)^2. \tag{1}$$

The theory of this degree–based topological index, introduced in the 1970s [9], is nowadays well elaborated [6–8,11].

Furtula and one of the present authors [4], recalled that in the formulas for total π–electron energy, reported in [9], in addition to M_1, also the sum of cubes of vertex degrees was encountered. This latter degree–based graph invariant did not attract any attention in mathematical chemistry literature for more than 40 years. In view of this, it was named *forgotten topological index*, and defined as

$$F = F(G) = \sum_{u \in V(G)} d(u)^3. \tag{2}$$

It can be shown that the F-index satisfies the identity

$$F(G) = \sum_{e=uv \in E(G)} [d(u)^2 + d(v)^2].$$

At this point, it needs to be mentioned that Zhang and Zhang [14] introduced the *first general Zagreb index* of a graph G as

$$M_1^\alpha = M_1^\alpha(G) = \sum_{u \in V(G)} d(u)^\alpha,$$

where α is an arbitrary real number. Evidently, the forgotten index is just the special case of the first general Zagreb index for $\alpha = 3$. In [14], all unicyclic graphs with the first three smallest and greatest values of M_1^α were characterized. Zhang et al. [13], determined all n-vertex bicyclic graphs, $n \geq 5$, with the first three smallest and greatest M_1^α when $\alpha > 1$. They also characterized the greatest and the first three smallest values of the first general Zagreb index when $0 < \alpha < 1$. Tong et al. [12], characterized all tricyclic graphs with the greatest, the second and third greatest values of M_1^α, and the tricyclic graphs with the smallest, the second and third smallest values of this index. These results are automatically applicable to the F-index. The aim of the present work is to extend the considerations to graphs with cyclomatic number $\gamma > 3$.

Until now, there are very few researches concerned solely with the F-index. Furtula et al. [5], among other results, proved that for triangle–free graphs $2F \leq M_1^2$. Abdo et al. [1] studied n-vertex trees with maximal values of the forgotten index. They proved that if $n - 2$ is divisible by 3, then the maximum value of the forgotten index is $22n - 42$ and when $3 \nmid n - 2$, then the maximum forgotten index will be $22(n-1) - 21x + x3$, where x is uniquely determined by $2 \leq x \leq 3$ and $n - 1 - x \equiv 0 \ (mod\ 3)$. Anyway, because of the close analogy between the first Zagreb index and the forgotten index, one may expect that in the majority of cases, the graphs extremal with respect to M_1 will also be extremal with respect to F. The truly interesting results would then be the specification of cases in which these two indices have a (significantly) different behavior. We also refer to [2,3] for more information on this topic.

From Eqs. (1) and (2) it is evident that two graphs with equal degree sequence necessarily have equal first Zagreb indices and equal forgotten indices. Bearing this in mind, it is purposeful to partition each set $C^\gamma(n)$ into equivalence classes, each class pertaining to a particular degree sequence. All elements of such an equivalence class have equal M_1 and equal F indices. Because we are aiming at finding graphs (i.e., the respective equivalence classes) with smallest F-values, we will consider only a few selected such classes, those in which many vertex degrees are equal to two. These equivalence classes are listed in Tables $1-10$ in the subsequent section.

In Tables $1-10$ are listed the equivalence classes (Eq.Cl.) of the sets $C^{\gamma}(n)$ that are of interest for the present considerations. The value of n is assumed to be sufficiently large, so that each equivalence class is non-empty. In order to facilitate the analysis, in the last column of each table, expression for the F-index of the elements of the respective equivalence class is given.

2. MAIN RESULTS

The aim of this section is to characterize the graphs (i.e., the respective equivalence classes) in which the F-index assumes the first few smallest values. We do this for the sets $C^{i}(n)$ for $0 \leq i \leq 5$.

In order to achieve this goal, we first introduce a graph transformation that decreases the forgotten index.

Transformation A. Let G_1 be a graph with vertices v_1 and v_2, such that $\mathrm{d}_{\mathrm{G}_1}(\mathrm{v}_1) \geq 2$ and $\mathrm{d}_{\mathrm{G}_1}(\mathrm{v}_2) = 1$. Let G_2 be another graph and w its vertex. Construct the graph G from G_1 and G_2 by connecting the vertices w and v_1. Construct the graph G' so that $G' = \mathrm{G} - \mathrm{wv}_1 + \mathrm{wv}_2$.

Lemma 2.1. $F(G') < F(G)$.

Proof. $F(G) - F(G') = [(d_{G_1}(v_1) + 1)^3 + 1^3] - [d_{G_1}(v_1)^3 + 2^3] > 0$, as $d_{G_1}(v_1) \geq 2$. □

Remark 2.2. *Note that in the exactly same manner we get $M_1(G') < M_1(G)$. This implies that whichever result is deduced for the F-index using Lemma 2.1, an analogous result will also hold for the first Zagreb index.*

We now focus our attention to the case $\gamma = 0$, namely to trees, i.e., to the equivalence classes of the set $C^0(n)$, listed in Table 1. First we state an auxiliary result:

Lemma 2.3. *If T is a tree with n vertices, then*

$$n_1 = 2 + \sum_{i=3}^{\Delta(G)} (i-2)n_i \quad and \quad n_2 = n - 2 - \sum_{i=3}^{\Delta(G)} (i-1)n_i.$$

Proof. The proof follows from $\sum_{i=1}^{\Delta(G)} n_i = n$ and $\sum_{i=1}^{\Delta(G)} in_i = 2(n-1)$. □

Corollary 2.4. *There exists a tree T of order n with $2 \leq n_1(T) \leq 6$, if and only if T belongs to one of the equivalence classes given in Table 1.*

Proof. We distinguish the following five cases:

(1) $n_1(T) = 2$,
(2) $n_1(T) = 3$,
(3) $n_1(T) = 4$,
(4) $n_1(T) = 5$,
(5) $n_1(T) = 6$.

We present a proof for the case (1) whereas other cases are treated in a similar manner. Assume that $n_1(T) = 2$. Then by Lemma 2.3, there is a tree T with $n_1(T) = 2$ if and only if $\sum_{i=3}^{\Delta(G)}(i-2)n_i = 0$ if and only if $n_2(T) = n-2$ and $n_i(T) = 0$, for each $i \geq 3$. This leads to the proof. □

Theorem 2.5. *Let $T_1 \in N_1$, $T_2 \in N_2$, $T_3 \in N_4$, $T_4 \in N_7$, $T_5 \in N_3$, and $T_6 \in N_{12}$. If $n \geq 10$ and $\in C^0(n) \setminus \{T_1, T_2, \ldots, T_6\}$, then $F(T_1) < F(T_2) < F(T_3) < F(T_4) < F(T_5) < F(T_6) < F(T)$.*

Proof. From Table 1, one can see that $F(T_1) < F(T_2) < F(T_3) < F(T_4) < F(T_5) < F(T_6)$. If $n_1(T) = 5$ or 6, then the proof follows from the data in Table 1. If $n_1(T) \geq 7$, then by a repeated application of Transformation A, we obtain a tree T_s such that $n_1(T_s) = 6$. By Lemma 2.1, $F(T_s) < F(T)$ and by Table 1, $F(T_6) \leq F(T_s)$, which yields the result. □

Lemma 2.6. *If G is a connected unicyclic graph with n vertices, then*

$$n_1 = \sum_{i=3}^{\Delta(G)}(i-2)n_i \text{ and } n_2 = n - \sum_{i=3}^{\Delta(G)}(i-1)n_i.$$

Proof. The proof follows from $\sum_{i=1}^{\Delta(G)} n_i = n$ *and* $\sum_{i=1}^{\Delta(G)} in_i = 2n$. □

Corollary 2.7. *There is a connected unicyclic graph G of order n with $n_1(G) \leq 2$ if and only if G belongs to one of equivalence classes given in Table 2.*

Proof. We distinguish the following three cases:

(1) $n_1(G) = 0$,
(2) $n_1(G) = 1$,
(3) $n_1(G) = 2$.

In order to prove (1), assume that $n_1(G) = 0$. Then by Lemma 2.6, there exists a connected unicyclic graph G with $n_1(G) = 0$ if and only if $\sum_{i=3}^{\Delta(G)}(i-2)n_i = 0$. But, this is equivalent to the fact that if and only if $n_2(T) = n$ and $n_i(T) = 0$, for each $i \geq 3$. The proofs of the remaining cases are similar and are omitted. □

Theorem 2.8. *Let* $G_1 \in A_1$, $G_2 \in A_2$ *and* $G_3 \in A_4$. *If* $G \in C^1(n) \setminus \{G_1, G_2, G_3\}$ *and* $n \geq 5$, *then* $F(G_1) < F(G_2) < F(G_3) < F(G)$.

Proof. From Table 2, one can see that $F(G_1) < F(G_2) < F(G_3)$. If $n_1(G)= 2$, then Table 2 leads us to the proof. If $n_1(G) \geq 3$, then by a repeated application of Transformation A, we obtain a connected unicyclic graphs Q such that $n_1(Q) = 2$. By Lemma 2.1, we have $F(Q) < F(G)$. On the other hand, by the data given in Table 2, $F(G_3) \leq F(Q)$, which yields the result. □

Lemma 2.9. *If* G *is a connected bicyclic graph with* n *vertices, then*

$$n_1 = \sum_{i=3}^{\Delta(G)}(i-2)n_i - 2 \quad \text{and} \quad n_2 = n + 2 - \sum_{i=3}^{\Delta(G)}(i-1)n_i.$$

Proof. The proof follows from $\sum_{i=1}^{\Delta(G)} n_i = n$ and $\sum_{i=1}^{\Delta(G)} in_i = 2n + 2$. □

Corollary 2.10. *There exists a connected bicyclic graph* G *of order* n *with* $n_1(G) \leq 1$ *if and only if* G *belongs to one of the equivalence classes given in Table* 3.

Proof. We distinguish the following two cases:

(1) $n_1(G) = 0$,

(2) $n_1(G) = 1$.

In order to prove (1), assume that $n_1(G) = 0$. Then by Lemma 2.9, there exists a connected bicyclic graph G with $n_1(G) = 0$ if and only if $\sum_{i=3}^{\Delta(G)}(i-2)n_i = 2$. But the latter requirement is equivalent to one of the following two conditions:

1. $n_2(G) = n - 1$, $n_3(G) = 0$, $n_4(G) = 1$, and $n_i(G) = 0$, for each $i \geq 5$,
2. $n_2(G) = n - 2$, $n_3(G) = 2$, and $n_i(G) = 0$, for each $i \geq 4$.

The proof of case (2) is analogous, and we omit it. □

Theorem 2.11. *Let* $G_1 \in B_2$, $G_2 \in B_1$, *and* $G_3 \in B_5$. *If* $G \in C^2(n) \setminus \{G_1, G_2, G_3\}$ *and* $n \geq 7$, *Then* $F(G_1) < F(G_2) < F(G_3) < F(G)$.

Proof. From Table 3, we have $F(G_1) < F(G_2) < F(G_3)$. If $n_1(G) = 1$, then the theorem can be proven by Table 3. If $n_1(G) \geq 2$, then by repeated application of Transformation

A, we obtain a connected bicyclic graph, say Q, such that $n_1(Q) = 1$. By applying Lemma 2.1 we conclude that $F(Q) < F(G)$. On the other hand, by the data in Table 3, $F(G_3) \leq F(Q)$, which yields the result. □

Lemma 2.12. *If G is a connected tricyclic graph with n vertices, then*

$$n_1 = \sum_{i=3}^{\Delta(G)} (i-2)n_i - 4 \text{ and } n_2 = n + 4 - \sum_{i=3}^{\Delta(G)} (i-1)n_i.$$

Corollary 2.13. *There is a connected tricyclic graph G of order n with $n_1(G) \leq 2$ if and only if G belongs to one of the equivalence classes given in Tables 4, 5, or 6.*

Theorem 2.14. *Let $G_1 \in D_5$, $G_2 \in E_7$, $G_3 \in D_4$ and $G_4 \in F_{11}$. If $n \geq 11$ and $G \in C^3(n) \setminus \{G_1, G_2, G_3, G_4\}$. Then $F(G_1) < F(G_2) < F(G_3) < F(G_4) < F(G)$.*

Proof. From Tables 4, 5, and 6, one can see that $F(G_1) < F(G_2) < F(G_3) < F(G_4)$. The case of $n_1(G) \leq 2$ is a direct consequence of the data given Tables 4, 5, and 6. If $n_1(G) \geq 3$, then by repeated applications of Transformation A, we obtain a connected tricyclic graphs, for example Q, such that $n_1(Q) = 2$. By applying Lemma 2.1 we get that $F(Q) < F(G)$. Then the data given in Table 6 imply that $F(G_4) \leq F(Q)$, which yields the result. □

Lemma 2.15. *If G is a connected tetracyclic graph with n vertices, then*

$$n_1 = \sum_{i=3}^{\Delta(G)} (i-2)n_i - 6 \text{ and } n_2 = n + 6 - \sum_{i=3}^{\Delta(G)} (i-1)n_i.$$

Corollary 2.16. *There exists a connected tetracyclic graph G of order n with $n_1(G) \leq 1$ if and only if G belongs to one of the equivalence classes given in Tables 7 and 8.*

Theorem 2.17. *Let $G_1 \in H_{11}$ and $G_2 \in I_{15}$. If $n \geq 12$ and $G \in C^4(n) \setminus \{G_1, G_2\}$. Then $F(G_1) < F(G_2) < F(G)$.*

Proof. From Tables 7 and 8 one can see that $F(G_1) < F(G_2)$. If $n_1(G) = 0$ or $n_1(G) = 1$, then the data given in Tables *7* and *8* completes the proof.

If $n_1(G) \geq 2$, then by repeated applications of Transformation A, a connected tetracyclic graph Q is obtained for which $n_1(Q) = 1$. By Lemma 2.1, $F(Q) < F(G)$ and by Table *8*, $F(G_2) \leq F(Q)$, which yields the result. □

Lemma 2.18. *If G is a connected pentacyclic graph with n vertices, then*

$$n_1 = \sum_{i=3}^{\Delta(G)} (i-2)n_i - 8 \quad \text{and} \quad n_2 = n + 8 - \sum_{i=3}^{\Delta(G)} (i-1)n_i.$$

Corollary 2.19. *There exists a connected pentacyclic graph G of order n with $n_1(G) \leq 1$ if and only if G belongs to one of the equivalence classes given in Tables 9 and 10.*

Theorem 2.20. *Let $G_1 \in K_{22}$ and $G_2 \in L_{29}$. If $n \geq 16$ and $G \in C^5(n) \setminus \{G_1, G_2\}$. Then $F(G_1) < F(G_2) < F(G)$.*

Proof. From Tables 9 and 10, it can be seen that $F(G_1) < F(G_2)$. If $n_1(G) = 0$ or $n_1(G) = 1$, then Tables 9 and 10 lead us to the proof. If $n_1(G) \geq 2$, then by repeated applications of Transformation A, a connected pentacyclic graph Q can be constructed, such $that\, n_1(Q) = 1$. By Lemma 2.1, $F(Q) < F(G)$ and by the data in Table 10, $F(G_2) \leq F(Q)$, which proves the result. □

3. CONCLUDING REMARKS

In this paper the connected graphs with fixed number of vertices and cyclomatic number (i.e., the respective equivalence classes of such graphs) are determined, whose $F-$indices assume the smallest possible value. Since the $F-$index is defined in a similar manner as the first Zagreb index, cf. Eqs. (1) and (2), their properties are expected also very similar. In view of this, it is purposeful to compare the result for these two graph invariants. For the sake of completeness, we first state three relevant results as follows:

Theorem 3.1. *The characterization of n-vertex trees, n-vertex unicyclic, and n-vertex bicyclic graphs with the smallest, the second smallest and the third smallest first Zagreb index are as follows:*

1. *Li and Zhao* [10]: *Trees with degree sequence* $[2^{n-2}, 1^2], [3, 2^{n-4}, 1^3]$, *and* $[3^2, 2^{n-6}, 1^4]$ *have the smallest, second smallest, and third smallest values of the first Zagreb index among all n-vertex trees.*
2. *Zhang and Zhang* [14, Theorem 1]: *Let G be an $n-$vertex unicyclic graph, $n \geq 7$. Then $M_1(G)$ attains the smallest, the second smallest, and the third smallest value if and only if the degree sequence of G is* $[2^n], [3, 2^{n-2}, 1]$, *and* $[3^2, 2^{n-4}, 1^2]$, *respectively.*
3. *Zhang et al.* [13, Theorems 1 and 4]: *Suppose that G is a bicyclic graph on $n \geq 5$ vertices, L_1 denotes the set of such graphs with degree sequence* $[4, 2^{n-1}]$ *or* $[3^3, 2^{n-4}, 1]$ *and L_2 is the set of all n-vertex bicyclic graphs with degree sequence* $[4, 3, 2^{n-3}, 1]$ *or* $[3^4, 2^{n-6}, 1^2]$. *Then the first Zagreb index $M_1(G)$ attains the smallest, the second smallest and the third smallest value if and only if the degree sequence of G is* $[3^2, 2^{n-2}]$, *$G \in L_1$, and $G \in L_2$, respectively.*

By Theorem 2.8, the n-vertex unicyclic graphs with degree sequences $[2^n]$, $[3, 2^{n-2}, 1]$ and $[3^2, 2^{n-4}, 1^2]$ have the smallest, second smallest and third smallest values of forgotten index which are the same as the case of the first Zagreb index. On the other hand, by Theorem 2.11, n-vertex bicyclic graphs with degree sequences $[3^2, 2^{n-2}]$, $[4, 2^{n-1}]$ and $[3^3, 2^{n-4}, 1]$ have the smallest, second smallest, and third smallest values of the forgotten index. Thus, the bicyclic graphs with smallest value of the forgotten and first Zagreb index are the same, but these graph invariants attain their second and third smallest value in different classes of bicyclic graphs.

Table 1. Degree distributions of trees with $2 \leq n_1 \leq 6$.

Eq.Cl.	n_6	n_5	n_4	n_3	n_2	n_1	n_i $(i \geq 7)$	F
N_1	0	0	0	0	$n-2$	2	0	$8n-14$
N_2	0	0	0	1	$n-4$	3	0	$8n-2$
N_3	0	0	1	0	$n-5$	4	0	$8n+28$
N_4	0	0	0	2	$n-6$	4	0	$8n+10$
N_5	0	1	0	0	$n-6$	5	0	$8n+82$
N_6	0	0	1	1	$n-7$	5	0	$8n+40$
N_7	0	0	0	3	$n-8$	5	0	$8n+22$
N_8	1	0	0	0	$n-7$	6	0	$8n+166$
N_9	0	1	0	1	$n-8$	6	0	$8n+94$
N_{10}	0	0	2	0	$n-8$	6	0	$8n+70$
N_{11}	0	0	1	2	$n-9$	6	0	$8n+52$
N_{12}	0	0	0	4	$n-10$	6	0	$8n+34$

Table 2. Degree distributions of connected unicyclic graphs with $n_1 \leq 2$.

Eq.Cl.	n_4	n_3	n_2	n_1	n_i $(i \geq 5)$	F
A_1	0	0	n	0	0	$8n$
A_2	0	1	$n-2$	1	0	$8n+12$
A_3	1	0	$n-3$	2	0	$8n+42$
A_4	0	2	$n-4$	2	0	$8n+24$

Table 3. Degree distributions of connected bicyclic graphs with $n_1 \leq 1$.

Eq.Cl.	n_5	n_4	n_3	n_2	n_1	n_i $(i \geq 6)$	F
B_1	0	1	0	$n-1$	0	0	$8n + 56$
B_2	0	0	2	$n-2$	0	0	$8n + 38$
B_3	1	0	0	$n-2$	1	0	$8n + 110$
B_4	0	1	1	$n-3$	1	0	$8n + 68$
B_5	0	0	3	$n-4$	1	0	$8n + 58$

Table 4. Degree distributions of connected tricyclic graphs with $n_1 = 0$.

Eq.Cl.	n_6	n_5	n_4	n_3	n_2	n_1	n_i $(i \geq 7)$	F
D_1	1	0	0	0	$n-1$	0	0	$8n + 208$
D_2	0	1	0	1	$n-2$	0	0	$8n + 136$
D_3	0	0	2	0	$n-2$	0	0	$8n + 112$
D_4	0	0	1	2	$n-3$	0	0	$8n + 94$
D_5	0	0	0	4	$n-4$	0	0	$8n + 76$

Table 5. Degree distributions of connected tricyclic graphs with $n_1 = 1$.

Eq.Cl.	n_7	n_6	n_5	n_4	n_3	n_2	n_1	n_i $(i \geq 8)$	F
E_1	1	0	0	0	0	$n-2$	1	0	$8n + 328$
E_2	0	1	0	0	1	$n-3$	1	0	$8n + 220$
E_3	0	0	1	1	0	$n-3$	1	0	$8n + 166$
E_4	0	0	1	0	2	$n-4$	1	0	$8n + 148$
E_5	0	0	0	2	1	$n-4$	1	0	$8n + 124$
E_6	0	0	0	1	3	$n-5$	1	0	$8n + 106$
E_7	0	0	0	0	5	$n-6$	1	0	$8n + 88$

Table 6. Degree distributions of connected tricyclic graphs with $n_1 = 2$.

Eq.Cl.	n_8	n_7	n_6	n_5	n_4	n_3	n_2	n_1	n_i $(i \geq 9)$	F
F_1	1	0	0	0	0	0	$n-3$	2	0	$8n + 490$
F_2	0	1	0	0	0	1	$n-4$	2	0	$8n + 340$
F_3	0	0	1	0	1	0	$n-4$	2	0	$8n + 250$
F_4	0	0	1	0	0	2	$n-5$	2	0	$8n + 232$
F_5	0	0	0	2	0	0	$n-4$	2	0	$8n + 220$
F_6	0	0	0	1	1	1	$n-5$	2	0	$8n + 178$
F_7	0	0	0	1	0	3	$n-6$	2	0	$8n + 160$
F_8	0	0	0	0	3	0	$n-5$	2	0	$8n + 154$
F_9	0	0	0	0	2	2	$n-6$	2	0	$8n + 136$
F_{10}	0	0	0	0	1	4	$n-7$	2	0	$8n + 118$
F_{11}	0	0	0	0	0	6	$n-8$	2	0	$8n + 100$

Table 7. Degree distributions of connected tetracyclic graphs with $n_1 = 0$.

Eq.Cl.	n_8	n_7	n_6	n_5	n_4	n_3	n_2	n_1	n_i $(i \geq 9)$	F
H_1	1	0	0	0	0	0	$n-1$	0	0	$8n + 504$
H_2	0	1	0	0	0	1	$n-2$	0	0	$8n + 354$
H_3	0	0	1	0	1	0	$n-2$	0	0	$8n + 264$
H_4	0	0	1	0	0	2	$n-3$	0	0	$8n + 246$
H_5	0	0	0	2	0	0	$n-2$	0	0	$8n + 234$
H_6	0	0	0	1	1	1	$n-3$	0	0	$8n + 192$
H_7	0	0	0	1	0	3	$n-4$	0	0	$8n + 174$
H_8	0	0	0	0	3	0	$n-3$	0	0	$8n + 168$
H_9	0	0	0	0	2	2	$n-4$	0	0	$8n + 150$
H_{10}	0	0	0	0	1	4	$n-5$	0	0	$8n + 132$
H_{11}	0	0	0	0	0	6	$n-6$	0	0	$8n + 114$

Table 8. Degree distributions of connected tetracyclic graphs with $n_1 = 1$.

Eq.Cl.	n_9	n_8	n_7	n_6	n_5	n_4	n_3	n_2	n_1	n_i $(i \geq 10)$	F
I_1	1	0	0	0	0	0	0	$n-2$	1	0	$8n + 714$
I_2	0	1	0	0	0	0	1	$n-3$	1	0	$8n + 516$
I_3	0	0	1	0	0	1	0	$n-3$	1	0	$8n + 384$
I_4	0	0	1	0	0	0	2	$n-4$	1	0	$8n + 366$
I_5	0	0	0	1	1	0	0	$n-3$	1	0	$8n + 318$
I_6	0	0	0	1	0	1	1	$n-4$	1	0	$8n + 276$
I_7	0	0	0	1	0	0	3	$n-5$	1	0	$8n + 258$
I_8	0	0	0	0	2	0	1	$n-4$	1	0	$8n + 246$
I_9	0	0	0	0	1	2	0	$n-4$	1	0	$8n + 222$
I_{10}	0	0	0	0	1	1	2	$n-5$	1	0	$8n + 204$
I_{11}	0	0	0	0	1	0	4	$n-6$	1	0	$8n + 186$
I_{12}	0	0	0	0	0	3	1	$n-5$	1	0	$8n + 180$
I_{13}	0	0	0	0	0	2	3	$n-6$	1	0	$8n + 162$
I_{14}	0	0	0	0	0	1	5	$n-7$	1	0	$8n + 144$
I_{15}	0	0	0	0	0	0	7	$n-8$	1	0	$8n + 125$

Table 9. Degree distributions of connected pentacyclic graphs with $n_1 = 0$.

Eq.Cl.	n_{10}	n_9	n_8	n_7	n_6	n_5	n_4	n_3	n_2	n_1	n_i $(i \geq 11)$	F
K_1	1	0	0	0	0	0	0	0	$n-1$	0	0	$8n + 992$
K_2	0	1	0	0	0	0	0	1	$n-2$	0	0	$8n + 740$
K_3	0	0	1	0	0	0	1	0	$n-2$	0	0	$8n + 560$
K_4	0	0	1	0	0	0	0	2	$n-3$	0	0	$8n + 542$
K_5	0	0	0	1	0	1	0	0	$n-2$	0	0	$8n + 452$
K_6	0	0	0	1	0	0	1	1	$n-3$	0	0	$8n + 410$
K_7	0	0	0	1	0	0	0	3	$n-4$	0	0	$8n + 392$
K_8	0	0	0	0	2	0	0	0	$n-2$	0	0	$8n + 416$
K_9	0	0	0	0	1	1	0	1	$n-3$	0	0	$8n + 344$
K_{10}	0	0	0	0	1	0	2	0	$n-3$	0	0	$8n + 320$
K_{11}	0	0	0	0	1	0	1	2	$n-4$	0	0	$8n + 302$
K_{12}	0	0	0	0	1	0	0	4	$n-5$	0	0	$8n + 284$
K_{13}	0	0	0	0	0	2	1	0	$n-3$	0	0	$8n + 290$
K_{14}	0	0	0	0	0	2	0	2	$n-4$	0	0	$8n + 272$
K_{15}	0	0	0	0	0	1	2	1	$n-4$	0	0	$8n + 248$
K_{16}	0	0	0	0	0	1	1	3	$n-5$	0	0	$8n + 230$
K_{17}	0	0	0	0	0	1	0	5	$n-6$	0	0	$8n + 212$
K_{18}	0	0	0	0	0	0	4	0	$n-4$	0	0	$8n + 224$
K_{19}	0	0	0	0	0	0	3	2	$n-5$	0	0	$8n + 206$
K_{20}	0	0	0	0	0	0	2	4	$n-6$	0	0	$8n + 188$
K_{21}	0	0	0	0	0	0	1	6	$n-7$	0	0	$8n + 170$
K_{22}	0	0	0	0	0	0	0	8	$n-8$	0	0	$8n + 152$

Table 10. Degree distributions of connected pentacyclic graphs with n_1 = 1

Eq.Cl.	n_{11}	n_{10}	n_9	n_8	n_7	n_6	n_5	n_4	n_3	n_2	n_1	n_i ($i \geq 12$)	F
L_1	1	0	0	0	0	0	0	0	0	$n-2$	1	0	$8n + 1316$
L_2	0	1	0	0	0	0	0	0	1	$n-3$	1	0	$8n + 1004$
L_3	0	0	1	0	0	0	0	1	0	$n-3$	1	0	$8n + 770$
L_4	0	0	1	0	0	0	0	0	2	$n-4$	1	0	$8n + 752$
L_5	0	0	0	1	0	0	1	0	0	$n-3$	1	0	$8n + 614$
L_6	0	0	0	1	0	0	0	1	1	$n-4$	1	0	$8n + 572$
L_7	0	0	0	1	0	0	0	0	3	$n-5$	1	0	$8n + 554$
L_8	0	0	0	0	1	1	0	0	0	$n-3$	1	0	$8n + 536$
L_9	0	0	0	0	1	0	1	0	1	$n-4$	1	0	$8n + 464$
L_{10}	0	0	0	0	1	0	0	2	0	$n-4$	1	0	$8n + 440$
L_{11}	0	0	0	0	1	0	0	1	2	$n-5$	1	0	$8n + 422$
L_{12}	0	0	0	0	1	0	0	0	4	$n-6$	1	0	$8n + 404$
L_{13}	0	0	0	0	0	2	0	0	1	$n-4$	1	0	$8n + 428$
L_{14}	0	0	0	0	0	1	1	1	0	$n-4$	1	0	$8n + 374$
L_{15}	0	0	0	0	0	1	0	2	1	$n-5$	1	0	$8n + 332$
L_{16}	0	0	0	0	0	1	0	1	3	$n-6$	1	0	$8n + 314$
L_{17}	0	0	0	0	0	1	0	0	5	$n-7$	1	0	$8n + 296$
L_{18}	0	0	0	0	0	0	3	0	0	$n-4$	1	0	$8n + 344$
L_{19}	0	0	0	0	0	0	2	1	1	$n-5$	1	0	$8n + 302$
L_{20}	0	0	0	0	0	0	2	0	3	$n-6$	1	0	$8n + 284$
L_{21}	0	0	0	0	0	0	1	3	0	$n-5$	1	0	$8n + 278$
L_{22}	0	0	0	0	0	0	1	2	2	$n-6$	1	0	$8n + 260$
L_{23}	0	0	0	0	0	0	1	1	4	$n-7$	1	0	$8n + 242$
L_{24}	0	0	0	0	0	0	1	0	7	$n-8$	1	0	$8n + 243$
L_{25}	0	0	0	0	0	0	0	4	1	$n-6$	1	0	$8n + 236$
L_{26}	0	0	0	0	0	0	0	3	3	$n-7$	1	0	$8n + 218$
L_{27}	0	0	0	0	0	0	0	2	5	$n-8$	1	0	$8n + 200$
L_{28}	0	0	0	0	0	0	0	1	7	$n-9$	1	0	$8n + 182$
L_{29}	0	0	0	0	0	0	0	0	9	$n-10$	1	0	$8n + 164$

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