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Neighbourly Irregular Derived Graphs

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ABSTRACT

A connected graph G is said to be neighbourly irregular graph if no two adjacent vertices of G have same degree. In this paper, we obtain neighbourly irregular derived graphs such as semitotal-point graph, *k-th* semitotal-point graph, semitotal-line graph, paraline graph, quasi-total graph and quasivertex-total graph and also neighbourly irregular of some graph products.

1. INTRODUCTION AND PRELIMINARIES

In this paper, we are concerned with finite, simple, connected graph G with vertex set $V(G)=\{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)=\{e_1, e_2, \ldots, e_m\}$. If v_i and v_j are vertices of G, then the edge connecting them will be denoted by v_iv_j. The *degree* of a vertex v in G is denoted by $d_G(v)$. The *complement* of G, denoted by $\overline{G}$, is a graph which has the same vertex set as G, in which two vertices are adjacent if and only if they are not adjacent in G and $d_{\overline{G}}(v) = n - 1 - d_G(v)$ holds for all $v \in V(G)$. Definitions not given here may be found in [4].

A graph G is said to be *regular* if all its vertices have the same degree. A connected graph G is said to be *highly irregular* if each neighbor of any vertex has different degree [1]. The graph G is said to be *neighbourly irregular graph*, abbreviated as *NI* graph, if no

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two adjacent vertices of G have the same degree. This concept was introduced by Bhragsam and Ayyaswamy [2]. In [2, 12], authors constructed NI graphs of order n for a given n and a partition of n with distinct parts and proved some properties of NI graphs related to graphoidal covering number, gracefulness, ply number, lace number, clique graph, minimal edge covering and studied the neighbourly irregularity of some graph products.

The *line graph* $L(G)$ of a graph G is the graph with vertex set as the edge set of G and two vertices of $L(G)$ are adjacent whenever the corresponding edges in G have a vertex in common. The *subdivision graph* $S(G)$ of a graph G whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if one is a vertex of G and other is an edge of G incident with it.

2. DERIVED GRAPHS

In this paper we considered the following graphs derived from the parent graph G:

1. The ***semitotal-point graph*** $T_2(G)$ as the graph [8] whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if (i) they are adjacent vertices of G or (ii) one is a vertex of G and other is an edge of G incident with it. If u is a vertex of G, then $d_{T_2(G)}(u) = 2d_G(u)$. If e is an edge of G, then $d_{T_2(G)}(e) = 2$.
2. The ***k-th semitotal-point graph*** $T_2^k(G)$ of G [6] is the graph obtained by adding k vertices to each edge of G and joining them to the endvertices of the respective edge. Obviously, this is equivalent to adding k triangles to each edge of G.
3. The ***semitotal-line graph*** $T_1(G)$ as the graph [8] whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if (i) they are adjacent edges of G or (ii) one is a vertex of G and other is an edge of G incident with it. If u is a vertex of G, then $d_{T_1(G)}(u) = d_G(u)$. If $e=uv$ is an edge of G, then $d_{T_1(G)}(e) = d_G(u) + d_G(v)$.
4. The ***paraline graph*** $PL(G)$ is a line graph of subdivision graph of G.
5. The ***quasi-total graph*** $P(G)$ as the graph [9] whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if (i) they are nonadjacent vertices of G or (ii) they are adjacent edges of G or (iii) one is a vertex of G and other is an edge of G incident with it. If u is a vertex of G, then $d_{P(G)}(u) = n - 1$. If $e = uv$ is an edge of G, then $d_{P(G)}(u) = d_G(u) + d_G(v)$.
6. The ***quasivertex-total graph*** $Q(G)$ as the graph [7] whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if (i) they are adjacent vertices of G or (ii) they are nonadjacent vertices of G (iii) they are adjacent edges of G or (iv) one is a vertex of G and other is an edge of G incident with it. If u is a vertex of G, then $d_{Q(G)}(u) = n - 1 + d_G(u)$. If $e = uv$ is an edge of G, then $d_{Q(G)}(e) = d_G(u) + d_G(v)$.

In Figure 1 self-explanatory examples of these derived graphs are depicted.

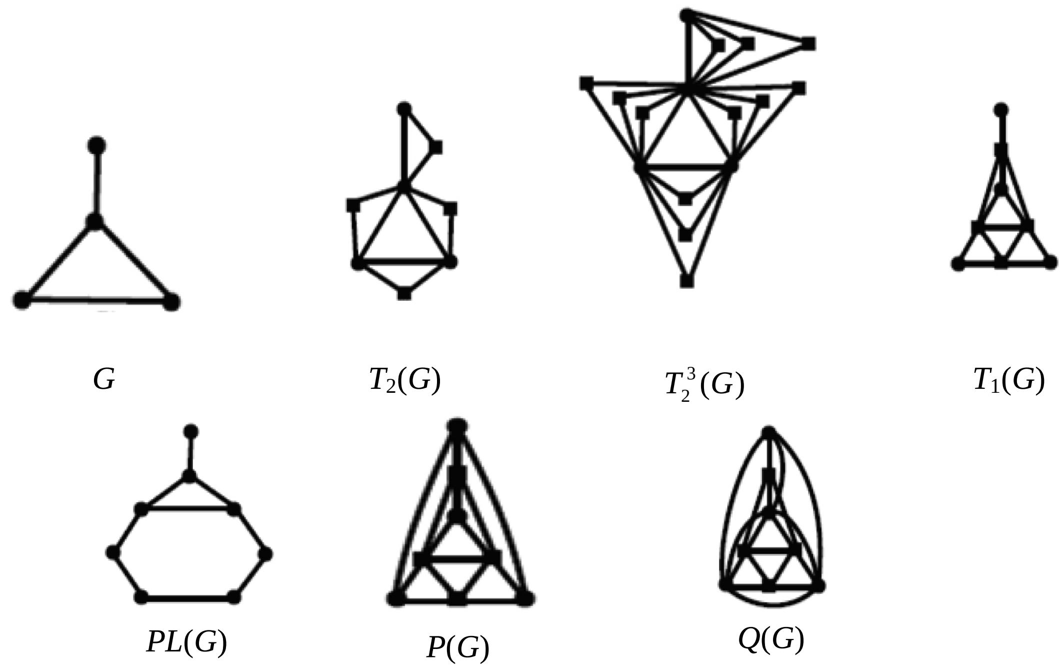

Figure 1. Various graphs derived from the graph G and $T_2^3(G)$ is *k-th* semitotal-point graph of G for $k = 3$.

The vertices of derived graphs depicted in Figure 1 except from the paraline graph *PL*, corresponding to the vertices of the parent graph *G*, are indicated by circles. The vertices of these graphs corresponding to the edges of the parent graph *G* are indicated by squares. In this paper we obtain neighbourly irregular derived graphs.

Theorem 2.1 [12] Let G be a graph. The subdivision graph S(G) is NI if and only if G does not have any vertex of degree two.

Theorem 2.2 [12] For any graph G, its line graph L(G) is NI graph if and only if N(u) contains all vertices of different degree for all $u \in V(G)$.

Theorem 2.3 [2] If G is NI graph, then $\overline{G}$ is not NI graph.

Theorem 2.4 [12] If G is NI graph, then L(G) is not NI graph.

Theorem 2.5 [12] For each integer $k \geq 1$, there exist a graph G with maximum degree $\Delta(G) = k$ such that L(G) is NI graph.

3. RESULTS

Theorem 3.1 For any graph G, the semitotal-point graph $T_2(G)$ is NI if and only if G is NI graph and no vertex of degree one is in G.

Proof. Suppose G is NI graph and no vertex of degree one is in G. In $T_2(G)$, let e = xy be an edge. Then x, y ∈ V(G) or x ∈ V(G) and y ∈ E(G).

(a) x, y ∈ V(G). Since $d_G(x) \neq d_G(y)$, $d_{T_2(G)}(x) = 2d_G(x) \neq 2d_G(y) = d_{T_2(G)}(y)$.

(b) x ∈ V(G) and y ∈ E(G). Since no vertex of degree is one in G and $d_{T_2(G)}(y) = 2$, $d_{T_2(G)}(x) = 2d_G(x) \neq 2 = d_{T_2(G)}(y)$. Thus from all the cases $T_2(G)$ is NI graph.

Conversely, suppose G is not NI graph. Then $d_G(x) = d_G(y)$ for some vertices x and y are adjacent in G. So, $d_{T_2(G)}(x) = d_{T_2(G)}(y)$. A contradiction to $T_2(G)$ is NI graph. Suppose $d_G(v) = 1$ for some v ∈ V(G). Let e = vy be an edge in $T_2(G)$. Then $d_{T_2(G)}(v) = 2d_G(v) = 2 = d_{T_2(G)}(y)$. Again a contradiction to $T_2(G)$ is NI graph. □

Theorem 3.2 For any graph G, the k^{th} semitotal-point graph is NI if and only if G is NI graph and $k \geq 2$.

Proof. The proof of this theorem is similar to the proof of the Theorem 3.1, so is omitted. □

Theorem 3.3 For any graph G, its $T_1(G)$ is NI if and only if L(G) is NI graph.

Proof. Suppose L(G) is NI graph. In $T_1(G)$, let e = xy be an edge. Then x, y ∈ E(G) or x ∈ V(G) and y ∈ E(G).

(a) x, y ∈ E(G). Let $x = v_iv_j$ and $y = v_iv_k$, so that x and y are adjacent in $T_1(G)$. Since L(G) is NI graph, we have $d_{L(G)}(x) \neq d_{L(G)}(y)$, $d_G(v_i) + d_G(v_j) - 2 \neq d_G(v_i) + d_G(v_k) - 2$ or $d_G(v_i) + d_G(v_j) \neq d_G(v_i) + d_G(v_k)$. Therefore $d_{T_1(G)}(x) \neq 2d_{T_1(G)}(y)$.

(b) x ∈ V(G) and y ∈ E(G). Let $e = xy = v_ie_j$ for some $v_i \in V(G)$ and $e_j \in E(G)$. Therefore $d_{T_1(G)}(x) = d_{T_1(G)}(v_i) = d_G(v_i)$ and $d_{T_1(G)}(y) = d_{T_1(G)}(e_j) = d_G(v_i) + d_G(v_k)$ where $e_j = v_iv_k \neq d_G(v_i)$ as $d_G(v_k) \neq 0 = d_G(x) = d_{T_1(G)}(x)$. Therefore for every pair of adjacent vertices in $T_1(G)$ have different degree. Thus $T_1(G)$ is NI graph.

Conversely, suppose L(G) is not NI graph. Then $d_{L(G)}(e_i) = d_{L(G)}(e_j)$ for some $e_i = v_rv_s$ and $e_j = v_rv_k$ are adjacent vertices in L(G). Hence, $d_G(v_r) + d_G(v_s) - 2 = d_G(v_r) + d_G(v_k) - 2$, $d_G(v_r) + d_G(v_s) = d_G(v_r) + d_G(v_k)$. Therefore $d_{T_1(G)}(e_i) = d_{T_1(G)}(e_j)$. A contradiction to $T_1(G)$ is NI graph. □

From Theorems 2.4, 2.5 and 3.3, we have the following corollaries.

Corollary 3.4 If G is NI graph, then $T_1(G)$ is not NI graph.

Corollary 3.5 For each integer $k \geq 1$, there exists a graph G with maximum degree $\Delta(G) = k$ such that $T_1(G)$ is NI graph.

Theorem 3.6 For any graph $G \neq K_2$, the paraline graph PL(G) is not NI graph.

Proof. Let v be a vertex of degree at least two in G. Then neighbourhood of v in S(G) has at least two vertices of degree two. By Theorem 2.2, L(S(G))=PL(G) is not NI graph. □

Theorem 3.7. For any graph $G \neq K_2$, the quasi-total graph P(G) is not NI graph.

Proof. Let $G \neq K_2$ be a graph. We have the following cases:

Case 1. If G is not a complete graph, then there exist at least two vertices $u, v \in V(G)$ such that $d_{P(G)}(u) = d_{P(G)}(v) = n - 1$. Therefore P(G) is not NI graph.

Case 2. If G is a complete graph, then there exist at least two edges $e_i, e_j \in E(G)$ such that $d_{P(G)}(e_i) = d_{P(G)}(e_j)$. Therefore P(G) is not NI graph. □

Theorem 3.8 For any graph G with n vertices, the quasivertex-total graph Q(G) is NI if and only if G, $\overline{G}$ and L(G) all are NI graphs and $\Delta(G) \neq n - 1$.

Proof. Suppose G, $\overline{G}$ and L(G) all are NI graphs. In Q(G), let e = xy be an edge, then $x, y \in V(G)$ or $x, y \in V(\overline{G})$ or $x, y \in E(G)$ or $x \in V(G)$ and $y \in E(G)$.

(a) $x, y \in V(G)$. Since $d_G(x) \neq d_G(y)$, $d_{Q(G)}(x) = n - 1 + d_G(x) \neq n - 1 + d_G(y) = d_{Q(G)}(y)$.

(b) $x, y \in V(\overline{G})$. Since $d_{\overline{G}}(x) \neq d_{\overline{G}}(y)$, $d_{Q(G)}(x) = n - 1 + d_G(x) \neq n - 1 + d_G(y) = d_{Q(G)}(y)$.

(c) $x, y \in E(G)$. Let $x = v_iv_j$ and $y = v_iv_k$. So that x and y are adjacent in Q(G). Therefore $d_{Q(G)}(x) = d_G(v_i) + d_G(v_j)$ and $d_{Q(G)}(x) = d_G(v_i) + d_G(v_k)$. But $d_{L(G)}(x) \neq d_{L(G)}(y)$ as L(G) is NI graph, $d_{L(G)}(x) = d_G(v_i) + d_G(v_j) - 2$ and $d_{L(G)}(y) = d_G(v_i) + d_G(v_k) - 2$. Therefore $d_{Q(G)}(x) \neq d_{Q(G)}(y)$.

(d) $x \in V(G)$ and $y \in E(G)$. Let $e = xy = v_ie_j$ for some $v_i \in V(G)$ and $e_j \in E(G)$. Then $d_{Q(G)}(y) = d_{Q(G)}(e_j) = d_{L(G)}(e_j) + 2$ where $e_j = v_iv_j = d_G(v_i) + d_G(v_j) \neq n - 1 + d_G(v_i)$ as $\Delta(G) \neq n - 1 \neq d_{Q(G)}(x)$. Thus in all the cases Q(G) is NI graph.

Conversely, suppose Q(G) is NI graph. We have to prove that G, $\overline{G}$ and L(G) are all NI graphs. If G is not NI graph, then there exists an edge $e_k = v_iv_j$ in G such that $d_G(v_i) = d_G(v_j)$. Therefore $n - 1 + d_G(v_i) = n - 1 + d_G(v_j)$. So, $d_{Q(G)}(v_i) = d_{Q(G)}(v_j)$. A contradiction to Q(G) is NI graph. Suppose $\overline{G}$ is not NI graph, then there exists an edge $e_k = v_iv_j$ in $\overline{G}$ such that $d_{\overline{G}}(v_i) = d_{\overline{G}}(v_j)$. Therefore $n - 1 + d_G(v_i) = n - 1 + d_G(v_j)$ and so $d_{Q(G)}(v_i) = d_{Q(G)}(v_j)$. A contradiction to Q(G) is NI graph.

Suppose L(G) is not NI graph, then there exists two adjacent vertices $e_i = v_rv_s$ and $e_j = v_rv_k$ in L(G) with $d_{L(G)}(e_i) = d_{L(G)}(e_j)$. Thus $d_G(v_r) + d_G(v_s) - 2 = d_G(v_r) + d_G(v_k) - 2$. Hence $d_G(v_r) + d_G(v_s) = d_G(v_r) + d_G(v_k)$ and so $d_{Q(G)}(e_i) = d_{Q(G)}(e_j)$. Again a contradiction to Q(G) is NI graph. Suppose $\Delta(G) = n - 1 = d_G(v)$ and let e = uv be an edge. Then $d_{Q(G)}(e) = d_{Q(G)}(u)$. Again a contradiction to Q(G) is NI graph. □

From Theorems 2.3, 2.4 and 3.8 we have following result.

Theorem 3.9 There is no nontrivial graph G whose quasivertex-total graph Q(G) is NI graph.

4. NEIGHBOURLY IRREGULAR GRAPH PRODUCTS

The corona [10] of two graphs G and H is the graph obtained by taking one copy of G, |V(G)| copies of H and joining each i-th vertex of G to every vertex in the i-th copy of H. The edge corona [5] of two graphs G and H denoted by $G \lozenge H$ is obtained by taking one copy of G and |E(G)| copies of H and joining each end vertices of i-th edge of G to every vertex in the i-th copy of H.

Theorem 4.1 Let G and H be nontrivial graphs. Then $G \lozenge H$ is NI graph if and only if both G and H are NI graphs and, G does not have pendent vertex or $\Delta(H) < |V(H)| - 1$, where $\Delta(H)$ is the maximum degree of the vertices of H.

Proof. To prove the result, we have to present some notations. Let G′ be the copy of G and H_i be the i-th copy of H in $G \lozenge H$, $1 \le i \le |E(G)|$. A vertex of $G \lozenge H$ corresponding to the vertex u in H is denoted by u′. Also, we denote a vertex of $G \lozenge H$ corresponding to the vertex v in G by v′.

Let G and H be NI graphs and, G does not have pendent vertex or $\Delta(H) < |V(H)| - 1$. Then it is clear that $G \lozenge H$ is NI graph.

Conversely, let G and H be two nontrivial graphs and $G \lozenge H$ is NI graph. Suppose $u'v' \in E(G \lozenge H)$ such that $u', v' \in V(H_i)$, then $d_{G\lozenge H}(u') - d_{G\lozenge H}(v') = d_H(u) - d_H(v) \neq 0$ and so H is NI graph. Also, if $u'v' \in E(G \lozenge H)$ such that $u', v' \in V(G')$, then $d_{G\lozenge H}(u') - d_{G\lozenge H}(v') = (|V(H)| + 1)(d_G(u) - d_G(v)) \neq 0$. Thus, G is NI graph. On the other hand, if $u'v' \in E(G \lozenge H)$ such that $u' \in V(G')$, and $v' \in V(H_i)$, then $d_{G\lozenge H}(u') - d_{G\lozenge H}(v') = (|V(H)| + 1)$

$d_G(u) - (d_H(v) + 2) \neq 0$ and it shows that, G does not have pendent vertex or $\Delta(H) < |V(H)| - 1$. □

To present the next results, we need two definitions as follows: The cluster G{H} is obtained by taking one copy of G and |V(G)| copies of a rooted graph H, and by identifying the root of the i-th copy of H with the i-th vertex of G, i = 1, 2, …, |V(G)| [11].

Suppose G and H are graphs with disjoint vertex sets. Following Došlić [3], for given vertices $y \in V(G)$ and $z \in V(H)$ a splice of G and H by vertices y and z, $(G \cdot H)(y, z)$, is defined by identifying the vertices y and z in the union of G and H.

Theorem 4.2 Let G and H be graphs. Then G{H} is NI graph if and only if both G and $(H \cdot S_{d_G(u_i)})(r, x)$ are NI graphs, for each i = 1, 2, …, |V(G)|, where x is the vertex with maximum degree of the star $S_{d_G(u_i)}$ and r the root vertex of H.

Proof. Let G and $(H \cdot S_{d_G(u_i)})(r, x)$ be NI graphs, for each i = 1, 2, …, |V(G)|, where x is the vertex with maximum degree of the star $S_{d_G(u_i)}$ and r the root vertex of H. Then, it is clear that G{H} is NI graph.

Conversely, let G{H} be NI graph. Also, suppose $u'v' \in E(G\{H\})$ and u′, v′ are the vertices of G{H} corresponding to the vertices u, v in G, respectively. If u′ and v′ are vertices of a copy of G, then $d_{G\{H\}}(u') - d_{G\{H\}}(v') = d_G(u) - d_G(v) \neq 0$. So G is NI graph. On the other hand, suppose $u'v' \in E(G\{H\})$ and u′, v′ are the vertices of $G\{H\} \cap H_i$ corresponding to the vertices u, v in H, respectively. Then, it is not difficult to see that $dG\{H\}(u') - dG\{H\}(v') \neq 0$ if and only if

$$d_{(H \cdot S_{d_G(u_i)})(r,x)}(u) - d_{(H \cdot S_{d_G(u_i)})(r,x)}(v) \neq 0 .$$

So, $(H \cdot S_{d_G(u_i)})(r, x)$ is NI graph. □

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