Iranian J. Math. Chem. 8 (4) December (2017) 403 − 411

Iranian Journal of Mathematical Chemistry

Journal homepage: ijmc.kashanu.ac.ir

The Uniqueness Theorem for Inverse Nodal Problems with a Chemical Potential

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ARTICLE INFO

Article History:

Received 3 February 2016
Accepted 16 April 2016
Published online 5 November 2016
Academic Editor: Ali Reza Ashrafi

Keywords:

Boundary Value Problem
Inverse Nodal Problem
Eigenvalues
Nodal Points

ABSTRACT

In this paper, an inverse nodal problem for a second-order differential equation having a chemical potential on a finite interval is investigated. First, we estimate the nodal points and nodal lengths of differential operator. Then, we show that the potential can be uniquely determined by a dense set of nodes of the eigenfunctions.

1 INTRODUCTION

There are many problems in mathematics, chemistry, physics and some engineering sciences which are connected to the second-order differential equations. For example, in the process of the formation of *methyliodide* (CH_3I) by the biological and photochemical production mechanisms in a biogeochemical module, the following equation appears:

$$\frac{dc}{dt} = P - S + F_{air-Sea} + \frac{\partial}{\partial z}(A_\nu \frac{dc}{dz}), \tag{1}$$

which describes the evolution of methyl iodide concentration (c [mmolm^{-3}]) over time under production (P), degradation (S), air–sea exchange (F), as well as turbulent vertical diffusion (A_ν–diffusion coefficient) (see [26]). Using the separation of variables technique we can transform the equation (1) to the following second-order differential equation:

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DOI: 10.22052/ijmc.2016.39228

$$y'' + \left(\lambda - \frac{A}{x^2} + q(x)\right)y = 0, \tag{2}$$

where λ is the spectral parameter, A is a real number, the potential $q(x)$ is real-valued. Equation (2) has a singularity at the endpoint $x = 0$. For other examples, in quantum chemistry or quantum mechanics, we refer to the quantum modeling of the *hydrogen atom*, or the *Hellman equation* to finding an approximation for the simplified description of complex systems, which can be transformed to (2) (see also [3, 4, 6, 13, 15, 17, 24]).

Inverse problems associated with the equation (2) with $A=0$ have various versions. The first version was studied by Borg and Levinson, and it is shown that the potential $q(x)$ can be uniquely determined from the given boundary condition and one possible reduced spectrum [5, 18]. For the second version, using two spectra λ_n and λ'_n, Marchenko uniquely determined the potential $q(x)$ and the corresponding boundary conditions [20]. Finally, Gelfand and Levitan proved that $q(x)$ uniquely determined by the spectral function [12].

Some inverse problems having singularities or turning points, and/or discontinuity conditions were studied by the above methods in many works (see [1, 2, 8-11, 16, 19, 23, 27]). Note that, in [22], we considered a second-order differential equation of Sturm-Liouville type having two turning points and singularities in a finite interval. Then, its asymptotic form of the solutions was studied, and obtained the infinite representation of the solutions of differential equation which plays an important role in investigating the corresponding inverse problem.

In later years, in some interesting works but without singularity, inverse problems were investigated using a new spectral data which are so-called *nodal points*, and their corresponding inverse problems are so-called *inverse nodal problems*. Mclaughlin seems to have been the first to consider this method for the one-dimensional Schrödinger equations [21]. For other works, see also [7, 14, 25].

In this work, we consider the inverse nodal problem associated with the singular differential equation (2) and the Dirichlet boundary condition

$$y(0) = 0 = y(1), \tag{3}$$

on the interval (0,1). We also assume that

$$q(x) \in x^{2-2k_0} \in L^1(0,1), \tag{4}$$

where k_0 is a member of {2,3,4,…}. The problem (2)-(3) has infinitely many nontrivial solutions. The values of λ for which there exist nontrivial solutions are so-called *eigenvalues*, and their corresponding nontrivial solutions $y(x,\lambda)$ are so-called *eigenfunctions*. All the eigenvalues are real and the set of the eigenvalues is countably infinite, and also the eigenvalues can be arranged in increasing order as follows

$$\lambda_1 < \lambda_2 < \lambda_3 < \ldots,$$

such that $\lambda_n \to \infty$ as $n \to \infty$. In the present paper, first, we obtain the asymptotic formula for the eigenvalues, the nodes of the eigenfunctions and the nodal lengths (Section 2). Then, we prove that the set of the nodal points of the boundary value problem (2)–(3) is dense in the interval (0,1) and the potential *q(x)* can be uniquely determined from this new kind of spectral data (see Section 3).

2 ASYMPTOTIC FORMULA FOR NODAL POINTS

We consider the boundary value problem $L=L(q(x))$ defined by (2)-(3). Assume that in (2),

$$A=\nu^2-\frac{1}{4},\ \nu=k_0-\frac{1}{2},\quad k_0\in\{2,3,4,...\}. \tag{5}$$

From [11], we know that the equation (2) has two solutions $y_1(x,\lambda)$ and $y_2(x,\lambda)$, which are linearly independent with respect to x, and also have the following asymptotic forms as $\lambda\to\infty$:

$$y_1(x,\lambda)=\lambda^{(k_0-1)/2}\left\{(-1)^{k_0-1}e^{i\sqrt{\lambda}x}[1]_0+e^{-i\sqrt{\lambda}x}[1]_0\right\}, \tag{6}$$

$$y_2(x,\lambda)=\frac{1}{4}i\lambda^{-k_0/2}\left\{-e^{i\sqrt{\lambda}x}[1]_0+(-1)^{k_0-1}e^{-i\sqrt{\lambda}x}[1]_0\right\}, \tag{7}$$

where $[1]_0=1+O((\sqrt{\lambda}x)^{-1})$. Therefore, the solution $y(x,\lambda)$ of the equation (2) under the condition $y(0)=0$ can be written as a linear combination of y_1 and y_2. Also, since the boundary value problem L is self-adjoint and y_1, y_2 are entire in λ, thus all of the eigenvalues of L are real and simple. In the case when k_0 is odd, it follows from (3), (7) that $y(x,\lambda)=y_2(x,\lambda)$ and the asymptotic form of the eigenvalues as follows

$$\sqrt{\lambda_n(q)}=n\pi+O\left(\frac{1}{n}\right). \tag{8}$$

Similarly, in the case when k_0 is even, we derive from (3), (6) that $y(x,\lambda)=y_1(x,\lambda)$ and also the eigenvalues of L may be calculated as (8).

For the boundary value problem L an analog of Sturm's oscillation theorem is true. More precisely, the eigenfunctions $y_n(x)=y(x,\lambda_n)$ has exactly $n-1$ (simple) zeros inside the interval (0,1), namely:

$$0<x_n^{(1)}<x_n^{(2)}<...<x_n^{(n-1)}<1.$$

The set

$$X_L:=\{x_n^{(j)}\},\quad n\ge 1,\quad j=\overline{1,n-1}, \tag{9}$$

is called the set of nodal points of the problem L. Also, let

$$I_n^{(j)}:=[x_n^{(j)},x_n^{(j+1)}]$$

be the j^{th} nodal domain of the n^{th} eigenfunction y_n, and let

$$\ell_n^{(j)}:=\left|I_n^{(j)}\right|=x_n^{(j+1)}-x_n^{(j)}$$

be the associated *nodal length*. Inverse nodal problems consist in recovering the potential $q(x)$ from the given set X_L of nodal points or from a certain its part.

Now, in the following theorem, we develop asymptotic expressions for nodal points $x_n^{(j)}$ and the nodal lengths $\ell_n^{(j)}$ ($n=1,2,3,\ldots$, $j=1,2\ldots,n-1$) at which y_n, the eigenfunction corresponding to the eigenvalue λ_n of the problem L, vanishes.

Theorem 1. We consider the equation (2) under Dirichlet boundary condition (3). Let $q(x)$ satisfies (4), then the nodal points of the problem L defined by (2)-(3) are

$$\begin{cases} x_n^{(j)} = \dfrac{j}{n} + O\left(\dfrac{1}{n}\right), \\ n = 1,2,3,\ldots, \quad j = 1,2,3,\ldots,n-1, \end{cases} \tag{10}$$

and the nodal lengths are

$$\ell_n^{(j)} = \frac{1}{n} + O\left(\frac{1}{n}\right).$$

Proof. Suppose $\nu=k_0-1/2$ and k_0 is odd. Then, by (7)-(8) and solving $y_2(x,\lambda_n)=0$, we approximate the nodal points of the form (10). Similarly, in the case when k_0 is even, using (6), (8) and from $y_1(x,\lambda_n) = 0$ we arrive at (10). Moreover,

$$\begin{aligned} \ell_n^{(j)} &= x_n^{(j+1)} - x_n^{(j)} \\ &= \left(\frac{j+1}{n} + O\left(\frac{1}{n}\right)\right) - \left(\frac{j}{n} + O\left(\frac{1}{n}\right)\right) \\ &= \frac{1}{n} + O\left(\frac{1}{n}\right). \end{aligned}$$

□

Theorem 1, specially the relation (10), provide the sufficient conditions for the uniqueness theorem in the next section.

3 THE UNIQUENESS THEOREM

In this section, we show that the set of the nodal points $x_n^{(j)}$ of the form (10) is dense in (0,1). Then, we prove a uniqueness theorem for the solution of the inverse nodal problem associated with the boundary value problem L.

First, we consider the equation

$$w''(x,\lambda) + \lambda\, w(x,\lambda) = 0, \quad 0 \le x \le 1, \tag{11}$$

with the boundary conditions

$$w(0,\lambda) = 0 = w(1,\lambda). \tag{12}$$

It is easily shown that the solution of the problem (11)-(12) is $w(x,\lambda) = \sin(\sqrt{\lambda}x)$. Furthermore, the exact eigenvalues of the problem L_0 defined by (11)-(12) are

$$\xi_n = n^2\pi^2, \tag{13}$$

and their corresponding eigenfunctions are

$$w_n(x) = w(x, \xi_n) = \sin(n\pi x). \tag{14}$$

Since for each $n \in \{2,3,4,\ldots\}$ there exist $k \in \{0,1,2,\ldots\}$ and $m \in \{1,2,\ldots,2^k\}$ such that $n=2^{k+1}-m+1$, so according to (13)-(14), the set

$$\left\{ (2^{k+1} - m + 1)^2\pi^2 \;\middle|\; k = 0,1,2,\ldots,\; m = 1,2,\ldots,2^k \right\},$$

consists of all eigenvalues of (11)-(12) except $\xi_1=\pi^2$. Moreover, the eigenfunction corresponding to the eigenvalue $\xi_n=(2^{k+1}-m+1)\pi^2$ is

$$w(x, \xi_n) = \sin((2^{k+1} - m + 1)\pi x),$$

so that $m/(2^{k+1}-m+1)$ is a zero of the eigenfunction $w_n(x)$. Therefore, the set of the nodal points of L_0 is

$$\begin{aligned} X_{L_0} &:= \left\{ \xi_n^j \right\}_{n \ge j,\, j=\overline{1,n-1}} \\ &= \left\{ \frac{m}{2^{k+1} - m + 1} \;\middle|\; k = 0,1,2,\ldots,\; m = 1,2,\ldots,2^k \right\} \cup \{0\}. \end{aligned} \tag{15}$$

Lemma 1. The set X_{L_0}, defined by (15), is dense in [0,1].

Proof. For each fixed $k \in \{0,1,2,\ldots\}$, we have

$$= \left\{ \frac{m}{2^{k+1} - m + 1} \;\middle|\; m = 1,2,\ldots,2^k \right\} = \left\{ \frac{1}{2^{k+1}}, \frac{2}{2^{k+1} - 1}, \frac{3}{2^{k+1} - 2}, \ldots, \frac{2^k}{2^k + 1} \right\}.$$

Moreover,

$$\frac{1}{2^{k+1}} - 0 = \frac{1}{2^{k+1}}, \qquad 1 - \frac{2^k}{2^k + 1} = \frac{1}{2^k + 1}, \tag{16}$$

and for $m=1,2,\ldots,2^k-1$,

$$\begin{aligned} \bar{\ell}_{m,k} &:= \frac{m+1}{2^{k+1} - (m+1) + 1} - \frac{m}{2^{k+1} - m + 1} \\ &= \frac{2^{k+1} + 1}{(2^{k+1} - m) - (2^{k+1} - m + 1)}. \end{aligned}$$

Hence, there exists a sufficiently large number $\bar{k}$ such that for each $k > \bar{k}$ we have

$$\bar{\ell}_{m,k} < \frac{1}{k+1}. \tag{17}$$

Now, let $\bar{x}_{m,k} := m/(2^{k+1} - m+1)$. Then, for each $x \in [0,1]$, there exists m $\in \{1,2,\ldots, 2^k - 1\}$ such that

$$x \in [0, \bar{x}_{1,k}] \ \vee \ x \in [\bar{x}_{m,k}, \bar{x}_{m+1,k}] \ \vee \ x \in [\bar{x}_{2^k,k}, 1]. \tag{18}$$

On the other hand, the right sides of equations (16) and (17) tend to zero as $k \to \infty$. This together with the equation (18) completes the proof. □

Theorem 2. The set of the nodal points of the boundary value problem L, X_L, is dense in the interval (0,1).

Proof. It follows from (15) that the nodal points $\xi_n^{(j)}$ of L_0 have the form

$$\xi_n^{(j)} = \frac{j}{n}, \qquad n \geq 2, \quad j - 1,2,3,\ldots, n-1.$$

Thus, using (10) we obtain

$$x_n^{(j)} = \xi_n^{(j)} + O\left(\frac{1}{n}\right). \tag{19}$$

By (19) and Lemma 1, we conclude that X_L is dense in (0,1). □

Now, we prove the main result of this section.

Theorem 3. Consider the boundary value problems defined by

$$y'' + \left(\lambda - \frac{A}{x^2} + q_i(x)\right) y = 0, \quad i = 1,2, \quad x \in (0,1), \tag{20}$$

and Dirichlet condition

$$y(0) = 0 = y(1). \tag{21}$$

Let q_1, q_2, satisfy the condition (4) and $x_n^{(j)}(q_1) = x_n^{(j)}(q_2)$. Then $q_1 = q_2$ (a.e.).

Proof. First, we consider the case when k_0 is odd, in (5). Let x be an arbitrary, fixed number in the interval [0,1]. Since the set of the nodal points X_L, defined in (9), is dense in the interval (0,1) by Theorem 2, it follows that there exists a subsequence $\{n_k\}$, $k=1,2,3,\ldots$, such that

$$\lim_{k \to \infty} x_{n_k}^{(j)} = x. \tag{22}$$

Let $\tilde{y}_i(x) = y_2(x, \lambda_{n_k}(q_i))$ be the solution of (20)-(21) with the potential $q_i(x)$. Then, using (20) we derive

$$\frac{d}{dx}(\tilde{y}_2\tilde{y}_1'-\tilde{y}_1\tilde{y}_2')(x)=\left\{\ q_1(x)-q_2(x)+\lambda_{n_k}(q_1)-\lambda_{n_k}(q_2)\ \right\}\tilde{y}_1(x)\tilde{y}_2(x)\,. \tag{23}$$

Integrating (23) from 0 to $x(j,n_k)=x_{n_k}^{(j)}:=x_{n_k}^{(j)}(q_1)=x_{n_k}^{(j)}(q_2)$, we get

$$(\tilde{y}_2\tilde{y}_1'-\tilde{y}_1\tilde{y}_2')(x)\,|_0^{x(j,n_k)}=\int_0^{x(j,n_k)}\left\{\ q_1(t)-q_2(t)+\lambda_{n_k}(q_1)-\lambda_{n_k}(q_2)\ \right\}\tilde{y}_1(t)\tilde{y}_2(t)dt\,. \tag{24}$$

Since $\tilde{y}_1(x(j,n_k))=\tilde{y}_2(x(j,n_k))=0$, the left side of (24) is equal to zero for each $k\in\{1,2,3,\ldots\}$. Thus,

$$\int_0^{x(j,n_k)}\left\{\ q_1(t)-q_2(t)+\lambda_{n_k}(q_1)-\lambda_{n_k}(q_2)\ \right\}\tilde{y}_1(t)\tilde{y}_2(t)dt=0\,,$$

for $k=1,2,3,\ldots$. We are done if we can show

$$\int_0^x(q_1(t)-q_2(t))dt=0\,.$$

For this goal, by (8) we have

$$\lambda_{n_k}(q_1)-\lambda_{n_k}(q_2)\to 0 \quad \text{as } k\to\infty\,.$$

Hence, together with (22) and (24) these results imply

$$\lim_{k\to\infty} n_k^2\pi^2\int_0^x(q_1(t)-q_2(t))\tilde{y}_1(t)\tilde{y}_2(t)dt=0\,. \tag{25}$$

Moreover, it follows from (7) that there exists a constant C such that for sufficiently large k, we have

$$\left|\tilde{y}_1(x)\tilde{y}_2(x)-(n_k\pi)^{-2}\sin^2(n_k\pi x)\right|<C(n_k\pi)^{-3}\,.$$

So,

$$n_k^2\pi^2\tilde{y}_1(x)\tilde{y}_2(x)\approx\sin^2(n_k\pi x)\,,\qquad k\to\infty\,. \tag{26}$$

Therefore, by (25)–(26) we get

$$\int_0^x(q_1(t)-q_2(t))dt=0\,. \tag{27}$$

Finally, since x was chosen arbitrary in the interval [0,1], together with (27) this yields $q_1=q_2$ (a.e.). In the case when k_0 is even, Theorem 3 can be proved similarly, by (6) and the same way as above. □

Theorem 3 shown that the solution of the inverse nodal problem associated with (2)−(3), the potential function *q(x)*, can be uniquely determined by a dense set of nodes of the eigenfunctions.

ACKNOWLEDGMENT This research is partially supported by the University of Kashan under grant number 464151/1.

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