A Note on Hyper–Zagreb Index of Graph Operations

B. BASAVANAGOUD AND S. PATIL

Department of Mathematics, Karnatak University, Dharwad – 580 003 Karnataka, India

Correspondence should be addressed to b.basavanagoud@gmail.com (B. Basavanagoud)

Received 15 October 2015; Accepted 29 November 2015

ACADEMIC EDITOR: ALI REZA ASHRAFI

ABSTRACT In this paper, the hyper-Zagreb index of the Cartesian product, composition and corona product of graphs are computed. These results correct some errors in G. H. Shirdel et al. [*Iranian J. Math. Chem.* **4** (2) (2013) 213–220].

KEYWORDS Hyper-Zagreb index • Zagreb index • graph operation.

1. INTRODUCTION

Throughout this paper, we consider only simple connected graphs. Let G be such a graph with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex $w \in V(G)$ is the number of vertices adjacent to w and is denoted by $d_G(w)$. We refer to [11] for unexplained terminology and notation.

In theoretical chemistry, the physico-chemical properties of chemical compounds are often modeled by means of molecular-graph-based structure-descriptors, which are also referred to as topological indices [10, 15]. The Zagreb indices are widely studied degree-based topological indices, and were introduced by Gutman and Trinajstic′ [9] in 1972. The first and the second Zagreb indices of a graph G are respectively defined as

$$M_1(G) = \sum_{u\in V(G)} d_G(u)^2 \text{ and } M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v).$$

The first Zagreb index can also be expressed as a sum over edges of G,

$$M_1(G) = \sum_{uv\in E(G)} [d_G(u) + d_G(v)].$$

Recently, G.H. Shirdel, H. Rezapour and A.M. Sayadi [14] introduced a new version of Zagreb index named hyper-Zagreb index which is defined for a graph G as

$$HM(G) = \sum_{uv\in E(G)} (d_G(u) + d_G(v))^2.$$

Some new results on the hyper-Zagreb index can be found in [7, 8].

The Cartesian product $G \times H$ of graphs G and H has the vertex set $V(G \times H) = V(G) \times V(H)$ and $(a,x)(b,y)$ is an edge of $G \times H$ if $a = b$ and $xy \in E(H)$, or $ab \in E(G)$ and $x = y$. If (a,x) is a vertex of $G \times H$, then $d_{G\times H}((a,x)) = d_G(a) + d_H(x)$.

The composition $G[H]$ of graphs G and H with disjoint vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$ is the graph with vertex set $V(G) \times V(H)$ and (a,x) is adjacent to (b,y) whenever a is adjacent to b or $a = b$ and x is adjacent to y. If (a,x) is a vertex of $G[H]$, then $d_{G[H]}((a,x)) = |V(H)|d_G(a) + d_H(x)$.

The corona product $G \circ H$ is defined as the graph obtained from G and H by taking one copy of G and $|V(G)|$ copies of H and then by joining with an edge each vertex of the i^{th} copy of H which is named (H, i) with the i^{th} vertex of G for $i = 1,2,\dots,|V(G)|$. If u is a vertex of $G \circ H$, then

$$d_{G\circ H}(u) = \begin{cases} d_G(u) + |V(H)| & if\ u \in V(G), \\ d_H(u) + 1 & if\ u \in V(H, i). \end{cases}$$

G. H. Shirdel et al. [14] computed the hyper-Zagreb index of some graph operations. However, the formulae of Theorem 2, Theorem 3, and Theorem 4 of their paper for computing the hyper-Zagreb index of Cartesian product, composition, and corona product are incorrect. In this paper, we give correct expressions for the hyper-Zagreb index of the Cartesian product, composition and corona product of graphs. Readers interested in more information on computing topological indices of graph operations can be referred to [1–6, 12, 13].

2. RESULTS

Theorem 2.1 *Let G and H be graphs. Then*
$HM(G \times H) = |V(G)|HM(H) + |V(H)|HM(G) + 12M_1(G)|E(H)| + 12M_1(H)|E(G)|$.

Proof. By definition of the hyper-Zagreb index, we have

$$\begin{aligned} HM(G \times H) &= \textstyle\sum_{(a,x)(b,y)\in E(G\times H)} [d_{G\times H}((a,x)) + d_{G\times H}((b,y))]^2 \\ &= \textstyle\sum_{a\in V(G)} \sum_{xy\in E(H)} [d_G(a) + d_H(x) + d_G(a) + d_H(y)]^2 \\ &+ \textstyle\sum_{x\in V(H)} \sum_{ab\in E(G)} [d_H(x) + d_G(a) + d_H(x) + d_G(b)]^2 \\ &= \textstyle\sum_{a\in V(G)} \sum_{xy\in E(H)} [2d_G(a) + d_H(x) + d_H(y)]^2 \\ &+ \textstyle\sum_{x\in V(H)} \sum_{ab\in E(G)} [2d_H(x) + d_G(a) + d_G(b)]^2 \\ &= \textstyle\sum_{a\in V(G)} \sum_{xy\in E(H)} [4d_G(a)^2 + (d_H(x) + d_H(y))^2 + 4d_G(a)(d_H(x) + d_H(y))] \\ &+ \textstyle\sum_{x\in V(H)} \sum_{ab\in E(G)} [4d_H(x)^2 + (d_G(a) + d_G(b))^2 + 4d_H(x)(d_G(a) + d_G(b))] \\ &= 4|E(H)|M_1(G) + |V(G)|HM(H) + 8|E(G)|M_1(H) \\ &+ 4|E(G)|M_1(H) + |V(H)|HM(G) + 8|E(H)|M_1(G). \end{aligned}$$

□

As an application of Theorem 2.1, we list explicit formulae for the hyper-Zagreb index of the rectangular grid $P_r \times P_s$, C_4 −nanotube $P_r \times C_q$, and C_4 −nanotorus $C_p \times C_q$. The formulae follow from Theorem 2.1 by using the expressions $M_1(P_n) = 4n - 6$, $n > 1$; $M_1(C_n) = 4n$; $HM(P_n) = 16n - 30$, $n > 2$ and $HM(C_n) = 16n$.

Corollary 2.2 $HM(P_r \times P_s) = 128rs - 150r - 150s + 144, r, s > 2$;

$$HM(P_r \times C_q) = 128rq - 150q, r > 2; HM(C_p \times C_q) = 128pq.$$

Theorem 2.3 *Let G and H be graphs. Then*

$$\begin{aligned} HM(G[H]) &= |V(H)|^4 HM(G) + |V(G)|HM(H) \\ &+ 12|V(H)|^2|E(H)|M_1(G) + 10|V(H)||E(G)|M_1(H) + 8|E(H)|^2|E(G)|. \end{aligned}$$

Proof. Using the definition of the hyper-Zagreb index, we have

$$HM(G[H]) = \sum_{(a,x)(b,y)\in E(G[H])} [d_{G[H]}((a,x)) + d_{G[H]}((b,y))]^2$$
$$= \sum_{x\in V(H)} \sum_{y\in V(H)} \sum_{ab\in E(G)} [|V(H)|d_G(a) + d_H(x) + |V(H)|d_G(b) + d_H(y)]^2$$
$$+ \sum_{a\in V(G)} \sum_{xy\in E(H)} [|V(H)|d_G(a) + d_H(x) + |V(H)|d_G(a) + d_H(y)]^2$$
$$= \sum_{x\in V(H)} \sum_{y\in V(H)} \sum_{ab\in E(G)} [|V(H)|^2(d_G(a) + d_G(b))^2 + d_H(x)^2 + d_H(y)^2$$
$$+2d_H(x)d_H(y) + 2|V(H)|(d_G(a) + d_G(b))(d_H(x) + d_H(y))]$$
$$+\sum_{a\in V(G)} \sum_{xy\in E(H)} [4|V(H)|^2 d_G(a)^2 + (d_H(x) + d_H(y))^2$$
$$+4|V(H)|d_G(a)(d_H(x) + d_H(y))]$$
$$= |V(H)|^4 HM(G) + |V(H)||E(G)|M_1(H) + |V(H)||E(G)|M_1(H) + 8|E(H)|^2|E(G)|$$
$$+2|V(H)|^2 M_1(G)(2|E(H)| + 2|E(H)|) + 4|V(H)|^2|E(H)|M_1(G) + |V(G)|HM(H)$$
$$+8|V(H)||E(G)|M_1(H).$$

□

As an application of Theorem 2.3, we present formulae for the hyper-Zagreb index of the fence graph $P_n[K_2]$ and the closed fence graph $C_n[K_2]$.

Corollary 2.4 $HM(P_n[K_2]) = 500n - 816$, $n > 2$; $HM(C_n[K_2]) = 500n$.

Theorem 2.5 *Let G and H be graphs. Then*

$$HM(G \circ H) = HM(G) + |V(G)|HM(H) + 5|V(H)|M_1(G) + 5|V(G)|M_1(H) +$$
$$4|V(H)|^2|E(G)| + 4|V(G)||E(H)| + 8|E(G)||E(H)| + |V(G)||V(H)|(|V(H)| + 1)^2$$
$$+ 4(|V(H)| + 1)(|E(G)||V(H)| + |E(H)||V(G)|).$$

Proof. By definition of the hyper-Zagreb index, we have

$$HM(G \circ H) = \sum_{uv\in E(G\circ H)} [d_{G\circ H}(u) + d_{G\circ H}(v)]^2$$
$$= \sum_{uv\in E(G)} [d_G(u) + |V(H)| + d_G(v) + |V(H)|]^2$$
$$+ \sum_{uv\in E(H)} \sum_{i=1}^{|V(G)|} [d_H(u) + 1 + d_H(v) + 1]^2$$
$$+ \sum_{u\in V(G)} \sum_{v\in V(H)} [d_G(u) + |V(H)| + d_H(v) + 1]^2.$$

It is easy to see that

$$\sum_{uv\in E(G)} [d_G(u) + d_G(v) + 2|V(H)|]^2 = \sum_{uv\in E(G)} [(d_G(u) + d_G(v))^2 + 4|V(H)|^2 \quad (2.1)$$
$$+4|V(H)|(d_G(u) + d_G(v))] = HM(G) + 4|V(H)|^2|E(G)| + 4|V(H)|M_1(G).$$

$$\sum_{uv\in E(H)} \sum_{i=1}^{|V(G)|} [d_H(u) + d_H(v) + 2]^2 = \sum_{uv\in E(H)} \sum_{i=1}^{|V(G)|} [(d_H(u) + d_H(v))^2 + 4$$
$$+4(d_H(u) + d_H(v))] = |V(G)|\big(HM(H) + 4|E(H)| + 4M_1(H)\big). \quad (2.2)$$

$$\sum_{u\in V(G)} \sum_{v\in V(H)} [d_G(u) + d_H(v) + |V(H)| + 1]^2 = \sum_{u\in V(G)} \sum_{v\in V(H)} [d_G(u)^2 + d_H(v)^2$$
$$+2d_G(u)d_H(v) + (|V(H)| + 1)^2 + 2(|V(H)| + 1)(d_G(u) + d_H(v))]$$
$$= |V(H)|M_1(G) + |V(G)|M_1(H) + 8|E(G)||E(H)| + |V(G)||V(H)|(|V(H)| + 1)^2$$

$$+4(|V(H)|+1)(|E(G)||V(H)|+|E(H)||V(G)|). \quad (2.3)$$

By adding Eqs. (2.1), (2.2), and (2.3) the proof is completed. □

Using Theorem 2.5, we can compute the hyper-Zagreb index of the k −thorny cycle $C_n \circ \overline{K}_k$.

Corollary 2.6 $HM(C_n \circ \overline{K}_k) = 16n + 25nk + 10nk^2 + nk^3$.

ACKNOWLEDGEMENT. This research is supported by UGC–UPE (Non–NET)–Fellowship, K. U. Dharwad, No. KU/Sch/UGC–UPE/2014–15/897, dated: 24 Nov 2014.

REFERENCES

[1] H. Abdo, D. Dimitrov, The total irregularity of graphs under graph operations, *Miskolc Math. Notes* **15**(1) (2014) 3–17.
[2] A. R. Ashrafi, T. Došlić, A. Hamzeh, The Zagreb coindices of graph operations, *Discrete Appl. Math.* **158** (2010) 1571–1578.
[3] M. Azari, Sharp lower bounds on the Narumi-Katayama index of graph operations, *Appl. Math. Comput.* **239** (2014) 409–421.
[4] M. Azari, A. Iranmanesh, Some inequalities for the multiplicative sum Zagreb index of graph operations, *J. Math. Inequal.* **9** (3) (2015) 727–738.
[5] M. Azari, A. Iranmanesh, Chemical graphs constructed from rooted product and their Zagreb indices, *MATCH Commun. Math. Comput. Chem.* **70** (2013) 901–909.
[6] K. C. Das, A. Yurttas, M. Togan, A. S. Cevik, I. N. Cangul, The multiplicative Zagreb indices of graph operations, *J. Inequal. Appl.* 2013 2013:90.
[7] M. R. Farahani, The hyper-Zagreb index of TUSC_4C_8(S) nanotubes, *Int. J. Eng. Technol. Res.* **3** (1) (2015) 1–6.
[8] M. R. Farahani, Computing the hyper-Zagreb index of hexagonal nanotubes, *J. Chem. Mat. Res.* **2** (1) (2015) 16–18.
[9] I. Gutman, N. Trinajstić, Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons, *Chem. Phys. Lett.* **17** (1972) 535–538.
[10] I. Gutman, O. E. Polansky, Mathematical Concepts in Organic Chemistry, *Springer, Berlin* 1986.
[11] F. Harary, *Graph Theory*, Addison–Wesley, Reading, Mass. 1969.
[12] M. H. Khalifeh, H. Yousefi-Azari, A. R. Ashrafi, The first and second Zagreb indices of some graph operations, *Discrete Appl. Math.* **157** (2009) 804–811.
[13] M. H. Khalifeh, H. Yousefi-Azari, A. R. Ashrafi, The hyper-Wiener index of graph operations, *Comput. Math. Appl.* **56** (2008) 1402–1407.
[14] G. H. Shirdel, H. Rezapour, A.M. Sayadi, The hyper-Zagreb index of graph operations, *Iranian J. Math. Chem.* **4** (2) (2013) 213–220.
[15] N. Trinajstić, *Chemical Graph Theory*, CRC Press, Boca Raton, FL 1992.