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Research Paper

The Generalized Energy of Matrix

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Abstract

The adjacency matrix is one of the fundamental tools in graph theory, providing a link between a graph's combinatorial structure and its algebraic properties. In this paper, we present a generalization of the adjacency matrix for simple weighted graphs based on an arbitrary weight function, while preserving the essential structure of the classical weighted adjacency matrix. Furthermore, this framework extends naturally to encompass new graph-theoretical concepts, such as graph energy, and provides a comprehensive and flexible foundation for the deeper investigation of simple weighted graphs.

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1 Introduction

In spectral graph theory, the notion of graph energy was first introduced in the context of chemical graph theory by Gutman and Trinajstić [1], and later developed into a general algebraic invariant of graphs [2, 3]. The concept of graph energy and its variants have been extensively studied in recent decades, particularly in relation to adjacency matrices and their eigenvalues [4, 5]. The spectral properties of graphs are closely connected with fundamental results in matrix theory and algebraic graph theory [6–8]. In particular, bounds on eigenvalues and spectral radius play a crucial role in understanding structural properties of graphs [9–12]. Also, several extensions of classical graph energy have been proposed, including degree-based energies and weighted adjacency matrices [13–15]. Moreover, spectral radius and related invariants have been investigated for various classes of graphs, including planar graphs and graphs with bounded genus [16], highlighting deep connections between combinatorial structure and spectral behavior. Although the classical adjacency matrix is well-suited for unweighted graphs, many practical networks involve weighted interactions. In communication networks, for example, edge weights may represent the number of calls, communication durations, or data volumes exchanged between users. Similarly, in transportation networks, weights may correspond to travel times, distances, or traffic capacities. In such situations, the binary structure of the classical adjacency matrix is often insufficient to capture the additional information carried

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by edge weights. Motivated by these developments, we consider a more general framework in which the edge weights are determined by an arbitrary weight function. This approach unifies several previously studied weighted adjacency matrices and allows their spectral properties to be investigated within a common setting. A further motivation for weighted graphs comes from telecommunication networks. Such networks can be represented by a star graph in which the central vertex corresponds to a communication center and the remaining vertices correspond to users. Edge weights may encode quantities such as communication frequency, duration, or data volume. Since stronger interactions correspond to larger edge weights and, as shown later, larger graph energy, the energy of a weighted graph may be viewed as a spectral indicator of interaction intensity within the network. In this paper, we define the generalized adjacency matrix and investigate its fundamental spectral properties. We then introduce the corresponding notion of graph energy and establish several results that extend classical properties of graph energy to the weighted setting. Before presenting our main results, we recall several standard definitions and preliminary concepts that will be used throughout the paper. Let $G = (V, E)$ be a simple undirected graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$, where $n = |V|$ denotes the order of G and $m = |E|$ denotes the size of G. The adjacency matrix of G, denoted by $A(G) = [a_{ij}]_{n\times n}$, is the symmetric matrix defined by

$$a_{ij} = \begin{cases} 1, & \text{if } v_iv_j \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

The degree of a vertex v_i, denoted by $d_i = d(v_i)$, is the number of edges incident with v_i. The maximum degree of G is denoted by $\Delta(G)$, or Δ for brevity. The degree matrix of G is the diagonal matrix $D(G) = [d_{ij}]_{n\times n}$, where $d_{ii} = d_i = d(v_i)$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $A(G)$. These eigenvalues satisfy the well-known identities

$$\sum_{i=1}^{n} \lambda_i = 0, \quad \sum_{i=1}^{n} \lambda_i^2 = 2m, \quad \sum_{i=1}^{n} \lambda_i^3 = 6t, \tag{1}$$

where t denotes the number of triangles in G [17]. The energy of a graph G, introduced in[1], is defined as:

$$En(G) = \sum_{i=1}^{n} |\lambda_i|,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the adjacency matrix $A(G)$.

2 Main results

Consider a graph $G = (V, E)$ with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$. Let $f : V \times V \to \mathbb{R}$ be an arbitrary symmetric function, that is $f(u, v) = f(v, u)$ for every $u, v \in V$ and it satisfies $f(u, v) > 0$ when u and v are adjacent and $f(u, v) = 0$ otherwise. Throughout this paper, f will denote such a graph function unless otherwise stated.

Associated with f, we define the matrix

$$A_f(G) = [a_{ij}]_{n\times n} = [(A_f(G))_{ij}]_{n\times n},$$

where $a_{ij} = (A_f(G))_{ij} = f(v_i, v_j)$. This matrix is called the f-adjacency matrix of the graph G.

For each vertex v_i, $i = 1, 2, ..., n$, the quantity $\tilde{d}_{v_i} = \tilde{d}_i = \sum_{k=1}^{n} a_{ik}$ is called the f-degree of v_i. The maximum f-degree among all vertices of G is denoted by $\Delta_f(G) = \Delta_f$.

We also define the diagonal matrix $D_f(G) = [\tilde{d}_{ij}]_{n\times n}$, where $\tilde{d}_{ii} = \tilde{d}_{v_i}$, and refer to it as the f-degree matrix of G.

Let $\eta_1, \eta_2, \ldots, \eta_n$ be the eigenvalues of $A_f(G)$. The energy of G with respect to the weight function f is defined by

$$En(G, f) = \sum_{i=1}^{n} |\eta_i|.$$

Furthermore, the quantity $\rho_f(G) = \max\{|\eta_i| : 1 \leq i \leq n\}$ is called the spectral radius of G associated with the matrix $A_f(G)$.

Let us illustrate the above definitions with a simple example. Consider a weighted triangular graph K_3, where the vertex set is $V = \{v_1, v_2, v_3\}$ with a weight function f defined as follows: $f(v_1, v_2) = f(v_2, v_3) = 1$ and $f(v_1, v_3) = 2$ ($f(v_i, v_j)$ represents the number of messages exchanged between two telecommunication centers, v_i and v_j, within a specific time period). Therefore, the associated adjacency matrix is

$$A_f(G) = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}.$$

The eigenvalues of $A_f(G)$ are $\eta_1 = -2$, $\eta_2 = 1 - \sqrt{3}$ and $\eta_3 = 1 + \sqrt{3}$. Therefore, the energy of G with respect to the weight function(graph function) f is

$$En(G, f) = | - 2| + |1 - \sqrt{3}| + |1 + \sqrt{3}| = 2(1 + \sqrt{3}).$$

This energy is greater than the classical graph energy of the corresponding unweighted graph, namely the complete graph K_3. This implies that the underlying network effectively captures and reflects the intensity of interactions among users. The following well-known result will be useful in our subsequent analysis.

Lemma 2.1. *([6]). Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$. Then, for every positive integer k, the eigenvalues of A^k are $\lambda_1^k, \lambda_2^k, \cdots, \lambda_n^k$. Moreover, the trace of A^k, that is, the sum of its diagonal entries, satisfies $tr(A^k) = \Sigma_{i=1}^{n} \lambda_i^k$.*

Theorem 2.2. *Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Suppose that $\eta_1, \ldots, \eta_n$ are the eigenvalues of the matrix $A_f(G) = [a_{ij}]_{n\times n}$, then the following results hold:*

(i) $\sum_{i=1}^{n} \eta_i = 0$.

(ii) $\sum_{i=1}^{n} \eta_i^2 = 2M$, *where* $M = \sum_{uv\in E(G)} f^2(u, v)$.

(iii) $\sum_{i=1}^{n} \eta_i^3 = 6S$, *where* $S = \sum_{\{uv,vw,wu\}\subseteq E(G)} f(u, v)f(v, w)f(w, u)$.

Proof. (i) Since every diagonal entry of $A_f(G)$ is zero, by Lemma 2.1, we have:

$$\sum_{i=1}^{n} \eta_i = tr(A_f(G)) = 0.$$

(ii) Again using Lemma 2.1 and the fact that the matrix $A_f(G)$ is symmetric, we obtain:

$$\begin{aligned}\sum_{i=1}^{n}\eta_i^2 &= tr(A_f^2(G)) = \sum_{i=1}^{n}(A_f^2(G))_{ii}\\ &= \sum_{k=1}^{n}a_{1k}a_{k1}+\cdots+\sum_{k=1}^{n}a_{nk}a_{kn} = \sum_{k=1}^{n}a_{1k}^2+\cdots+\sum_{k=1}^{n}a_{nk}^2\\ &= \sum_{k=1}^{n}f^2(v_1,v_k)+\cdots+\sum_{k=1}^{n}f^2(v_n,v_k) = 2\sum_{i<j}f^2(v_i,v_j)\\ &= 2\sum_{uv\in E(G)}f^2(u,v) = 2M.\end{aligned}$$

(iii) For the sum of the cubes of eigenvalues of the symmetric matrix $A_f(G)$, we have:

$$\begin{aligned}\sum_{i=1}^{n}\eta_i^3 &= tr(A_f^3(G)) = \sum_{k=1}^{n}(A_f^3(G))_{kk}\\ &= (A_f^3(G))_{11}+\cdots+(A_f^3(G))_{nn} = \sum_{k=1}^{n}a_{1k}(A_f^2(G))_{k1}+\cdots+\sum_{k=1}^{n}a_{nk}(A_f^2(G))_{kn}\\ &= (\sum_{1\le k\le n}a_{11}a_{1k}a_{k1}+\cdots+\sum_{1\le k\le n}a_{1n}a_{1k}a_{kn})+\cdots+(\sum_{1\le k\le n}a_{n1}a_{1k}a_{kn}+\cdots+\sum_{1\le k\le n}a_{nn}a_{nk}a_{kn})\\ &\underline{\underline{a_{ik}=a_{ki}}}\quad 2\sum_{1\le p<q\le n}a_{1p}a_{1q}a_{pq}+\cdots+2\sum_{1\le p<q\le n}a_{np}a_{nq}a_{pq}\\ &= 2\sum_{i=1}^{n}\sum_{1\le p<q\le n}a_{ip}a_{iq}a_{pq} = 3\times 2\sum_{\{v_iv_p,v_iv_q,v_pv_q\}\subseteq E(G)}a_{ip}a_{iq}a_{pq}\\ &= 6\sum_{\{v_iv_p,v_iv_q,v_pv_q\}\subseteq E(G)}a_{ip}a_{iq}a_{pq} = 6\sum_{\{uv,vw,wu\}\subseteq E(G)}f(u,v)f(v,w)f(w,u) = 6S.\end{aligned}$$

■

Observe that $m = |E(G)| = 0$ if and only if $M = 0$. Therefore, Theorem 2.2 (ii) immediately yields the following result.

Corollary 2.3. *For every graph G, $En(G,f) \ge 0$. Moreover, $En(G,f) = 0$ if and only if $M = 0$.*

It is worth emphasizing that, although Theorem 2.2 is elementary, it provides a useful tool for determining eigenvalues and deriving spectral quantities such as the energy of a graph.

Example 2.4. Let G_p be the graph obtained from the path $v_1 - v_2 - v_3$ by attaching p pendant vertices $u_1,\dots,u_p$ to v_1 and p pendant vertices $w_1,\dots,w_p$ to v_3. Define a weight function f on G_p by $f(v_1,v_2) = f(v_2,v_3) = a$, and $f(u_i,v_1) = f(w_i,v_3) = b$, for each $i = 1,2,\dots,p$.
We now determine the energy of G_p. The adjacency matrix $A_f(G_p)$, with rows and columns indexed by $u_1,\dots,u_p,v_1,v_2,v_3,w_1,\dots,w_p$ is given by

$$A_f(G_p) = \begin{bmatrix}\mathbf{0} & \mathbf{B_1} & \mathbf{0}\\ \mathbf{B_1^T} & \mathbf{C} & \mathbf{B_2^T}\\ \mathbf{0} & \mathbf{B_2} & \mathbf{0}\end{bmatrix}_{(2p+3)\times(2p+3)},$$

where,

$$\mathbf{0} = [0]_{p\times p},\ \mathbf{B_1} = \begin{bmatrix} b & 0 & 0 \\ b & 0 & 0 \\ \vdots & \vdots & \vdots \\ b & 0 & 0 \end{bmatrix}_{p\times 3},\ \mathbf{B_2} = \begin{bmatrix} 0 & 0 & b \\ 0 & 0 & b \\ \vdots & \vdots & \vdots \\ 0 & 0 & b \end{bmatrix}_{p\times 3} \text{ and } \mathbf{C} = \begin{bmatrix} 0 & a & 0 \\ a & 0 & a \\ 0 & a & 0 \end{bmatrix}.$$

The column spaces corresponding to the vertex sets $\{u_1,\ldots,u_p\}$, $\{v_1,v_2,v_3\}$ and $\{w_1,\ldots,w_p\}$ have ranks $1, 2$ and 1, respectively. Hence,

$$rank\,(A_f(G_p)) = 1+2+1 = 4.$$

It follows that

$$null\,(A_f(G_p)) = 2p+3-rank\,(A_f(G_p)) = 2p+3-4 = 2p-1.$$

Therefore, 0 is an eigenvalue of $A_f(G_p)$ with multiplicity $2p-1$.
On the other hand, we have:

$$A_f(G_p)\mathbf{X} = b\sqrt{p}\,\mathbf{X} \quad \text{and} \quad A_f(G_p)\mathbf{Y} = -b\sqrt{p}\,\mathbf{Y},$$

where, $\mathbf{X} = \left[\overbrace{1,1,\ldots,1}^{p\text{ times}}, b\sqrt{p}, 0, -b\sqrt{p}, \overbrace{-1,-1,\ldots,-1}^{p\text{ times}}\right]^T$, and

$\mathbf{Y} = \left[\overbrace{1,1,\ldots,1}^{p\text{ times}}, -b\sqrt{p}, 0, b\sqrt{p}, \overbrace{-1,-1,\ldots,-1}^{p\text{ times}}\right]^T$.

Therefore, $\pm b\sqrt{p}$ are eigenvalues of $A_f(G_p)$. Let α and β denote the remaining two eigenvalues. By Theorem 2.2,

$$\alpha+\beta+b\sqrt{p}-b\sqrt{p} = 0 \quad \text{and} \quad \alpha^2+\beta^2+2b^2p = 2\left(2pb^2+2a^2\right).$$

It follows that

$$\alpha = \sqrt{pb^2+2a^2} \text{ and } \beta = -\sqrt{pb^2+2a^2} \text{ or } \alpha = -\sqrt{pb^2+2a^2} \text{ and } \beta = \sqrt{pb^2+2a^2}.$$

In particular, if $b>0$, then

$$\begin{aligned} En(G_p,f) &= |b\sqrt{p}| + |-b\sqrt{p}| + |\sqrt{pb^2+2a^2}| + |-\sqrt{pb^2+2a^2}| \\ &= 2b\sqrt{p} + 2\sqrt{pb^2+2a^2}. \end{aligned}$$

Proposition 2.5. *If G has connected components $G_1,\ldots,G_t$, then*

$$En(G,f) = En(G_1,f)+\cdots+En(G_t,f).$$

Proof.

$$det(\eta I - A_f(G)) = det\begin{bmatrix} \eta I - A_f(G_1) & 0 & 0 & \cdots & 0 \\ 0 & \eta I - A_f(G_2) & 0 & \cdots & 0 \\ 0 & 0 & \eta I - A_f(G_3) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \eta I - A_f(G_t) \end{bmatrix}.$$

Therefore, $det(\eta I - A_f(G)) = \prod_{i=1}^{t} det(\eta I - A_f(G_i)) = \prod_{i=1}^{n_1}(\eta - \eta_i^{(1)}) \cdots \prod_{i=1}^{n_t}(\eta - \eta_i^{(t)})$, where $|V(G_i)| = n_i$ and $\eta_1^{(i)}, ..., \eta_{n_i}^{(i)}$ are the eigenvalues of $A_f(G_i)$.
Thus, $En(G, f) = \sum_{i=1}^{t} |\eta_i| = \sum_{i=1}^{n_1} |\eta_i^{(1)}| + \cdots + \sum_{i=1}^{n_t} |\eta_i^{(t)}| = En(G_1, f) + \cdots + En(G_t, f)$. ■

As an immediate consequence of the proposition, isolated vertices do not affect the energy.

Corollary 2.6. *If the graph G_1 is obtained from the graph G by removing all isolated vertices, then $En(G, f) = En(G_1, f)$.*

It is important to note that, we emphasize that whenever M and S appear,they respectively denote the quantities $\sum_{uv \in E(G)} f^2(u, v)$ and $\sum_{\{uv,vw,wu\} \subseteq E(G)} f(u, v)f(v, w)f(w, u)$, unless explicitly stated otherwise.

Theorem 2.7. *For every graph G of order n and size m, we have:*

$$En(G, f) \leq \sqrt{2Mn}.$$

Moreover, if $f(u, v) = c > 0$, for every $uv \in E(G)$, then, equality holds if and only if $\Delta \leq 1$.

Proof. By Cauchy-Schwarz inequality, we have $[En(G, f)]^2 = \left(\sum_{i=1}^{n} |\eta_i|\right)^2 \leq n(\sum_{i=1}^{n} |\eta_i|^2) = 2Mn$. For the equality case, since $A_f(G) = cA(G)$, it follows that for each $1 \leq i \leq n$, $\eta_i = c\lambda_i$, where λ_i are the eigenvalues of $A(G)$. Consequently,

$$En(G, f) = \sum_{i=1}^{n} |\eta_i| = \sum_{i=1}^{n} |c\lambda_i| = cEn(G),$$

$$M = \sum_{uv \in E(G)} f^2(u, v) = \sum_{i=1}^{m} c^2 = mc^2,$$

thus, $cEn(G) \leq \sqrt{2mc^2n}$. It follows that $En(G) \leq \sqrt{2mn}$.

In view of [17, Th. 5.4.1], the equality in the preceding inequality holds if and only if $\Delta(G) \leq 1$. ■

Proposition 2.8. *Let $S_n = (V, E)$ be the stare graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{v_1v_2, v_1v_3, \ldots, v_1v_n\}$. Then, we have*

$$En(S_n, f) = 2\sqrt{M}.$$

Proof. Let $f(v_1, v_i) = w_i$ for any $i = 2, 3, \ldots, n$. Then, the matrix $A_f(S_n)$ takes the form

$$A_f(S_n) = \begin{bmatrix} 0 & w_2 & w_3 & \cdots & w_n \\ w_2 & 0 & 0 & \cdots & 0 \\ w_3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_n & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Thus,

$$\begin{aligned} det\,(\eta I - A_f(S_n)) &= det \begin{bmatrix} \eta & -w_2 & -w_3 & \cdots & -w_n \\ -w_2 & \eta & 0 & \cdots & 0 \\ -w_3 & 0 & \eta & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -w_n & 0 & 0 & \vdots & \eta \end{bmatrix} \\ &= w_2 det \begin{bmatrix} -w_2 & -w_3 & \cdots & -w_n \\ 0 & \eta & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \eta \end{bmatrix} + \eta det \begin{bmatrix} \eta & -w_3 & \cdots & -w_n \\ -w_3 & \eta & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -w_n & 0 & \vdots & \eta \end{bmatrix} \\ &= -w_2^2 \eta^{n-2} + \eta det \begin{bmatrix} \eta & -w_3 & \cdots & -w_n \\ -w_3 & \eta & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -w_n & 0 & \vdots & \eta \end{bmatrix}. \end{aligned}$$

By repeating the above process, we obtain:

$$det\,(\eta I - A_f(S_n)) = \eta^{n-2}\left(\eta^2 - w_2^2 - w_3^2 - \cdots - w_n^2\right),$$

thus, the spectrum of $A_f(S_n)$ consists of the eigenvalue 0 with multiplicity $n-2$ and $\pm\sqrt{w_2^2 + \cdots + w_n^2}$. It follows that

$$\begin{aligned} En(S_n, f) &= (n-2)|0| + \left|\sqrt{w_2^2 + \cdots + w_n^2}\right| + \left|-\sqrt{w_2^2 + \cdots + w_n^2}\right| \\ &= 2\sqrt{w_2^2 + \cdots + w_n^2} = 2\sqrt{f^2(v_1, v_2) + \cdots + f^2(v_1, v_n)} = 2\sqrt{M}. \end{aligned}$$

■

Observe that the energy of S_n increases with edge weights. Furthermore,

$$En(S_n, f) = 2\sqrt{M} \geq 2\sqrt{n-1} = En(G).$$

Theorem 2.9. *For every graph G of order n and size m, we have*

$$En(G, f) \geq 2\sqrt{M}.$$

If $f(u, v) = c > 0$, for every $uv \in E(G)$, then equality holds if and only if G is a complete bipartite graph.

Proof. By applying Theorem 2.2, we get:

$$0 = \left(\sum_{i=1}^{n} \eta_i\right)^2 = \sum_{i=1}^{n} \eta_i^2 + 2\sum_{i<j} \eta_i \eta_j = 2M + 2\sum_{i<j} \eta_i \eta_j,$$

thus,

$$M = |\sum_{i<j} \eta_i \eta_j| \leq \sum_{i<j} |\eta_i||\eta_j|.$$

It follows that

$$\begin{aligned}(En(G,f))^2 &= \left(\sum_{i=1}^{n}|\eta_i|\right)^2 = \sum_{i=1}^{n}\eta_i^2 + 2\sum_{i<j}|\eta_i||\eta_j| \\ &= 2M + 2\sum_{i<j}|\eta_i||\eta_j| \geq 2M + 2M = 4M.\end{aligned}$$

Now, we verify the equality condition. Similar to the proof of Theorem 2.7, we have $En(G) = 2\sqrt{m}$ and by [17, Th.5.4.2], this equality holds if and only if G is a complete bipartite graph. ■

We remark that the above lower bound is in fact sharp. For example, Proposition 2.8 shows that equality is attained.

Theorem 2.10. *For every connected graph G of order n and size m, if n_0 is the number of nonisolated vertices and $\frac{n_0}{2} \leq M$, then $En(G,f) \leq 2M$.*

Proof. For $n = 1$, the inequality holds. Let $n \geq 2$. First, suppose that the graph G has no isolated vertex. In this case, $n_0 = n$. By Theorem 2.7, we get:

$$En(G,f) \leq \sqrt{2Mn} \leq \sqrt{2M(\frac{n}{2})2} \leq \sqrt{4M^2} = 2M.$$

Now, we prove the theorem when the graph G has isolated vertices. Suppose the graph G_1 is obtained from G by removing all isolated vertices. By the first part of the theorem and by Corollary 2.6, we obtain:

$$En(G,f) = En(G_1,f) \leq 2M.$$

■

It is noteworthy that both the upper bound of this theorem and the lower bound from Theorem 2.9 are simultaneously sharp.
To show this, it suffices to consider the graph from Proposition 2.8 with the condition $M = 1$.

Lemma 2.11. *If $G = (V,E)$ is a graph of order n and η_1 is the largest eigenvalue of the $A_f(G) = [a_{ij}]_{n\times n}$, then we have:*

$$\eta_1 \geq \frac{2M'}{n}, \tag{2}$$

where $M' = \sum_{uv\in E(G)} f(u,v)$.

Proof. According to the Rayleigh quotient for a symmetric matrix $A_f(G)$, we have:

$$\frac{X^T A_f(G) X}{X^T X} \leq \eta_1,$$

where $X = [x_{ij}]_{n\times 1} \neq \mathbf{0}$. Now, we replace the X with $[1]_{n\times 1}$, we get:

$$\eta_1 \geq \frac{X^T A_f(G) X}{X^T X} = \frac{\sum_{1\leq i,j\leq n} a_{ij}}{n} = \frac{2\sum_{uv\in E(G)} f(u,v)}{n} = \frac{2M'}{n}.$$

■

Note that from now on, we set $M' = \sum_{uv\in E(G)} f(u,v)$.

Theorem 2.12. *For every graph G with $n \geq 2$ vertices, where $\frac{M'^2}{M} \geq \frac{n}{2}$, we have:*

$$En(G,f) \leq \frac{2M'}{n} + \sqrt{(n-1)\left[2M - \left(\frac{2M'}{n}\right)^2\right]}.$$

Proof. Let $\eta_1, \eta_2, \ldots, \eta_n$ be the eigenvalues of $A_f(G)$, where η_1 is the greatest eigenvalue. Applying the Cauchy-Schwarz inequality, we get:

$$\left(\sum_{i=2}^{n} |\eta_i|\right)^2 \leq (n-1)\sum_{i=2}^{n} \eta_i^2 = (n-1)\left(\sum_{i=1}^{n} \eta_i^2 - \eta_1^2\right) = (n-1)\left(2M - \eta_1^2\right). \tag{3}$$

Therefore,

$$En(G,f) = \sum_{i=1}^{n} |\eta_i| = \eta_1 + \sum_{i=2}^{n} |\eta_i| \leq \eta_1 + \sqrt{(n-1)\left(2M - \eta_1^2\right)}.$$

Now, by examining the possible variations of η_1, we find the maximum value of the right-hand side of the above expression. This is done by differentiating with respect to η_1, which leads to an upper bound for $En(G,f)$.

Consider the function $g(x) = x + \sqrt{(n-1)\left(2M - x^2\right)}$ with domain $\sqrt{\frac{2M}{n}} \leq x \leq \sqrt{2M}$.

The derivation of the function is given as, $\frac{d}{dx}g(x) = 1 - \frac{x\sqrt{n-1}}{\sqrt{2M-x^2}}$.

We now show that the derivative of the function g is non-positive on the specified domain.

It follows from $x \geq \sqrt{\frac{2M}{n}}$ that $x\sqrt{n-1} \geq \sqrt{2M - x^2}$.

Thus,

$$\frac{d}{dx}g(x) = 1 - \frac{x\sqrt{n-1}}{\sqrt{2M-x^2}} \leq 0.$$

On the other hand, from (3) and $\frac{M'^2}{M} \geq \frac{n}{2}$, the following results are obtained

$$\eta_1 \leq \sqrt{2M}, \qquad \text{and} \qquad \frac{2M'}{n} \geq \sqrt{\frac{2M}{n}}.$$

By the last two relations and Lemma 2.11, we have:

$$\sqrt{2M} \geq \eta_1 \geq \frac{2M'}{n} \geq \sqrt{\frac{2M}{n}}.$$

Since the function g is decreasing on this domain, it follows that $g(\eta_1) \leq g\left(\frac{2M'}{n}\right)$, which completes the proof. ■

Notably, for a graph G, it is known from a recent theorem that if $f(u,v) = 1$ for every edge $uv \in E(G)$, then [17, Th.5.4.3], follows:

$$En(G) \leq \frac{2m}{n} + \sqrt{(n-1)\left[2m - \left(\frac{2m}{n}\right)^2\right]},$$

where $\frac{2m}{n} \geq 1$.

Theorem 2.13. *Let G be a graph with n vertices. If η is an arbitrary eigenvalue of $A_f(G) = [a_{ij}]_{n\times n}$, then we have $\eta \leq \Delta_f$, where $\Delta_f = max\{\sum_{i=1}^{n} a_{ij} : j = 1, 2, ..., n\}$.*

Proof. For $\eta = 0$, it is obvious. Let $\eta \neq 0$. So, there is a nonzero vector $X = [x_i]_{n\times 1}$ such that $A_f X = \eta X$. Let $Max\{|x_i| : i = 1, 2, ..., n\} = |x_t| > 0$. On the other hand, we have

$$\Delta_f \geq \sum_{j=1}^{n} a_{ij} = \tilde{d}_i, \qquad \text{and} \qquad \sum_{j=1}^{n} a_{tj}x_j = \eta x_t.$$

It follows that

$$\Delta_f |x_t| \geq \tilde{d}_t |x_t| = (a_{t1} + \cdots + a_{tn})\,|x_t| = (|a_{t1}| + \cdots + |a_{tn}|)|x_t| \geq |a_{t1} + \cdots + a_{tn}||x_t| = |\eta||x_t|.$$

■

Theorem 2.14. *Let $G = (V, E)$ be a graph of order n with a non-empty edge set E, then we have*

$$En(G, f) \leq \frac{n}{2}\left(\Delta_f + \frac{M}{M'}\right).$$

Proof. Suppose $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_n$ are eigenvalues of the matrix $A_f(G)$. By Theorem 2.2, we have:

$$\sum_{i=1}^{n}(|\eta_i| - |\eta_1|)^2 = \sum_{i=1}^{n}\eta_i^2 - 2|\eta_1|\sum_{i=1}^{n}|\eta_1| + n\eta_1^2 = 2M - 2|\eta_1|En(G, f) + n\eta_1^2 \geq 0,$$

thus,

$$En(G, f) \leq \frac{n}{2}|\eta_1| + \frac{M}{|\eta_1|}.$$

On the other hand, by Lemma 2.11 and Theorem 2.13, we have:

$$\frac{2M'}{n} \leq \eta_1 = |\eta_1| \leq \Delta_f.$$

Therefore,

$$En(G, f) \leq \frac{n}{2}\Delta_f + \frac{n}{2M'} \times M = \frac{n}{2}\left(\Delta_f + \frac{M}{M'}\right).$$

■

Theorem 2.15. *Let $G = (V, E)$ be a graph of order n with a non-empty edge set E. If for every $uv \in E(G)$, $f(u, v) \geq 1$ holds, then*

$$En(G, f) \leq M(d^{-1} + 1),$$

where $d = \frac{2M'}{n}$.

Proof. We note that

$$\sum_{i=1}^{n}(|\eta_i| - d)^2 = \sum_{i=1}^{n}\eta_i^2 - 2d\sum_{i=1}^{n}|\eta_i| + nd^2 = 2M - 2dEn(G, f) + nd^2 \geq 0.$$

Therefore,

$$En(G, f) \leq \frac{M}{d} + \frac{nd}{2}.$$

Since $f(u,v) \geq 1$ holds, for every $uv \in E(G)$, then we have $M' \leq M$. Thus,

$$En(G,f) \leq \frac{M}{d} + \frac{nd}{2} = \frac{M}{d} + M' \leq \frac{M}{d} + M = M(d^{-1}+1).$$

■

In what follows, we present some lower bounds for the graph energy.

Theorem 2.16. *For every graph G with n vertices and $m \geq 1$ edges, we have:*

$$En(G,f) \geq \frac{2M}{\rho_f} \geq \frac{2M}{\Delta_f}.$$

Proof. Suppose $\eta_1, \eta_2, \ldots, \eta_n$ are the eigenvalues of matrix $A_f(G)$. By Theorem 2.2 and $\rho_f = \rho_f(G) = Max\{|\eta_i|\colon\ i = 1, 2, ..., n\}$, we can write

$$2M = \sum_{i=1}^{n} \eta_i^2 = \sum_{i=1}^{n} |\eta_i|^2 \leq \sum_{i=1}^{n} |\eta_i| \rho_f(G) = \rho_f(G) \sum_{i=1}^{n} |\eta_i| = \rho_f(G) En(G,f).$$

Thus, by Theorem 2.13,

$$En(G,f) \geq \frac{2M}{\rho_f} \geq \frac{2M}{\Delta_f},$$

which completes the proof. ■

Theorem 2.17. *For every graph G with n vertices and $m \geq 1$ edges, we have:*

$$En(G,f) \geq \frac{6S}{\rho_f^2} \geq \frac{6S}{\Delta_f^2}.$$

Proof. By Theorems 2.2 and 2.13, we can write

$$6S = \sum_{i=1}^{n} \eta_i^3 \leq \sum_{i=1}^{n} |\eta_i|^3 = \sum_{i=1}^{n} |\eta_i||\eta_i|^2 \leq \rho_f^2(G) \sum_{i=1}^{n} |\eta_i| = \rho_f^2(G) En(G,f) \leq \Delta_f^2(G) En(G,f).$$

It follows that

$$En(G,f) \geq \frac{6S}{\rho_f^2} \geq \frac{6S}{\Delta_f^2}.$$

■

3 Some examples with symmetric division degree function

In this section, we present several examples and applications of the generalized adjacency matrix and the corresponding graph energy introduced in the previous sections, with a particular emphasis on the symmetric division degree function[18]. By studying this function in detail, we obtain explicit formulas and bounds for the energy of several well-known classes of graphs, including complete graphs, bipartite graphs, paths, and friendship graphs.

The symmetric division degree function is especially useful as it reflects the relative significance of vertices through their degrees, providing a more balanced view of the underlying graph structure. Furthermore, we derive both upper and lower bounds, as well as exact expressions for the energy of specific graph families, illustrating the effectiveness of the proposed generalized framework. Through these concrete examples, we connect the theoretical developments to practical computations, offering a clearer insight into the relationship between graph structure, spectral properties, and graph energy.

Definition 3.1. Let $G = (V, E)$ be a graph. A graph function f defined by

$$f(u, v) = \frac{d(u)}{d(v)} + \frac{d(v)}{d(u)},$$

for every edge $uv \in E(G)$, and $f(u, v) = 0$ otherwise. This function is called *symmetric division degree function* of G, and is denoted by **ssd**.

Lemma 3.2. *If $G = (V, E)$ is a graph and for every $uv \in E(G)$, $f(u, v) = a > 0$, then*

$$En(G, f) = aEn(G),$$

Proof. It is obvious that $A_f(G) = aA(G)$. So, the eigenvalues satisfy $\eta_i = a\lambda_i$ for $i = 1, 2, \ldots, n$, where η_i and λ_i denote the eigenvalues of the matrices $A_f(G)$ and $A(G)$, respectively. Consequently,

$$En(G, f) = \sum_{i=1}^{n} |\eta_i| = \sum_{i=1}^{n} |a\lambda_i| = a \sum_{i=1}^{n} |\lambda_i| = aEn(G).$$

■

Corollary 3.3. *Let $K_{m,n}$, K_n and C_n denote the complete graph of order n, the complete bipartite graph and the cycle graph of order n respectively. Then, we have:*

(i) $En(K_{m,n}, ssd) = \left(\frac{m}{n} + \frac{n}{m}\right) En(K_{m,n}) = 2\left(\frac{m}{n} + \frac{n}{m}\right)\sqrt{mn}$.

(ii) *If G is a regular graph, then $En(G, sdd) = 2En(G)$.*

(iii) $En(K_n, ssd) = 2En(K_n) = 4(n-1)$.

(iv) $En(C_n, sdd) = 2En(C_n)$.

Proposition 3.4. *([3]). The energy of the n-vertex path is*

$$En(P_n) = \begin{cases} -2 + 2\csc\frac{\pi}{2n+2}, & \text{if } n \text{ is even,} \\ -2 + 2\cot\frac{\pi}{2n+2}, & \text{if } n \text{ is odd.} \end{cases}$$

Let $M_n = [m_{ij}]_{n\times n}$ be a matrix with real entries. The eigenvalues of the matrix $M_n M_n^T$ are denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$. The square roots of these eigenvalues, $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_n}$, are referred to as the singular values of the matrix M_n. Following the concept of matrix energy introduced by Nikiforov [5], we define the Nikiforov energy of a matrix M_n as the sum of its singular values. This energy is denoted by $\varepsilon(M_n)$.

It should be noted that $En(M_n)$ is not necessarily equal to $\varepsilon(M_n)$, unless M_n is symmetric and the mean of its eigenvalues is zero. For example $En(A_f(G)) = \varepsilon(A_f(G))$.

Theorem 3.5. ***[Ky Fan's theorem]*** *The following inequality holds for all square matrices A, B of the same size:*

$$\varepsilon(A + B) \leq \varepsilon(A) + \varepsilon(B).$$

Proposition 3.6. *For the path graph P_n of order n, we have:*

(i) $En(P_n, ssd) \leq 4\left(1.5 + \csc\frac{\pi}{2n-2}\right)$ *if n is even.*

(ii) $En(P_n, ssd) \leq 4\left(1.5 + \cot\frac{\pi}{2n-2}\right)$ *if n is odd.*

Proof. By Corollary 3.3, $En(P_2, ssd) = 4$ and $En(P_3, ssd) = 5\sqrt{2}$. Now let $4 \leq n$. The matrix $A_{ssd}(P_n)$ is as follows:

$$A_{ssd}(P_n) = \begin{bmatrix} 0 & 2.5 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 2.5 & 0 & 2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 2 & 0 & 2.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 2.5 & 0 \end{bmatrix}.$$

The matrix $A_{ssd}(P_n)$ is represented as the sum of two matrices B and C as follow:

$$B = \begin{bmatrix} 0 & 2.5 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 2.5 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 2.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 2.5 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

By Ky Fan's theorem, $\varepsilon(A_{ssd}(P_n)) = \varepsilon(B + C) \leq \varepsilon(B) + \varepsilon(C)$.
Note that the matrices $A_{ssd}(P_n)$, B, and C are symmetric, and the average of the eigenvalues of each matrix is equal to zero. It follows that, $En(P_n, ssd) = En(A_{ssd}(P_n)) \leq En(B) + En(C)$. We now proceed to determine $En(B)$ and $En(C)$.

By elementary properties of determinant, $det(\lambda I - B) = \lambda^{n-4}(\lambda^2 - 2.5^2)^2$. As a result $En(B) = 4 \times 2.5 = 10$.
Matrix C can be interpreted as the adjacency matrix of a weighted path graph P_n, where the weights of the first and last edges are zero, and all other edge weights are equal to 2. Based on this structure, we conclude that the matrix C corresponds to a graph consisting of two components K_1 and K_1, along with a component P_{n-2}, with the graph function $g(u, v) = 2$ for all $uv \in E(P_n)$. Then, by Lemma 3.2 and Corollary 2.6,

$$En(C) = En(P_{n-2}, g) = 2En(P_{n-2}) = \begin{cases} -4 + 4\csc\frac{\pi}{2n-2}, & \text{if } n \text{ is even}, \\ -4 + 4\cot\frac{\pi}{2n-2}, & \text{if } n \text{ is odd}. \end{cases}$$

Thus,

$$En(P_n, ssd) \leq En(B)+En(C) = \begin{cases} 10 - 4 + 4\csc\frac{\pi}{2n-2}, & \text{if } n \text{ is even}, \\ 10 - 4 + 4\cot\frac{\pi}{2n-2}, & \text{if } n \text{ is odd}. \end{cases} = \begin{cases} 4\left[1.5 + \csc\frac{\pi}{2n-2}\right], & \text{if } n \text{ is even}, \\ 4\left[1.5 + \cot\frac{\pi}{2n-2}\right], & \text{if } n \text{ is odd}. \end{cases}$$

■

Let n be an odd natural number greater than one. A friendship graph, denoted by F_n, is a simple graph consisting of $\frac{n-1}{2}$ triangles that share exactly one common vertex. This graph has order n and size $\frac{3(n-1)}{2}$.

Proposition 3.7. *For the friendship graph F_n, we have:*

$$En(F_n, ssd) \leq 2\left[n-1+\sqrt{n-1}\left(\frac{2}{n-1}+\frac{n-1}{2}\right)\right].$$

Proof. We construct two matrices B and C as follows:
In $A_{ssd}(F_n)$ we replace each entry equal to $\frac{2}{n-1}+\frac{n-1}{2}$ with 0, and denote the matrix obtained by B. Similarly, we replace each entry equal to 2 in the $A_f(F_n)$ with 0, and denote the resulting matrix by C. It is evident that $A_{ssd}(F_n) = B + C$. On the other hand, the characteristic polynomials of the B and C matrices can be readily expressed as follows:

$$det(\lambda I - B) = \lambda(\lambda^2-4)^{\frac{n-1}{2}}, \quad \text{and} \quad det(\lambda I - C) = \lambda^{n-2}\left[\lambda^2-(n-1)\left(\tfrac{2}{n-1}+\tfrac{n-1}{2}\right)^2\right].$$

By Ky Fan's theorem, we can write

$$\begin{aligned} En(F_n, ssd) = \varepsilon\left(A_{ssd}(F_n)\right) = \varepsilon(B+C) &\leq \varepsilon(B)+\varepsilon(C) \\ &= En(B)+En(C) = 2(n-1)+2\sqrt{n-1}\left(\frac{2}{n-1}+\frac{n-1}{2}\right) \\ &= 2\left[n-1+\sqrt{n-1}\left(\frac{2}{n-1}+\frac{n-1}{2}\right)\right]. \end{aligned}$$

■

Proposition 3.8. *If T is the graph corresponding to the molecular C_nH_{2n+2}, then:*

$$En(T, sdd) \leq \begin{cases} 4.25\left[4\sqrt{3}+2(n-2)\sqrt{2}\right]+2\left[-2+2\csc\frac{\pi}{2n+2}\right], & \text{if } n \text{ is even,} \\ 4.25\left[4\sqrt{3}+2(n-2)\sqrt{2}\right]+2\left[-2+2\cot\frac{\pi}{2n+2}\right], & \text{if } n \text{ is odd.} \end{cases}$$

Proof. In the molecular graph of C_nH_{2n+2}, we label the carbon and hydrogen atoms as $v_1, \cdots, v_n$ and $v_{n+1}, \cdots, v_{3n+2}$, respectively, from left to right. We construct the adjacency matrix $A_{sdd}(T)$ so that its rows from top to bottom and its columns from left to right are labeled according to this ordering. We then define two matrices, X and Y, as follows:
in the adjacency matrix $A_{sdd}(T)$, we set all entries equal to 2 to zero to obtain X, and we set all entries equal to 4.25 to zero to obtain Y. Therefore, $A_{sdd}(T) = X + Y$.
The characteristic polynomial of the X readily be expressed as:

$$det(\lambda I - X) = \lambda^{n+2}(\lambda^2-34.25^2)^2(\lambda^2-24.25^2)^{n-2}.$$

Therefore, $\varepsilon(X) = En(X) = 4.5[4\sqrt{3}+2(n-2)\sqrt{2}]$.
On the other hand,

$$\varepsilon(Y) = En(Y) = En(2A(P_n)) = \begin{cases} -4+4\csc\frac{\pi}{2n+2}, & \text{if } n \text{ is even,} \\ -4+4\cot\frac{\pi}{2n+2}, & \text{if } n \text{ is odd.} \end{cases}$$

By ky Fan's theorem, we have:

$$En(T, sdd) = \varepsilon(A_{sdd}(T)) \leq \varepsilon(X)+\varepsilon(Y) = \begin{cases} 4.25\left[4\sqrt{3}+2(n-2)\sqrt{2}\right]+2\left[-2+2\csc\frac{\pi}{2n+2}\right], & \text{if } n \text{ is even,} \\ 4.25\left[4\sqrt{3}+2(n-2)\sqrt{2}\right]+2\left[-2+2\cot\frac{\pi}{2n+2}\right], & \text{if } n \text{ is odd.} \end{cases}$$

■

The **corona product** of graph G_1 with n_1 vertices and graph G_2, denoted by $G_1 \diamondsuit G_2$, is obtained by taking one copy of G_1 and n_1 copies of G_2, then joining each vertex of the i-th copy of G_2 to the i-th vertex of G_1.

Example 3.9. Assume that $R = K_n \diamondsuit H$, where K_n is a complete graph of order n, and H is an edge-less graph of order $n-1$. Then, we have:

$$En(R, sdd) \leq n\sqrt{n-1}\left[\frac{1}{n-1}+4(n-1)\right]+4(n-1).$$

Proof. We label the vertices of the complete graph K_n as $v_1, v_2, \ldots, v_n$ and the n-th copy of H as $v_{n+1}, v_{n+2}, \ldots, v_{n^2}$,in order. Additionally, we construct the adjacency matrix $A_{sdd}(R)$ such that its rows are labeled from top to bottom and its columns from left to right in the order $v_1, v_2, \ldots, v_{n^2}$.
In the $A_{sdd}(R)$ we replace each element equal to 2 with 0 and denote the matrix obtained by B and similarly we replace each entry equal to $\frac{1}{2(n-1)}+2(n-1)$ with 0 and denote the resulting matrix by C. It is evident that $A_{sdd}(R) = B + C$. With the straightforward calculation, the energies of matrices B and C can be expressed as follows:

$$\varepsilon(B) = En(B) = n\left[\frac{1}{2(n-1)}+2(n-1)\right]En(S_n) = n\left[\frac{1}{2(n-1)}+2(n-1)\right]\times 2\sqrt{n-1},$$

and

$$\varepsilon(C) = En(C) = 2En(K_n) = 4(n-1).$$

By Ky Fan's Theorem, we have:

$$En(R, sdd) = \varepsilon\left(A_{sdd}(R)\right) \leq \varepsilon(B)+\varepsilon(C) = n\sqrt{n-1}\left[\frac{1}{n-1}+4(n-1)\right]+4(n-1).$$

■

Proposition 3.10. *Let $G = K_{n,n}^{uv}$ be the graph obtained from $K_{n,n}$ and K_2 by joining every vertex in one part of $K_{n,n}$ to the vertex u of K_2, and every vertex in the other part to the other vertex v of K_2. Then*
(i) *The eigenvalues of $A(K_{n,n}^{uv})$ are 0 with multiplicity $2n$ and $\pm(n+1)$.*
(ii) *$En(K_{n,n}^{uv}, sdd) = 4(n+1)$.*

Proof. (i) Suppose $X = \{u_1, ..., u_n\}$ is the set of vertices of one part and $V_1 = \{v_1, ..., v_n\}$ is the set of vertices of the other part of the graph $K_{n,n}$. We write the elements of matrix $G = K_{n,n}^{uv}$ such that the rows are labeled from top to bottom and the columns from left to right, sequentially with $u_1, ..., u_n, v_1, ..., v_n, u, v$.
In $det(\lambda I - A(K_{n,n}^{uv}))$, first, multiply the row corresponding to vertex v by -1 and add it to the rows corresponding to the vertices in set V_2. Next, do the same for vertex u , multiplying its

corresponding by -1 and adding it to the rows corresponding to the vertices in set Y. Finally, factor out λ from each row corresponding to a vertex in set $V_1 \cup V_2$. So,

$$det(\lambda I - A(K_{n,n}^{uv}) = \lambda^{2n}(\lambda - \alpha)(\lambda - \beta).$$

where α and β are the eigenvalues of $A(K_{n,n}^{uv})$.
Using Equation (1) we have:

$$\sum_{i=1}^{n} \lambda_i = \alpha + \beta = 0, \qquad \text{and} \qquad \sum_{i=1}^{n} \lambda_i^2 = \alpha^2 + \beta^2 = 2(n^2 + 2n + 1) = 2(n+1)^2.$$

Therefore,

$$\alpha = (n+1) \quad , \quad \beta = -(n+1) \quad \text{or} \quad \alpha = -(n+1) \quad , \quad \beta = (n+1).$$

(ii) Since the graph $K_{n,n}^{uv}$ is $(n+1)$-regular, by Corollary 3.3,

$$En(K_{n,n}^{uv}, sdd) = 2En(K_{n,n}^{uv}) = 2 \times 2(n+1) = 4(n+1).$$

■

Proposition 3.11. *For the graph $K_{n,n}^{uv}$, the following relation holds:*

$$En(K_{n,n}^{uv}, sdd) = 4(n+1).$$

Proof. Since the graph $K_{n,n}^{uv}$ is $(n+1)$-regular, by Corollary 3.3,

$$En(K_{n,n}^{uv}, sdd) = 2En(K_{n,n}^{uv}) = 2 \times 2(n+1) = 4(n+1).$$

■

Proposition 3.12. *Consider the complete bipartite graph $K_{m,n}$, where $U_1 = \{v_1, ..., v_m\}$ is one part and $U_2 = \{u_1, ..., u_n\}$ is the other part of $K_{m,n}$. If f is a graph function of the graph $K_{m,n}$ with $f(v_i, u_j) = a_i$ for $i = 1, ..., m$ and $j = 1, ..., n$, then*

$$En(K_{m,n}, f) = 2\sqrt{n\,(a_1^2 + \cdots + a_m^2)}\,,$$

especially,

$$En(K_{m,n}, sdd) = 2\left(\frac{m}{n} + \frac{n}{m}\right)\sqrt{mn}.$$

Proof. The matrix $A_f(K_{m,n})$ can be written as follows:

$$A_f(K_{m,n}) = \begin{bmatrix} 0 & W \\ W^T & 0 \end{bmatrix}, \quad \text{where} \quad W = \begin{bmatrix} 0 & W \\ W^T & 0 \end{bmatrix}.$$

On the other hand, we have:

$$rank(A_f(K_{m,n})) = rank(W) + rank(W^T) = 1 + 1 = 2, \quad \text{and}$$

$$null(A_f(K_{m,n})) = dim(A_f(K_{m,n})) - rank(A_f(K_{m,n})) = m + n - 2.$$

It follows from the recent equality that 0 is an eigenvalue of the matrix $A_f(K_{m,n})$ with multiplicity $m + n - 2$.

Now suppose that α and β are the other eigenvalues of the matrix $A_f(K_{m,n})$ such that $\alpha \geq \beta$. By Equation (1) we have:

$$\alpha + \beta = 0, \quad \text{and} \quad \alpha^2 + \beta^2 = 2 \sum_{uv \in E(G)} f^2(u,v) = 2n \left(a_1^2 + \cdots + a_m^2\right),$$

thus,

$$\alpha = \sqrt{n\,(a_1^2 + \cdots + a_m^2)}, \quad \text{and} \quad \beta = -\sqrt{n\,(a_1^2 + \cdots + a_m^2)}.$$

It follows that

$$En(K_{m,n}, f) = 2\sqrt{n\,(a_1^2 + \cdots + a_m^2)}\ .$$

Consequently, from the above relation, we have

$$En(K_{m,n}, ssd) = 2\sqrt{nm\left(\frac{m}{n} + \frac{n}{n}\right)^2} = 2\left(\frac{m}{n} + \frac{n}{m}\right)\sqrt{mn}.$$

■

Corollary 3.13. *Consider the star graph $S_n = (V, E)$ with vertex set $V = \{v_1, v_2, \ldots, v_n\}$, and central vertex v_1. Define a graph function f of this star graph by $f(v_1v_i) = a_i$ for $i = 2, ..., n$, and $f(v_i, v_j) = 0$ for $i, j \in \{2, 3, \ldots, n\}$, then we have:*

$$En(S_n, f) = 2\sqrt{a_2^2 + \cdots + a_n^2}\ .$$

Proposition 3.14. *Let $G = (V, E)$ be a $k-$regular graph of order n and size m. Then, we have:*

$$En(G, ssd) \leq n(k+1).$$

Proof. For each $v \in V(G)$, it is not difficult to check that

$$\Delta_f = 2k, \qquad M = 4m, \qquad \text{and} \quad M' = 2m.$$

Therefore, by Theorem 2.14, we obtain:

$$En(G, f) \leq \frac{n}{2}(\Delta_f + \frac{M}{M'}) = \frac{n}{2}(2k + \frac{4m}{2m}) = n(k+1).$$

■

Proposition 3.15. *Let $P_{n\geq 4}$ denote a path graph of order n. Then, we have:*

$$\frac{2}{9}(8n+1) \leq En(P_n, ssd) \leq \frac{n}{2}\left(\frac{13n-4}{2n-1}\right).$$

Proof. By Theorems 2.14 and 2.16, we have:

$$\frac{n}{2}(\Delta_{ssd} + \frac{M}{M'}) \geq En(P_n, ssd) \geq \frac{2M}{\Delta_{ssd}}.$$

On the other hand,

$$\Delta_{ssd} = 4.5\ , \quad M = 2 \times (2.5)^2 + (n-3) \times 4 = 4n + 0.5, \quad \text{and}$$

$$M' = 2 \times 2.5 + (n-3) \times 2 = 2n - 1.$$

Therefore,

$$\begin{aligned} \frac{n}{2}\left(\Delta_{ssd} + \frac{M}{M'}\right) &= \frac{n}{2}\left(4.5 + \frac{4n+0.5}{2n-1}\right) = \frac{n}{2}\left(\frac{13n-4}{+2n-1}\right) \geq En(P_n, sdd) \\ &\geq \frac{2M}{\Delta_{ssd}} = \frac{1+8n}{4.5} = \frac{2}{9}(8n+1). \end{aligned}$$

■

4 Concluding remarks

In this paper, we introduced a generalized adjacency matrix for weighted simple graphs, extending the classical adjacency matrix by incorporating edge weights through a symmetric weight function f. This framework preserves important spectral features while providing greater flexibility for the study of weighted graph structures. Based on this matrix, we investigated the corresponding graph energy and established several fundamental properties, bounds, and exact formulas.

A central aspect of our study was the symmetric division degree function $f(u,v) = \frac{d_u}{d_v} + \frac{d_v}{d_u}$, which naturally reflects the relationship between the degrees of adjacent vertices. Using this function, we derived explicit expressions for the energy of several important graph classes, including complete graphs, complete bipartite graphs, paths, and friendship graphs. We also obtained bounds for broader families of graphs, such as molecular graphs and bridge-connected bipartite graphs. These results show that the proposed energy generalizes the classical graph energy while retaining many of its desirable spectral properties.

Overall, the proposed framework provides a unified approach to studying weighted graph energies and highlights the influence of structural characteristics on spectral behavior. We hope that the ideas developed here will stimulate further investigations into generalized graph matrices, weighted graph invariants, and their applications in spectral graph theory and network analysis.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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