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Research Paper

Extremal α-Sum of the Randić Index and its Reciprocal in Small k-Uniform Hypertrees

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Abstract

For $\alpha > 0$ we study a Randić-type functional on k-uniform hypergraphs,

$$\Phi_\alpha(H) = \sum_{e \in E(H)} \left(P_H(e)^{-\alpha/2} + P_H(e)^{\alpha/2} \right), \quad P_H(e) = \prod_{v \in e} d_H(v),$$

which extends Gao's (R+RR) at $\alpha = 1$. We prove that splitting a (redundant) intersection strictly decreases Φ_α for every $\alpha > 0$, and deduce a sharp maximum with fixed order n: among simple connected k-uniform hypergraphs on n vertices, Φ_α is uniquely maximized by the complete hypergraph $K_n^{(k)}$. We then study minimizers on small k-uniform hypertrees, where $m = |E(H)|$ denotes the number of hyperedges. For $m = 3$ hyperedges, there is a unique threshold $\alpha_0 \approx 6.1412175821$ where the minimizer switches from the linear hyperpath $P_{3,k}$ to the sunflower $S(3, 1, k)$ (with both extremal at $\alpha = \alpha_0$). For $m = 4$ no transition occurs: $P_{4,k}$ is the unique minimizer for all $\alpha > 0$. Finally, we show a barrier at $\alpha = 4$: for $\alpha > 4$ branching can decrease Φ_α, invalidating direct pendent-path merging.

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1 Introduction

Degree-based *topological indices* are classical descriptors in mathematical chemistry (QSAR/QSPR); see, e.g., [1]. Among them, the Randić index was introduced by Randić as a measure of molecular branching [2] and has motivated an extensive literature, including sharp extremal bounds and structural characterizations in broad graph classes; see, e.g., [3, 4]. A standard generaliza-

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tion introduces a real parameter and considers the *generalized Randić index*

$$R_\alpha(G) = \sum_{uv \in E(G)} \big(d_G(u) d_G(v)\big)^\alpha,$$

whose systematic study (extremal properties, variations, and the role of the parameter) is surveyed, for instance, in [5]. This family also includes the *reciprocal Randić index*

$$RR(G) = \sum_{uv \in E(G)} \sqrt{d_G(u) d_G(v)},$$

which is related to irregularity measures and other degree-based variants [6]. Moreover, studying jointly an index and its reciprocal, in particular via their sum, has been proposed as a systematic research direction [7].

When relevant interactions are not strictly binary, it is natural to model them using *hypergraphs* [8–10]. In the k-uniform setting, a direct extension of Randić-type indices is to weight each hyperedge e through the product

$$P_H(e) := \prod_{v \in e} d_H(v),$$

which leads to multiplicative functionals analogous to those for graphs. This "hypergraph approach" to degree-products has been used explicitly to define and bound generalized Randić-type indices in uniform hypergraphs; see [11]. Besides degree-product formulations, recent research has also considered spectral extensions of Randić-type concepts for uniform hypergraphs. In particular, Shirdel et al. [12] introduced the Randić matrix of a uniform hypergraph and studied the corresponding Randić energy, establishing spectral properties and energy bounds. More generally, degree-based indices for hypergraphs have received increasing attention in recent years; see, for example, [13]. These developments indicate that Randić-type quantities in hypergraphs can be studied from both combinatorial and spectral perspectives.

We also note that [11] discusses alternative modeling choices, including adjacency-based and graph-inspired formulations, which are closely related to the spectral study of hypergraphs; see also [14]. Recently, Gao [15] introduced for k-uniform hypergraphs the functional

$$(\mathrm{R} + \mathrm{RR})(H) = \sum_{e \in E(H)} \left(\frac{1}{\sqrt{P_H(e)}} + \sqrt{P_H(e)} \right),$$

and solved extremal problems in natural classes, in particular in *hypertrees* (in the sense of [16]), obtaining an optimal upper bound in the connected class with fixed order n.

In this paper, we study a parametric extension of that functional. For $\alpha > 0$ we define

$$\mathrm{R}_\alpha(H) = \sum_{e \in E(H)} P_H(e)^{-\alpha/2}, \quad \mathrm{RR}_\alpha(H) = \sum_{e \in E(H)} P_H(e)^{\alpha/2}, \quad \Phi_\alpha(H) := (\mathrm{R}_\alpha + \mathrm{RR}_\alpha)(H),$$

which recovers [15] at $\alpha = 1$. From the chemical perspective, α should be understood as a *tuning parameter* (rather than a universal physicochemical constant), with potential interest as a family of degree-based descriptors.

Our contributions are: (i) we prove that splitting redundant intersections strictly decreases Φ_α for all $\alpha > 0$ (Lemma 3.5), allowing us to extend the fixed-order upper bound and characterize the complete hypergraph as the unique maximizer (Theorem 4.2); (ii) for hypertrees

with $m = 3$ hyperedges, we show a threshold α_0 where the minimizer switches from $P_{3,k}$ to $S(3,1,k)$ (Theorem 5.7); (iii) for $m = 4$, we show that the minimizer is always $P_{4,k}$ for every $\alpha > 0$ (Theorem 7.6); and (iv) we exhibit a methodological barrier: for $\alpha > 4$, there are local configurations (already for graphs) where branching decreases Φ_α, preventing a direct extension of the pendent-path merging technique (Proposition 6.1).

Relation with Gao's work

The present paper is motivated by the recent work of Gao [15], who investigated the quantity

$$R(H) + RR(H),$$

for uniform hypertrees and established extremal results for the case $\alpha = 1$.

Several structural ideas used in the present paper originate from Gao's approach. In particular, the comparison of local transformations on hypertrees and the analysis of degree-product expressions are inspired by techniques already appearing in [15]. Consequently, some arguments may be viewed as parameterized extensions of known methods.

However, the main contributions of the present work are genuinely new. First, we replace the fixed exponent 1 by a continuous parameter $\alpha > 0$, leading to the family of functionals

$$\Phi_\alpha(H) = R_\alpha(H) + RR_\alpha(H).$$

This introduces a new analytic component that is absent from the framework of [15].

Second, for hypertrees with three hyperedges, we prove the existence of a critical threshold $\alpha_0 \approx 6.1412175821$, at which the extremal structure changes from the path $P_{3,k}$ to the hyperstar $S^\star_{3,k}$. No analogous threshold phenomenon appears in the case $\alpha = 1$.

Third, we identify a structural barrier at $\alpha = 4$, showing that the pendent-path merging principle underlying Gao's method cannot be extended uniformly to all values of α. This reveals a qualitative change in the behavior of Φ_α and explains why new techniques are required beyond the classical setting.

Finally, the complete classification of extremal k-uniform hypertrees with three and four hyperedges for arbitrary $\alpha > 0$ appears to be new.

Organization. Section 2 contains notation. Section 3 establishes analytic lemmas and the splitting operation. Section 4 proves upper bounds with fixed order n. Section 5 treats $m = 3$ and the threshold α_0. Section 6 discusses the barrier at $\alpha = 4$. Section 7 treats $m = 4$. We close with conclusions.

2 Preliminaries and notation

Definition 2.1 (Hyperedge contribution)**.** Let H be a k-uniform hypergraph and let $e \in E(H)$. Define

$$P_H(e) := \prod_{v \in e} d_H(v), \qquad \Phi_\alpha(H; e) := P_H(e)^{\alpha/2} + P_H(e)^{-\alpha/2}.$$

Then

$$\Phi_\alpha(H) = \sum_{e \in E(H)} \Phi_\alpha(H; e).$$

Definition 2.2 (Hypertree [16])**.** Let $H = (V, E)$ be a simple hypergraph. For $e \in E$ write $H - e := (V, E \setminus \{e\})$ and let $c(\cdot)$ denote the number of connected components (including isolated

vertices). We say that H is a *hypertree* if H is connected and, for every hyperedge $e \in E$, one has

$$c(H - e) = c(H) + |e| - 1.$$

Remark 1. If H is connected and k-uniform, then $c(H) = 1$ and the condition in Definition 2.2 is equivalent to $c(H - e) = k$ for every $e \in E(H)$.

Lemma 2.3 (Intersections in k-uniform hypertrees)**.** *Let T be a k-uniform hypertree. Then for any two distinct hyperedges $e \neq f$ of T, we have:*

$$|e \cap f| \leq 1.$$

Proof. Since T is k-uniform and connected, Remark 1 gives $c(T - e) = k$ for every hyperedge e.

Assume for a contradiction that there exist $e \neq f$ with $|e \cap f| \geq 2$, and choose $x \neq y$ in $e \cap f$. In the hypergraph $T - e$ the hyperedge f remains, so x and y still lie in the same connected component of $T - e$. In particular, after removing e, it is impossible for the k vertices of e to be distributed among k distinct components, so necessarily $c(T - e) \leq k - 1$, contradicting $c(T - e) = k$. ■

Definition 2.4 (Intersection graph)**.** Let $H = (V, E)$ be a hypergraph. The *intersection graph* $L(H)$ is the graph whose vertices are the hyperedges of H, where two distinct hyperedges $e, f \in E$ are adjacent if and only if $e \cap f \neq \varnothing$.

Definition 2.5 (Linear hyperpath and sunflower)**.** A *linear k-uniform hyperpath* is a hyperpath in which any two hyperedges have at most one vertex in common. We denote by $P_{m,k}$ the linear k-uniform hyperpath with m hyperedges.

Let $m > 0$ and $0 < h < k$. A *sunflower* $S(m, h, k)$ is the k-uniform hypergraph defined by a core set A of h vertices and pairwise disjoint sets $B_1, \ldots, B_m$ of size $k - h$, whose hyperedges are $A \cup B_i$ for $1 \leq i \leq m$.

In particular, $S(3, 1, k)$ is a sunflower with a one-vertex core; we call it the *hyperstar* and denote it by $S^{\star}_{3,k} := S(3, 1, k)$.

3 Analytic lemmas and a splitting operation

Lemma 3.1 (Basic monotonicity)**.** *Let $\alpha > 0$ and $f_\alpha(x) = x^\alpha + x^{-\alpha}$ for $x > 0$. Then f_α is strictly increasing on $[1, \infty)$.*

Proof. For $x > 0$,

$$f'_\alpha(x) = \alpha x^{\alpha-1} - \alpha x^{-\alpha-1} = \alpha x^{-\alpha-1}(x^{2\alpha} - 1).$$

If $x \geq 1$ then $x^{2\alpha} - 1 > 0$, hence $f'_\alpha(x) > 0$. ■

The following operation is the direct analogue (for Φ_α) of the key idea in Lemma 2.4 of [15]: splitting an intersection decreases the functional.

Definition 3.2 (Splitting an intersection)**.** Let H be a hypergraph and suppose two distinct hyperedges e_1, e_2 share a vertex $u \in e_1 \cap e_2$. Let u' be a new vertex (not in $V(H)$) and define H' by replacing u with u' in e_2, that is,

$$E(H') = \big(E(H) \setminus \{e_2\}\big) \cup \{(e_2 \setminus \{u\}) \cup \{u'\}\},$$

keeping all other hyperedges and vertices unchanged.

Definition 3.3 (Redundant intersection vertex)**.** Let $e_1 \neq e_2$ be two hyperedges with $e_1 \cap e_2 \neq \varnothing$. We say that $u \in e_1 \cap e_2$ is *redundant (for the pair e_1, e_2)* if

$$(e_1 \cap e_2) \setminus \{u\} \neq \varnothing,$$

that is, if e_1 and e_2 share at least one more vertex besides u.

Lemma 3.4 (Redundant splitting preserves connectedness)**.** *Let H be a connected hypergraph and let $e_1 \neq e_2$ be hyperedges that share a redundant vertex $u \in e_1 \cap e_2$ (in the sense of Definition 3.3). Let H' be obtained by splitting u in e_2 as in Definition 3.2. Then H' is connected.*

Proof. Let $e_2' = (e_2 \setminus \{u\}) \cup \{u'\}$. By redundancy there exists $w \in (e_1 \cap e_2) \setminus \{u\}$, hence $w \in e_1 \cap e_2'$. Thus, any walk in H using e_2 can be replaced by a walk in H' using e_2', and essential intersections are not lost. Moreover, the new vertex u' belongs to e_2' and is therefore connected to the rest via e_2' and w. Hence, H' is connected. ■

Lemma 3.5 (Splitting decreases Φ_α (an α-extension of Lemma 2.4 in [15]))**.** *Let H be a k-uniform hypergraph and let H' be obtained by the operation in Definition 3.2. Then for every $\alpha > 0$,*

$$\Phi_\alpha(H') < \Phi_\alpha(H).$$

Proof. Let $s = d_H(u) \geq 2$ and write $e_2' := (e_2 \setminus \{u\}) \cup \{u'\}$, so that $E(H') = (E(H) \setminus \{e_2\}) \cup \{e_2'\}$. In H' the vertex u decreases its degree from s to $s-1$, while the new vertex u' has degree 1. Thus, the only hyperedges whose degree-product may change are:

- all hyperedges $e \in E(H)$ with $u \in e$ (since $d_H(u)$ changes), and
- the hyperedge e_2, which is replaced by e_2'.

Let $e \in E(H)$ with $u \in e$ and $e \neq e_2$. Then all degrees in e remain the same except at u, hence

$$P_{H'}(e) = P_H(e) \cdot \frac{s-1}{s},$$

so, $P_{H'}(e) < P_H(e)$.

For the replaced hyperedge,

$$P_H(e_2) = d_H(u) \prod_{v \in e_2 \setminus \{u\}} d_H(v) = s \prod_{v \in e_2 \setminus \{u\}} d_H(v),$$

whereas in H',

$$P_{H'}(e_2') = d_{H'}(u') \prod_{v \in e_2 \setminus \{u\}} d_{H'}(v) = 1 \cdot \prod_{v \in e_2 \setminus \{u\}} d_H(v) = \frac{1}{s} P_H(e_2).$$

Thus, $P_{H'}(e_2') < P_H(e_2)$.

If $f \in E(H)$ satisfies $u \notin f$ and $f \neq e_2$, then degrees on f do not change and $\Phi_\alpha(H'; f) = \Phi_\alpha(H; f)$.

Finally, since every vertex degree is at least 1, we have $P_H(f) \geq 1$ for all $f \in E(H)$ and also $P_{H'}(f) \geq 1$ for all $f \in E(H')$. Therefore, we are in the interval $[1, \infty)$ where the function $x \mapsto x^{\alpha/2} + x^{-\alpha/2}$ is strictly increasing (Lemma 3.1). Hence, from $P_{H'}(e) < P_H(e)$ we obtain

$$\Phi_\alpha(H'; e) < \Phi_\alpha(H; e) \quad \text{for every affected } e \in E(H).$$

In particular, the contribution of e_2 decreases strictly, and no other contribution increases, so $\Phi_\alpha(H') < \Phi_\alpha(H)$. ■

Remark 2. Lemma 3.5 formalizes the heuristic that decreasing degree-products in hyperedges decreases Φ_α, which drives several extremal arguments in [15].

4 Upper bounds with fixed order n

Definition 4.1. The *complete k-uniform hypergraph* on n vertices, denoted by $K_n^{(k)}$, is the k-uniform hypergraph whose hyperedges are all k-subsets of its vertex set V (with $|V| = n$).

Theorem 4.2 (Maximum with fixed order n (an α-extension of Theorem 5.1 in [15]))**.** *Let H be a simple connected k-uniform hypergraph on n vertices, where $2 \le k \le n$ and $n \ge 3$. Then for every $\alpha > 0$,*

$$\Phi_\alpha(H) \le \binom{n}{k}\left[\binom{n-1}{k-1}^{\alpha k/2} + \binom{n-1}{k-1}^{-\alpha k/2}\right].$$

Equality holds if and only if $H \cong K_n^{(k)}$.

Proof. Since H is k-uniform, for each vertex $v \in V(H)$ the number of hyperedges that may contain v is at most the number of $(k-1)$-subsets of $V(H) \setminus \{v\}$, namely

$$d_H(v) \le \binom{n-1}{k-1}.$$

Therefore, for each hyperedge $e \in E(H)$,

$$P_H(e) = \prod_{v \in e} d_H(v) \le \binom{n-1}{k-1}^k.$$

Consider the function

$$g_\alpha(x) = x^{\alpha/2} + x^{-\alpha/2} \qquad (x \ge 1).$$

It is strictly increasing on $[1, \infty)$, since

$$g'_\alpha(x) = \frac{\alpha}{2}\, x^{\alpha/2-1} - \frac{\alpha}{2}\, x^{-\alpha/2-1} = \frac{\alpha}{2}\, x^{-\alpha/2-1}\left(x^\alpha - 1\right) > 0 \quad \text{for } x \ge 1.$$

As $P_H(e) \ge 1$, monotonicity yields for every $e \in E(H)$,

$$\Phi_\alpha(H; e) = P_H(e)^{\alpha/2} + P_H(e)^{-\alpha/2} \le \binom{n-1}{k-1}^{\alpha k/2} + \binom{n-1}{k-1}^{-\alpha k/2}.$$

Finally, a k-uniform hypergraph on n vertices has at most $\binom{n}{k}$ hyperedges, hence

$$\begin{aligned}\Phi_\alpha(H) &= \sum_{e \in E(H)} \Phi_\alpha(H; e) \\ &\le |E(H)|\Big[\binom{n-1}{k-1}^{\alpha k/2} + \binom{n-1}{k-1}^{-\alpha k/2}\Big] \\ &\le \binom{n}{k}\Big[\binom{n-1}{k-1}^{\alpha k/2} + \binom{n-1}{k-1}^{-\alpha k/2}\Big].\end{aligned}$$

For equality, we must have simultaneously:

- $|E(H)| = \binom{n}{k}$, and
- $P_H(e) = \binom{n-1}{k-1}^k$ for every $e \in E(H)$, forcing $d_H(v) = \binom{n-1}{k-1}$ for all $v \in V(H)$.

The first condition implies that $E(H)$ contains all k-subsets of $V(H)$, i.e. $H \cong K_n^{(k)}$. Conversely, if $H \cong K_n^{(k)}$, then $|E(H)| = \binom{n}{k}$ and every vertex satisfies $d_H(v) = \binom{n-1}{k-1}$, so equality holds. ■

Corollary 4.3 (A separate upper bound for RR_α)**.** *Let H be a simple connected k-uniform hypergraph on n vertices, where $2 \le k \le n$ and $n \ge 3$. Then, for every $\alpha > 0$,*

$$\mathrm{RR}_\alpha(H) = \sum_{e \in E(H)} \Big(\prod_{v \in e} d_H(v) \Big)^{\alpha/2} \le \binom{n}{k} \binom{n-1}{k-1}^{\alpha k/2}.$$

Equality holds if and only if $H \cong K_n^{(k)}$.

Proof. As observed in the proof of Theorem 4.2, every vertex of H satisfies

$$d_H(v) \le \binom{n-1}{k-1}.$$

Consequently, for every hyperedge $e \in E(H)$,

$$P_H(e) = \prod_{v \in e} d_H(v) \le \binom{n-1}{k-1}^k.$$

Since $\alpha > 0$, it follows that

$$\big(P_H(e)\big)^{\alpha/2} \le \binom{n-1}{k-1}^{\alpha k/2}.$$

Summing over all hyperedges yields $\mathrm{RR}_\alpha(H) \le |E(H)| \binom{n-1}{k-1}^{\alpha k/2}$. Since a simple k-uniform hypergraph on n vertices has at most $\binom{n}{k}$ hyperedges, we obtain

$$\mathrm{RR}_\alpha(H) \le \binom{n}{k} \binom{n-1}{k-1}^{\alpha k/2}.$$

Suppose now that equality holds. Then equality must occur in both inequalities above. In particular, $|E(H)| = \binom{n}{k}$. Since H is a simple k-uniform hypergraph on n vertices, this implies that every k-subset of $V(H)$ is a hyperedge. Hence $H \cong K_n^{(k)}$. Conversely, if $H \cong K_n^{(k)}$, then

$$|E(H)| = \binom{n}{k} \quad \text{and} \quad d_H(v) = \binom{n-1}{k-1},$$

for every vertex $v \in V(H)$. Therefore, $P_H(e) = \binom{n-1}{k-1}^k$ for every hyperedge e, and the above bound is attained with equality. ■

Corollary 4.4 (Recovery of Theorem 5.1 in [15])**.** *Under the hypotheses of Theorem 4.2, taking $\alpha = 1$ yields*

$$(\mathrm{R} + \mathrm{RR})(H) \le \binom{n}{k} \left[\binom{n-1}{k-1}^{k/2} + \binom{n-1}{k-1}^{-k/2} \right],$$

with equality if and only if $H \cong K_n^{(k)}$.

Proof. Substituting $\alpha = 1$ into Theorem 4.2 gives the stated bound, and the equality characterization is unchanged. ■

5 The case $m = 3$: structural classification of minimizers

In this section, we fix $k \geq 2$ and consider k-uniform hypertrees with exactly $m = 3$ hyperedges.

5.1 Two canonical candidates

Lemma 5.1 (Closed forms (an α-version of Lemmas 3.1–3.2 in [15], case $m = 3$)). *Let $\alpha > 0$. Then*

$$\Phi_\alpha(P_{3,k}) = 2\Big(2^{\alpha/2} + 2^{-\alpha/2}\Big) + \Big(2^{\alpha} + 2^{-\alpha}\Big),$$

and

$$\Phi_\alpha(S^\star_{3,k}) = 3\Big(3^{\alpha/2} + 3^{-\alpha/2}\Big).$$

Proof. In the linear hyperpath $P_{3,k}$, the two pendent hyperedges each contain exactly one vertex of degree 2 and $k-1$ pendent vertices, so their degree-product equals 2. The central hyperedge contains two vertices of degree 2 and $k-2$ pendent vertices, so its product is 4. Summing the three contributions gives the stated formula for $\Phi_\alpha(P_{3,k})$.

In the sunflower $S^\star_{3,k} = S(3, 1, k)$ there is a core vertex of degree 3, and each hyperedge has degree-product 3. Summing the three contributions gives the stated expression. ■

5.2 Reduction to the linear form when $\Delta(T) \leq 2$

Lemma 5.2 (Minimization under $\Delta(T) \leq 2$). *Let T be a k-uniform hypertree with $|E(T)| = 3$ and $\Delta(T) := \max_{v \in V(T)} d_T(v) \leq 2$. Then for every $\alpha > 0$,*

$$\Phi_\alpha(T) \geq \Phi_\alpha(P_{3,k}),$$

with equality if and only if $T \cong P_{3,k}$.

Proof. By Lemma 2.3, any two hyperedges of T intersect in at most one vertex. Since $|E(T)| = 3$ and T is connected, the intersection graph $L(T)$ (Definition 2.4) is a connected graph on three vertices, hence $L(T) \cong P_3$ or $L(T) \cong C_3$.

We claim that $L(T)$ cannot be a triangle under $\Delta(T) \leq 2$. Indeed, if $L(T) \cong C_3$, then every pair of hyperedges intersects. Since no vertex belongs to three hyperedges (because $\Delta(T) \leq 2$), the three intersections $e_1 \cap e_2$, $e_2 \cap e_3$, $e_3 \cap e_1$ must be distinct vertices. Let $u \in e_1 \cap e_2$ and $v \in e_2 \cap e_3$. In $T - e_2$ the hyperedges e_1 and e_3 remain and intersect, hence u and v lie in the same connected component of $T - e_2$. Thus two vertices of e_2 lie in one component of $T - e_2$, so $c(T - e_2) \leq k - 1$, contradicting that T is a k-uniform hypertree (Remark 1).

Therefore $L(T) \cong P_3$, i.e. there are hyperedges e_1, e_2, e_3 with $e_1 \cap e_2 \neq \varnothing$, $e_2 \cap e_3 \neq \varnothing$, and $e_1 \cap e_3 = \varnothing$. By Lemma 2.3, $|e_1 \cap e_2| = |e_2 \cap e_3| = 1$. This determines (up to isomorphism) exactly the linear hyperpath $P_{3,k}$: there are two vertices of degree 2 (one in $e_1 \cap e_2$ and one in $e_2 \cap e_3$), and all other vertices are pendent. Hence $T \cong P_{3,k}$ and the claim follows. ■

5.3 Reduction to the sunflower when a degree-3 vertex exists

Lemma 5.3 (Minimization under existence of degree 3). *Let T be a k-uniform hypertree with $|E(T)| = 3$ such that there exists $u \in V(T)$ with $d_T(u) = 3$. Then for every $\alpha > 0$,*

$$\Phi_\alpha(T) \geq \Phi_\alpha(S^\star_{3,k}),$$

with equality if and only if $T \cong S^\star_{3,k}$.

Proof. If there exists $u \in V(T)$ with $d_T(u) = 3$ and $|E(T)| = 3$, then u belongs to all three hyperedges of T. By Lemma 2.3, any two hyperedges intersect in at most one vertex, and since they already share u, their intersection is exactly $\{u\}$. Thus the three hyperedges meet only in the core vertex u, so $T \cong S(3,1,k) = S^{\star}_{3,k}$ and the statement follows. ■

5.4 Comparison and threshold in α

By Lemmas 5.2 and 5.3, any global minimizer (among k-uniform hypertrees with $m = 3$ hyperedges) must be isomorphic to $P_{3,k}$ or $S^{\star}_{3,k}$. Thus, the problem reduces to comparing the two expressions in Lemma 5.1.

Define the difference function

$$F(\alpha) := \Phi_\alpha(P_{3,k}) - \Phi_\alpha(S^{\star}_{3,k}) = 2\Big(2^{\alpha/2} + 2^{-\alpha/2}\Big) + \Big(2^{\alpha} + 2^{-\alpha}\Big) - 3\Big(3^{\alpha/2} + 3^{-\alpha/2}\Big).$$

Note that $F(0) = 0$ (interpreting the limit as $\alpha \downarrow 0$) and $F(\alpha) = F(-\alpha)$.

Lemma 5.4 (Convexity from $\alpha = 4$)**.** *The function F is strictly convex on $[4, \infty)$; in particular,*

$$F''(\alpha) > 0 \qquad \textit{for all } \alpha \geq 4.$$

Proof. Let $t = \alpha/2 \geq 2$. Then,

$$F(\alpha) = 2(2^t + 2^{-t}) + (4^t + 4^{-t}) - 3(3^t + 3^{-t}).$$

A direct differentiation gives

$$F''(\alpha) = \frac{(\log 2)^2}{2}(2^t + 2^{-t}) + (\log 2)^2(4^t + 4^{-t}) - \frac{3(\log 3)^2}{4}(3^t + 3^{-t}).$$

Define

$$D(t) = \frac{(\log 2)^2}{2}(2^t + 2^{-t}) + (\log 2)^2(4^t + 4^{-t}) - \frac{3(\log 3)^2}{4}(3^t + 3^{-t}).$$

Thus,

$$F''(\alpha) = D(\alpha/2).$$

Differentiating with respect to t, we obtain

$$D'(t) = (\log 2)^3 \left(2(4^t - 4^{-t}) + \frac{1}{2}(2^t - 2^{-t})\right) - \frac{3(\log 3)^3}{4}(3^t - 3^{-t}).$$

For $t \geq 2$,

$$4^t - 4^{-t} = 4^t(1 - 4^{-2t}) \geq \frac{255}{256}4^t,$$

and $3^t - 3^{-t} \leq 3^t$. Also,

$$2^t - 2^{-t} \geq 0.$$

Therefore,

$$D'(t) \geq 2\frac{255}{256}(\log 2)^3\, 4^t - \frac{3}{4}(\log 3)^3\, 3^t.$$

Since $t \geq 2$, we have

$$\left(\frac{4}{3}\right)^t \geq \left(\frac{4}{3}\right)^2 = \frac{16}{9}.$$

Hence,

$$D'(t) \geq \left(2\frac{255}{256}\frac{16}{9}(\log 2)^3 - \frac{3}{4}(\log 3)^3\right) 3^t.$$

Using the well-known estimates $\log 2 > 0.69$, and $\log 3 < 1.1$, we obtain:

$$2\frac{255}{256}\frac{16}{9}(\log 2)^3 - \frac{3}{4}(\log 3)^3 > 2\frac{255}{256}\frac{16}{9}(0.69)^3 - \frac{3}{4}(1.1)^3 > 0.$$

Consequently,

$$D'(t) > 0 \qquad \text{for all } t \geq 2.$$

Thus, D is strictly increasing on $[2, \infty)$.

It remains to check that $D(2) > 0$. We have

$$D(2) = \left(\frac{17}{8} + \frac{257}{16}\right)(\log 2)^2 - \frac{3}{4}\left(9 + \frac{1}{9}\right)(\log 3)^2.$$

Using again

$$\log 2 > \frac{69}{100}, \qquad \log 3 < \frac{11}{10},$$

we get

$$D(2) > \left(\frac{17}{8} + \frac{257}{16}\right)\left(\frac{69}{100}\right)^2 - \frac{3}{4}\left(9 + \frac{1}{9}\right)\left(\frac{11}{10}\right)^2 > 0.$$

Therefore, $D(t) > 0$ for all $t \geq 2$. Since

$$F''(\alpha) = D(\alpha/2),$$

we conclude that

$$F''(\alpha) > 0 \qquad \text{for all } \alpha \geq 4.$$

Hence, F is strictly convex on $[4, \infty)$. ■

Lemma 5.5. *For every $0 < \alpha \leq 4$, one has $F(\alpha) < 0$.*

Proof. Put $t = \dfrac{\alpha}{2}$. Then $0 < t \leq 2$, and $F(\alpha) = G(t)$, where

$$G(t) = 2(2^t + 2^{-t}) + (4^t + 4^{-t}) - 3(3^t + 3^{-t}).$$

Let $s = t \log 2$, $r = \dfrac{\log 3}{\log 2}$. Then, $0 < s \leq 2 \log 2 = \log 4$, and

$$G(t) = 4\cosh s + 2\cosh(2s) - 6\cosh(rs).$$

Thus $G(t) < 0$ is equivalent to

$$D(s) := 3\cosh(rs) - 2\cosh s - \cosh(2s) > 0.$$

We shall prove that $D(s) > 0$ for $0 < s \leq \log 4$. Since

$$r = \frac{\log 3}{\log 2} > \frac{19}{12},$$

and since $\cosh x$ is increasing for $x > 0$, we have

$$\cosh(rs) > \cosh\left(\frac{19}{12}s\right).$$

Therefore,

$$D(s) > 3\cosh\left(\frac{19}{12}s\right) - 2\cosh s - \cosh(2s).$$

For $0 < s \leq \log 4$, Taylor's formula with remainder gives

$$\cosh\left(\frac{19}{12}s\right) \geq 1 + \frac{1}{2}\left(\frac{19}{12}s\right)^2 + \frac{1}{24}\left(\frac{19}{12}s\right)^4 + \frac{1}{720}\left(\frac{19}{12}s\right)^6,$$

while

$$\cosh s \leq 1 + \frac{s^2}{2} + \frac{s^4}{24} + \frac{\cosh(\log 4)}{720}s^6 + \frac{\cosh(\log 4)}{40320}s^8,$$

and

$$\cosh(2s) \leq 1 + \frac{(2s)^2}{2} + \frac{(2s)^4}{24} + \frac{(2s)^6}{720} + \frac{\cosh(2\log 4)}{40320}(2s)^8.$$

Since

$$\cosh(\log 4) = \frac{17}{8}, \qquad \cosh(2\log 4) = \frac{257}{32},$$

we obtain $D(s) > P(s)$, where

$$P(s) = \frac{73}{96}s^2 + \frac{5905}{165888}s^4 - \frac{4177051}{143327232}s^6 - \frac{2747}{53760}s^8.$$

Writing $z = s^2$, this becomes

$$P(s) = z\left(\frac{73}{96} + \frac{5905}{165888}z - \frac{4177051}{143327232}z^2 - \frac{2747}{53760}z^3\right).$$

Because $0 < s \leq \log 4$, we have $0 < z < 2$. Hence,

$$\begin{aligned}\frac{73}{96} + \frac{5905}{165888}z - \frac{4177051}{143327232}z^2 - \frac{2747}{53760}z^3 &\geq \frac{73}{96} - \frac{4177051}{143327232}\cdot 2 - \frac{2747}{53760}\cdot 4 \\ &= \frac{1248444367}{2508226560} > 0.\end{aligned}$$

Therefore $P(s) > 0$, and consequently

$$D(s) > 0, \qquad \text{for } 0 < s \leq \log 4.$$

Thus,

$$G(t) < 0, \qquad \text{for } 0 < t \leq 2.$$

Equivalently,

$$F(\alpha) < 0, \qquad \text{for } 0 < \alpha \leq 4.$$

■

Lemma 5.6 (Existence and uniqueness of the threshold α_0)**.** *There exists a unique real number $\alpha_0 > 4$ such that $F(\alpha_0) = 0$. Moreover, $F(\alpha) < 0$ for $0 < \alpha < \alpha_0$, and $F(\alpha) > 0$ for $\alpha > \alpha_0$.*

Proof. By Lemma 5.5,

$$F(\alpha) < 0 \qquad \text{for every } 0 < \alpha \leq 4.$$

In particular, $F(4) < 0$. On the other hand,

$$\lim_{\alpha \to \infty} F(\alpha) = +\infty,$$

since the dominant term of $F(\alpha)$ is 2^α, which grows faster than $3^{\alpha/2}$.
Therefore, by continuity, F has at least one zero in the interval $(4, \infty)$.

Now, by Lemma 5.4, the function F is strictly convex on $[4, \infty)$. Consequently, F' is strictly increasing on $[4, \infty)$.
Since $F(4) < 0$ and $F(\alpha) \to +\infty$ as $\alpha \to \infty$, strict convexity implies that F can intersect the horizontal axis at most once on $(4, \infty)$. Hence there exists a unique number $\alpha_0 \in (4, \infty)$, such that $F(\alpha_0) = 0$.

Finally, since $F(\alpha) < 0$ on $(0, 4]$, and α_0 is the unique zero of F in $(4, \infty)$, it follows that

$$F(\alpha) < 0 \qquad \text{for } 0 < \alpha < \alpha_0.$$

Moreover, because F is continuous and has no further zeros for $\alpha > \alpha_0$, we obtain

$$F(\alpha) > 0 \qquad \text{for } \alpha > \alpha_0.$$

Therefore, α_0 is the unique threshold satisfying the stated sign change, and in particular $\alpha_0 > 4$. ■

Remark 3. [Numerical value] The equation $F(\alpha) = 0$ is equivalent to

$$2\left(2^{\alpha/2} + 2^{-\alpha/2}\right) + \left(2^{\alpha} + 2^{-\alpha}\right) = 3\left(3^{\alpha/2} + 3^{-\alpha/2}\right).$$

Its unique positive solution is $\alpha_0 \approx 6.1412175821$.

Theorem 5.7 (Threshold in α for the minimizer when $m = 3$)**.** *Let $k \geq 2$ and let T be a k-uniform hypertree with $|E(T)| = 3$. Let α_0 be the threshold from Lemma 5.6. Then:*

- *if $0 < \alpha < \alpha_0$, then $\Phi_\alpha(T) \geq \Phi_\alpha(P_{3,k})$ with equality if and only if $T \cong P_{3,k}$;*
- *if $\alpha > \alpha_0$, then $\Phi_\alpha(T) \geq \Phi_\alpha(S^\star_{3,k})$ with equality if and only if $T \cong S^\star_{3,k}$;*
- *if $\alpha = \alpha_0$, then both attain the same value and are exactly the two minimizers.*

Proof. By Lemmas 5.2 and 5.3, any global minimizer must be isomorphic to $P_{3,k}$ or $S^\star_{3,k}$. The comparison is governed by the sign of $F(\alpha) = \Phi_\alpha(P_{3,k}) - \Phi_\alpha(S^\star_{3,k})$. By Lemma 5.6, $F(\alpha) < 0$ iff $0 < \alpha < \alpha_0$, and $F(\alpha) > 0$ iff $\alpha > \alpha_0$. The statement follows. ■

6 A methodological threshold at $\alpha = 4$ for pendent-path merging

In [15], the minimum for $m \geq 3$ is enforced by transformations that essentially compare attaching two pendent paths versus attaching a single longer one (Lemmas 2.5-2.7 in [15]). In this section, we show that this principle cannot, in general, be extended to Φ_α when $\alpha > 4$.

To simplify and capture the phenomenon, it suffices to consider the case $k = 2$ (graphs). Let G be a connected graph and let $u \in V(G)$ with $\deg_G(u) = p \geq 1$. Let $G_u(1, 1)$ be the graph obtained from G by attaching two new leaves to u, and let $G_u(2)$ be the graph obtained by attaching a path of length 2 to u (a new vertex x adjacent to u and a leaf adjacent to x). This is the same type of local operation underlying Lemma 2.5 in [15].

Proposition 6.1 (Failure of pendent merging for $\alpha > 4$)**.** *For every $\alpha > 4$ there exist $p \geq 1$ and a connected graph G with a vertex u of degree p such that*

$$\Phi_\alpha(G_u(1,1)) < \Phi_\alpha(G_u(2)).$$

In particular, the pendent-path merging argument used for $\alpha = 1$ does not extend directly to the whole range $\alpha > 0$.

Proof. Let $t = \dfrac{\alpha}{2}$. Since $\alpha > 4$, we have $t > 2$. Recall that for graphs,

$$\Phi_\alpha(G) = \sum_{xy \in E(G)} \Big((\deg x \, \deg y)^t + (\deg x \, \deg y)^{-t} \Big).$$

Take $G = K_{1,p}$, with center u. In $G_u(1,1)$, the center has degree $p+2$, and every edge has degree-product $p+2$. Hence,

$$\Phi_\alpha(G_u(1,1)) = (p+2)\big((p+2)^t + (p+2)^{-t}\big).$$

In $G_u(2)$, the original p edges have degree-product $p+1$, the new edge incident with u has degree-product $2(p+1)$, and the terminal edge has degree-product 2. Therefore,

$$\Phi_\alpha(G_u(2)) = p\big((p+1)^t + (p+1)^{-t}\big) + (2(p+1))^t + (2(p+1))^{-t} + 2^t + 2^{-t}.$$

Then,

$$\begin{aligned}\Delta_t(p) = (p+2)\big((p+2)^t + (p+2)^{-t}\big) - p\big((p+1)^t + (p+1)^{-t}\big) \\ - (2(p+1))^t - (2(p+1))^{-t} - 2^t - 2^{-t}.\end{aligned}$$

We now estimate $\Delta_t(p)$ as $p \to \infty$. Since t is fixed, Taylor's formula gives

$$(p+2)^{t+1} = p^{t+1}\left(1 + \frac{2}{p}\right)^{t+1} = p^{t+1} + 2(t+1)p^t + O(p^{t-1}),$$

and

$$p(p+1)^t = p^{t+1}\left(1 + \frac{1}{p}\right)^t = p^{t+1} + tp^t + O(p^{t-1}).$$

Similarly,

$$(2(p+1))^t = 2^t p^t \left(1 + \frac{1}{p}\right)^t = 2^t p^t + O(p^{t-1}).$$

The remaining negative-power terms satisfy

$$(p+2)^{1-t} = o(p^t), \qquad p(p+1)^{-t} = o(p^t), \qquad (2(p+1))^{-t} = o(p^t),$$

and the constant term $2^t + 2^{-t}$ is also $o(p^t)$. Consequently,

$$\Delta_t(p) = \big(t + 2 - 2^t\big)p^t + o(p^t).$$

Since $t > 2$, we have $2^t > t + 2$. Indeed, the function $h(t) = 2^t - t - 2$ satisfies $h(2) = 0$ and

$$h'(t) = 2^t \log 2 - 1 \geq 4 \log 2 - 1 > 0, \qquad (t \geq 2).$$

Thus, $t + 2 - 2^t < 0$. It follows that $\Delta_t(p) < 0$ for all sufficiently large p. Hence, for such p,

$$\Phi_\alpha(G_u(1,1)) < \Phi_\alpha(G_u(2)).$$

This proves the claim.

■

Remark 4. The critical value $t = 2$ (i.e. $\alpha = 4$) appears structurally via the equation $t + 2 = 2^t$ governing the sign of the leading term. This identifies $\alpha = 4$ as a methodological threshold for pendent-merging arguments when studying Φ_α.

7 The case $m = 4$: no threshold and the minimizer is always $P_{4,k}$

In this section, we fix $k \geq 2$ and consider k-uniform hypertrees with exactly $m = 4$ hyperedges. As before, write $f_t(x) = x^t + x^{-t}$ with $t = \alpha/2 > 0$.

7.1 Three canonical candidates and a branched linear variant

We define three canonical shapes (all k-uniform):

- the linear hyperpath $P_{4,k}$;
- the sunflower $S(4, 1, k)$ (core a single vertex of degree 4), denoted $S^{\star}_{4,k}$;
- the *broom* $B_{4,k}$: three hyperedges share a vertex u (of degree 3), and a fourth hyperedge is attached to one of those three via a vertex v (of degree 2), with $v \neq u$, and no further intersections.

Moreover, within the subclass $\Delta(T) \leq 2$ there is a branched realization:

- the *linear edge-star* $E_{4,k}$: one central hyperedge is adjacent to the other three via three distinct vertices (each of degree 2), and the three leaves are pairwise disjoint.

Lemma 7.1 (Closed forms for the four models)**.** *Let $\alpha > 0$ and $t = \alpha/2$. Then:*

$$\Phi_\alpha(P_{4,k}) = 2f_t(2) + 2f_t(4), \qquad \Phi_\alpha(S^{\star}_{4,k}) = 4f_t(4),$$

$$\Phi_\alpha(E_{4,k}) = 3f_t(2) + f_t(8), \qquad \Phi_\alpha(B_{4,k}) = f_t(2) + 2f_t(3) + f_t(6).$$

Proof. Compute the degree-product $P_T(e) = \prod_{v \in e} d_T(v)$ for each hyperedge and sum contributions. In $P_{4,k}$ there are two hyperedges with product 2 (the ends) and two with product 4 (the internal ones). In $S^{\star}_{4,k}$ each hyperedge contains the core vertex of degree 4, so the product is 4 for all four hyperedges. In $E_{4,k}$ the three leaf hyperedges have product 2, while the central hyperedge contains three vertices of degree 2 and thus has product 8. In $B_{4,k}$: two of the central three hyperedges have product 3, the central hyperedge that also contains v has product $3 \cdot 2 = 6$, and the final leaf hyperedge has product 2. ∎

7.2 Structural classification for $m = 4$

Lemma 7.2 (Structural classification for $m = 4$)**.** *Let T be a k-uniform hypertree with $|E(T)| = 4$. Then T is isomorphic to one of the four models $P_{4,k}$, $E_{4,k}$, $B_{4,k}$, or $S^{\star}_{4,k}$.*

Proof. By Lemma 2.3, any two distinct hyperedges of T intersect in at most one vertex.

We distinguish according to the maximum degree $\Delta(T)$.

Case 1: $\Delta(T) = 4$.

There exists a vertex u contained in all four hyperedges. Since any two hyperedges already intersect in u, Lemma 2.3 implies that they cannot share any additional vertex. Hence, every pair of hyperedges intersects exactly in $\{u\}$, and therefore $T \cong S^{\star}_{4,k}$.

Case 2: $\Delta(T) = 3$.

There exists a vertex u contained in exactly three hyperedges, say e_1, e_2, e_3. Again by Lemma 2.3,

$$e_i \cap e_j = \{u\}, \qquad (1 \leq i < j \leq 3).$$

Since T is connected, the fourth hyperedge e_4 must intersect at least one of e_1, e_2, e_3. Moreover, e_4 cannot contain u, for otherwise u would belong to all four hyperedges and we would have $\Delta(T) = 4$. Thus e_4, meets one of e_1, e_2, e_3 in a unique vertex $v \neq u$.

Since no pair of hyperedges can intersect in more than one vertex, this configuration is uniquely determined up to isomorphism and yields the broom $B_{4,k}$.

Case 3: $\Delta(T) \leq 2$.

No vertex belongs to three hyperedges. Consider the intersection graph $L(T)$. Since T is connected, $L(T)$ is connected and has four vertices.

We claim that $L(T)$ contains no cycle. Suppose otherwise that $L(T)$ contains a cycle

$$e_1 e_2 \cdots e_r e_1.$$

For each i, choose a vertex

$$x_i \in e_i \cap e_{i+1},$$

where indices are taken modulo r. By Lemma 2.3, these intersection vertices are distinct.

Now remove the hyperedge e_1. The vertices x_r and x_1, both belonging to e_1, remain connected through the chain

$$e_r, e_{r-1}, \ldots, e_2.$$

Hence, the deletion of e_1 does not disconnect all vertices of e_1, contradicting the acyclic character of a hypertree. Therefore, $L(T)$ is acyclic.

Since $L(T)$ is connected and acyclic on four vertices, it is a tree. Consequently,

$$L(T) \cong P_4, \qquad \text{or} \qquad L(T) \cong K_{1,3}.$$

If $L(T) \cong P_4$, then the four hyperedges form a linear chain, and therefore $T \cong P_{4,k}$.

If $L(T) \cong K_{1,3}$, then one hyperedge intersects each of the other three, while those three are pairwise disjoint. Since $\Delta(T) \leq 2$, the three intersections occur at distinct vertices of the central hyperedge. This is precisely the configuration $E_{4,k}$.

Thus, T is isomorphic to one of $P_{4,k}$, $E_{4,k}$, $B_{4,k}$, $S^{\star}_{4,k}$. ■

7.3 Final comparisons: $P_{4,k}$ beats all others

Lemma 7.3 (Comparison $P_{4,k}$ vs. $S^{\star}_{4,k}$)**.** *For every $\alpha > 0$, one has $\Phi_\alpha(P_{4,k}) < \Phi_\alpha(S^{\star}_{4,k})$.*

Proof. By Lemma 7.1,

$$\Phi_\alpha(S^{\star}_{4,k}) - \Phi_\alpha(P_{4,k}) = 4f_t(4) - (2f_t(2) + 2f_t(4)) = 2(f_t(4) - f_t(2)) > 0,$$

since f_t is strictly increasing on $[1, \infty)$ (Lemma 3.1) and $4 > 2$. ■

Lemma 7.4 (Comparison $P_{4,k}$ vs. $E_{4,k}$)**.** *For every $\alpha > 0$, one has $\Phi_\alpha(P_{4,k}) < \Phi_\alpha(E_{4,k})$.*

Proof. By Lemma 7.1,

$$\Phi_\alpha(E_{4,k}) - \Phi_\alpha(P_{4,k}) = \big(f_t(8) + f_t(2)\big) - 2f_t(4).$$

Writing $f_t(x) = 2\cosh(t \ln x)$, the function $x \mapsto f_t(e^x) = 2\cosh(tx)$ is strictly convex on $\mathbb{R}$. Since $\ln 4$ is the midpoint of $\ln 2$ and $\ln 8$, strict convexity gives $2f_t(4) < f_t(2) + f_t(8)$, hence the difference is positive. ■

Lemma 7.5 (Comparison $P_{4,k}$ vs. $B_{4,k}$)**.** *For every $\alpha > 0$, one has $\Phi_\alpha(P_{4,k}) < \Phi_\alpha(B_{4,k})$.*

Proof. Let $t = \alpha/2 > 0$ and set $X = 2^t$, $Y = 3^t$. From Lemma 7.1 and $f_t(x) = x^t + x^{-t}$,

$$\Phi_\alpha(B_{4,k}) - \Phi_\alpha(P_{4,k}) = \big(- f_t(2) + 2f_t(3) + f_t(6) - 2f_t(4)\big).$$

In terms of X, Y this equals

$$-(X + X^{-1}) + 2(Y + Y^{-1}) + (XY + (XY)^{-1}) - 2(X^2 + X^{-2}).$$

A direct factorization yields

$$\Phi_\alpha(B_{4,k}) - \Phi_\alpha(P_{4,k}) = -\frac{(X^2Y - 1)\big(X(2X + 1) - Y(X + 2)\big)}{X^2Y}.$$

Since $X^2Y > 1$ for $t > 0$, it suffices to prove

$$X(2X + 1) - Y(X + 2) < 0 \quad \Longleftrightarrow \quad \left(\frac{3}{2}\right)^t > \frac{2X + 1}{X + 2}.$$

As $X = 2^t$, the right-hand side simplifies to

$$\frac{2^{t+1} + 1}{2^t + 2} = 2 - \frac{3}{2^t + 2}.$$

Define

$$\varphi(t) := \left(\frac{3}{2}\right)^t - 2 + \frac{3}{2^t + 2}.$$

Then $\varphi(0) = 0$ and

$$\varphi'(t) = \left(\frac{3}{2}\right)^t \ln\left(\frac{3}{2}\right) - \frac{3 \cdot 2^t \ln 2}{(2^t + 2)^2}.$$

By AM–GM, $(2^t + 2)^2 \geq 8 \cdot 2^t$, hence

$$0 < \frac{3 \cdot 2^t \ln 2}{(2^t + 2)^2} \leq \frac{3 \ln 2}{8}.$$

Therefore,

$$\varphi'(t) \geq \ln\left(\frac{3}{2}\right) - \frac{3 \ln 2}{8} > 0,$$

so φ is strictly increasing and $\varphi(t) > 0$ for all $t > 0$. This implies $\Phi_\alpha(B_{4,k}) - \Phi_\alpha(P_{4,k}) > 0$. ■

Theorem 7.6 (Global minimizer for $m = 4$)**.** *Let $k \geq 2$ and $\alpha > 0$. Among all k-uniform hypertrees with $|E(T)| = 4$, the functional $\Phi_\alpha(T)$ is minimized uniquely by the linear hyperpath $P_{4,k}$.*

Proof. Let T be a k-uniform hypertree with $|E(T)| = 4$. By Lemma 7.2, T is isomorphic to one of $P_{4,k}$, $E_{4,k}$, $B_{4,k}$, or $S^\star_{4,k}$. By Lemmas 7.3 to 7.5, for every $\alpha > 0$ we have

$$\Phi_\alpha(P_{4,k}) < \Phi_\alpha(X) \quad \text{for all } X \in \{E_{4,k}, B_{4,k}, S^\star_{4,k}\}.$$

Hence $\Phi_\alpha(T) \geq \Phi_\alpha(P_{4,k})$, with equality if and only if $T \cong P_{4,k}$. ■

8 Conclusions

We considered the parametric family $\Phi_\alpha = \mathrm{R}_\alpha + \mathrm{RR}_\alpha$ on k-uniform hypergraphs (recovering the case $\alpha = 1$) and showed that the parameter α can change the extremal structure even for small hypertrees. For $m = 3$ hyperedges a threshold occurs: there exists a unique $\alpha_0 > 0$ such that $P_{3,k}$ is the minimizer for $0 < \alpha < \alpha_0$, whereas $S(3, 1, k)$ is the minimizer for $\alpha > \alpha_0$ (and both coincide at $\alpha = \alpha_0$). In contrast, for $m = 4$ there is no transition: the global minimizer is always $P_{4,k}$ for every $\alpha > 0$.

One of the most significant phenomena revealed by our analysis is the appearance of a structural barrier at $\alpha = 4$. Proposition 6.1 shows that the pendent-path merging principle, which is one of the key ingredients in Gao's approach [15], ceases to be valid once $\alpha > 4$. Consequently, arguments based on repeated merging transformations cannot be expected to provide a uniform extremal theory for the entire family Φ_α.

From a methodological perspective, this observation is important because it identifies a genuine obstruction rather than a technical limitation of the proof. The value $\alpha = 4$ emerges as a critical threshold for the pendent-path merging technique. Proposition 6.1 shows that this transformation, which is effective in the classical setting, ceases to be universally valid when $\alpha > 4$. Beyond this point, the term

$$(\deg u \, \deg v)^{\alpha/2},$$

dominates the reciprocal contribution, causing local transformations that decrease Φ_α for small values of α to become ineffective.

This suggests that the study of extremal problems for Φ_α naturally splits into different parameter regimes and that new structural methods may be required for the range $\alpha > 4$. Understanding these regimes for larger classes of uniform hypertrees appears to be an interesting direction for future research.

On the structural side, splitting redundant intersections strictly decreases Φ_α for all $\alpha > 0$, which extends the fixed-order upper bound and characterizes the complete hypergraph as the unique maximizer in the connected class.

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