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Research Paper

Vertex-Degree Function Index of Hypergraphs with Given Independence Number

Shashwath S Shetty[1] and K Arathi Bhat[1⋆]

[1]Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, India

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Abstract

Hypergraphs generalize graphs by permitting edges to link arbitrary numbers of vertices, making them increasingly important for modeling complex networks. Topological indices for hypergraphs offer critical insights into their structural properties and complexity, with broad applications in network theory, chemistry, and data science. This paper focuses on the vertex-degree function index of hypergraphs characterized by a fixed independence number. We establish bounds for this index, generalizing classical results from graph theory to hypergraphs. Specifically, we determine upper and lower bounds for the vertex-degree function index, assuming convexity in hypertrees and unicyclic hypergraphs and fixing their independence number.

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1 Introduction

A hypergraph $\mathcal{H}$ is an ordered pair $(\mathcal{V}, \mathcal{E})$, where the elements of the set $\mathcal{V}$ are called the vertices of $\mathcal{H}$, and $\mathcal{E}$ contains the collection of non-empty subsets of $\mathcal{V}$, called the hyperedges of $\mathcal{H}$. By *order* and *size* of a hypergraph, we mean the number of vertices and hyperedges in a hypergraph, respectively. In our study, by a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$, we mean an undirected hypergraph with $\mathcal{E}$ as a set (which forbids multiple hyperedges). We also restrict the hyperedges to being a set, with cardinality greater than one (which forbids self-loops). The *rank* of a hypergraph $\mathcal{H}$ is given by $rank(\mathcal{H}) = \max_{e\in\mathcal{E}} |e|$, while the co-rank of $\mathcal{H}$ is defined as $co\text{-}rank(\mathcal{H}) = \min_{e\in\mathcal{E}} |e|$. A hypergraph $\mathcal{H}$ is said to be k-uniform if $rank(\mathcal{H}) = k = co\text{-}rank(\mathcal{H})$. The deletion of a hyperedge $e \in \mathcal{E}$ from $\mathcal{H}$ is denoted by $\mathcal{H} - e$. Let $d_{\mathcal{H}}(u)$ or simply d_u (if the hypergraph under consideration is clear from the context), denote the degree of a vertex u in a hypergraph $\mathcal{H}$, which is defined as the cardinality of the set $\{e \in \mathcal{E} | u \in e\}$. A pendant vertex is a vertex of degree one.

*Corresponding author
E-mail addresses: shashwathsshetty01334@gmail.com (S. S. Shetty), arathi.bhat@manipal.edu (K. A. Bhat)
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A walk of length r in a hypergraph is an alternating sequence of vertices $v_j, 0 \leq j \leq r$ and hyperedges $e_j, 1 \leq j \leq r$, which starts and ends with a vertex, and $v_{j-1}, v_j \in e_j$ and $v_{j-1} \neq v_j$ for all $1 \leq j \leq r$. A walk is called a path if all the v_j's and e_i's are distinct, and a walk is called a cycle if all of them are distinct except $v_0 = v_r$. A hypergraph is said to be connected if there is a path between every pair of vertices in the hypergraph. A hypertree is a connected hypergraph with no cycles. A hyperedge e in a hypergraph is said to be a pendant hyperedge at a vertex $u \in e$ if all the vertices of e are pendant except u. A hypertree is said to be a hyperstar if every hyperedge in the hypertree is pendant. We denote a hyperstar on n vertices by $\mathcal{S}_n$ and a hyperstar having m hyperedges by $\mathcal{S}^{(m)}$. Two hyperedges are said to be adjacent if they have a vertex in common. A hyperpath is a hypertree in which each hyperedge is adjacent to at most two other hyperedges. A hypergraph is said to be linear if any two hyperedges in a hypergraph have at most one vertex in common. A hypercycle is a connected linear hypergraph in which every hyperedge is adjacent to exactly two other hyperedges.

A hypergraph $\mathcal{G} = (\mathcal{V}', \mathcal{E}')$ is said to be a sub-hypergraph of a hypergraph, $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ if and only if $\mathcal{V}' \subseteq \mathcal{V}$ and $\mathcal{E}' \subseteq \mathcal{E}$. An induced sub-hypergraph of a hypergraph $\mathcal{H}$ induced by $\mathcal{V}_1 \subseteq \mathcal{V}$ is a sub-hypergraph of $\mathcal{H}$ with vertex set as $\mathcal{V}_1$ and edge set as $\mathcal{E}_1 = \{e \in \mathcal{E} : e \subseteq \mathcal{V}_1\}$. We denote by $\mathcal{H}[\mathcal{V}_1]$, an induced sub-hypergraph of $\mathcal{H}$ induced by $\mathcal{V}_1 \subseteq \mathcal{V}$. A subset I of the vertex set $\mathcal{V}$ of the hypergraph $\mathcal{H}$ is said to be an independent set if $\mathcal{H}[I]$ is a totally disconnected graph. A maximum independent set in a hypergraph is an independent set with maximum cardinality. The cardinality of the maximum independent set in a hypergraph $\mathcal{H}$ is the independence number of $\mathcal{H}$, denoted by $\alpha(\mathcal{H})$. In general, it is trivial to note that the maximum independent set in a hypergraph is not unique. The independence number of a hypergraph is well studied in the literature [1–5]. For all other undefined terminology in hypergraphs, readers can refer to [6].

The vertex-degree function index has been extensively studied for graphs in [7–11]. Very recently, vertex-degree-based topological indices have been generalized from graphs to hypergraphs in [12–20] and directed hypergraphs [21]. The vertex-degree function index $\Phi : \mathbb{N} -> \mathbb{R}$ of a hypergraph $\mathcal{H}$ can be defined as:

$$\Phi(\mathcal{H}) = \sum_{u \in \mathcal{V}} \varphi(d_u).$$

In Section 2, we discuss some basic results on the independence number and the vertex-degree function index of hypergraphs. The upper and lower bound for the vertex-degree function index of hypertrees, unicyclic hypergraphs, and uniform hypergraphs has been discussed in Section 3.

2 Preliminary results

In this section, we discuss some preliminary results that will be used in proving the further results.

Proposition 2.1. *Let $\mathcal{H}$ be a hypergraph of order n and size m, and let* co-rank$(\mathcal{H}) \geq 2$. *Then the independence number of $\mathcal{H}$ is $n-1$ if and only if $\mathcal{H}$ has a vertex of degree m.*

Proof. If there exists a vertex $v \in \mathcal{V}(\mathcal{H})$ of degree m, then $\mathcal{V}(\mathcal{H}) \setminus \{v\}$ is a maximum independent set of $\mathcal{H}$. For the converse part, assume that the independence number of $\mathcal{H}$ is $n-1$. Then, there exists a vertex v such that the hypergraph induced by $\mathcal{V}(\mathcal{H}) \setminus \{v\}$ is a totally disconnected hypergraph. But the hypergraph induced by $\mathcal{V}$ contains all the m hyperedges of $\mathcal{H}$, which implies $d_v = m$. ■

Lemma 2.2. *Let $\mathcal{H}$ be a connected hypergraph with $m \geq 2$ hyperedges and* co-rank$(\mathcal{H}) \geq 2$. *Then there always exists a maximum independent set that contains all the pendant vertices of $\mathcal{H}$.*

Proof. Let I be a maximum independent set, i.e., $|I| = \alpha(\mathcal{H})$. Suppose that $e = \{v_1, \ldots, v_k\}$ is a hyperedge of cardinality k, such that $v_{i_1}, \ldots, v_{i_p}, 1 \leq i_r \leq k-1, 1 \leq r \leq p$ are the pendant vertices of e that do not belong to I and $v_{j_1}, \ldots, v_{j_q}, 1 \leq j_s \leq k-p, 1 \leq s \leq q$ are the non-pendant vertices of e that belong to I. Now, if $q \geq p$, then $I' = (I \setminus \{v_{j_1}, \ldots, v_{j_p}\}) \cup \{v_{i_1}, \ldots, v_{i_p}\}$ is also an independent set with cardinality $\alpha(\mathcal{H})$. If $p > q$, then $I'' = (I \setminus \{v_{j_1}, \ldots, v_{j_q}\}) \cup \{v_{i_1}, \ldots, v_{i_p}\}$ is an independent set with cardinality $\alpha(\mathcal{H}) + p - q > \alpha(\mathcal{H})$, a contradiction. ■

Proposition 2.3. *Let $\mathcal{H}$ be a hypergraph with co-rank$(\mathcal{H}) \geq 2$ and let e_p be a pendant hyperedge at a vertex u. Then, the independence number of the hypergraph, $\alpha(\mathcal{H} - e_p)$ is equal to $\alpha(\mathcal{H}) + 1$ or $\alpha(\mathcal{H})$.*

Proof. Since e_p is a pendant hyperedge, $|e_p| - 1$ vertices of e_p are contained in the maximum independent set of $\mathcal{H}$. In $\mathcal{H} - e_p$, the maximum independent set may or may not contain the non-pendant (in $\mathcal{H}$) vertex of e_p. ■

Lemma 2.4. *Let $\mathcal{T}$ (resp. $\mathcal{U}$) be a hypertree (resp. unicyclic hypergraph) of order n and size m with independence number $n - s$, $1 \leq s \leq \lfloor \frac{m}{2} \rfloor$. If $p(\mathcal{T})$ (resp. $p(\mathcal{U})$) denotes the number of pendant vertices in $\mathcal{T}$ (resp. $\mathcal{U}$), then*

- $n - 2s + 1 \leq p(\mathcal{T}) \leq n - s$,
- $n - 2s \leq p(\mathcal{U}) \leq n - s - 1$.

Proof. By using Lemma 2.2, we have that the number of pendant vertices is at most $n - s$ in a hypertree, and $n - s - 1$ in a unicyclic hypergraph with independence number $n - s$. A hypertree (resp. unicyclic hypergraph) with independence number $n - s$ can have at most $2s - 1$ (resp. $2s$) non-pendant vertices and hence the lower bound for the number of pendant vertices. ■

Lemma 2.5. *Let $y > 0$ and $x \geq y + 2\epsilon$ for some $\epsilon > 0$. If φ is strictly convex, then*

$$\varphi(x) + \varphi(y) > \varphi(x - \epsilon) + \varphi(y + \epsilon).$$

Proof. By using the Mean Value Theorem, there exists $a \in (x, x + \epsilon)$ such that $\epsilon\varphi'(a) = \varphi(x + \epsilon) - \varphi(x)$. Also, $F(x) := \varphi(x + \epsilon) - \varphi(x)$ is strictly increasing, as φ is strictly convex (and hence $\varphi'(a) > 0$). Hence, $x - \epsilon > y \implies F(x - \epsilon) > F(y)$, from which the result follows. ■

3 Extremal bounds

In the following definition, we define a hypergraph $\mathcal{H}^*$ that attains the maximum value of the vertex-degree function index, given the independence of the hypergraph.

Definition 3.1. Let $\mathcal{H}^* = (\mathcal{V}, \mathcal{E})$ be a k-uniform hypergraph with a partition of the vertex set $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$, such that the edge set of $\mathcal{H}^*$ is given by $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$, where $\mathcal{E}_1 = \{e \subseteq \mathcal{V}_1 : |e| = k\}$ and $\mathcal{E}_2 = \{e \subseteq \mathcal{V} : |e| = k \text{ and } e \cap \mathcal{V}_1 \neq \emptyset \neq e \cap \mathcal{V}_2\}$.

Proposition 3.2. *Let $\mathcal{H}$ be a k-uniform hypergraph of order n having independence number s. If φ is an increasing function, then*

$$\Phi(\mathcal{H}) \leq (n - s)\varphi(a) + s\varphi(b),$$

where $a = \sum_{i=0}^{k-2} \binom{s}{i}\binom{n-s}{k-1-i}$ *and* $b = \binom{n-s-1}{k-1} + \sum_{i=0}^{k-2} \binom{n-s}{i}\binom{s}{k-1-i}$. *Equality in the above bound is attained by the* k*-uniform hypergraph* $\mathcal{H}^*$ *defined in Definition 3.1.*

Proof. Let $\mathcal{H}$ be any hypergraph on n vertices with the maximum independent set I and $|I| = s$. Since φ is increasing, $\Phi(\mathcal{H})$ is maximized if the degree of each vertex is as maximum as possible. Partition the vertex set $\mathcal{V}$ into two parts: I and $\mathcal{V} \setminus I$. Now construct a hypergraph $\mathcal{H}'$ from $\mathcal{H}$ by adding all those hyperedges, 'inside the vertices $\mathcal{V} \setminus I$' and 'between the vertices of I and $\mathcal{V} \setminus I$ (each such hyperedge has a non-empty intersection with both $\mathcal{I}$ and $\mathcal{V} \setminus I$)', which are not in $\mathcal{H}$. Then, we can observe that $\alpha(\mathcal{H}) = s$, and the degree of any vertex w in I is given by

$$\binom{n-s}{k-1} + \ldots + \binom{s}{k-2}\binom{n-s}{1} = \sum_{i=0}^{k-2} \binom{s}{i}\binom{n-s}{k-1-i},$$

and degree of any vertex v in $\mathcal{V} \setminus I$ is equal to $\binom{n-s-1}{k-1} + \binom{s}{k-1} + \ldots + \binom{n-s}{k-2}\binom{s}{1} = \binom{n-s-1}{k-1} + \sum_{i=0}^{k-2} \binom{n-s}{i}\binom{s}{k-1-i}$. Since φ is increasing, we have:

$$\Phi(\mathcal{H}) \leq \Phi(\mathcal{H}') = s\varphi(d_{\mathcal{H}'}(w)) + (n-s)\varphi(d_{\mathcal{H}'}(v)).$$

Hence, this concludes the proof. ■

Remark 1. Proposition 3.2 can be restated to general hypergraphs, and the upper bound is given by

$$\Phi(\mathcal{H}) \leq (n-s)\varphi(2^{n-1}) + s\varphi(2^{n-1} - 2^{s-1}),$$

and similarly the extremal hypergraph.

3.1 Hypertrees

In the following definition, we define a hypertree $\mathcal{T}^*$, which attains the maximum value of the vertex-degree function index.

Definition 3.3. Let $\mathcal{T}^*$ be a hypertree of size m having independence number $n-s$, obtained from a hyperstar $\mathcal{S}^{(m-s+1)}, 1 \leq s \leq \lfloor \frac{m}{2} \rfloor$, by attaching $s-1$ hyperedges to any of the $m-s+1$ distinct pendant vertices (some of them may be contained in the same hyperedge) of $\mathcal{S}^{(m-s+1)}$.

Theorem 3.4. *Let* $\mathcal{T}$ *be a hypertree of order* n *and size* m *having independence number* $n-s$, $1 \leq s \leq \lfloor \frac{m}{2} \rfloor$. *Then, for a strictly convex function* φ,

$$\Phi(\mathcal{T}) \leq (n-s)\varphi(1) + (s-1)\varphi(2) + \varphi(m-s+1),$$

with equality if and only if $\mathcal{T}$ *is a hypertree with* $\alpha(\mathcal{T}) = n-s$, $1 \leq s \leq \lfloor \frac{m}{2} \rfloor$ *defined in Definition 3.3.*

Proof. Let $\mathcal{T}$ be a hypertree of order n and size $m \geq 2$ with independence number $n-s$.

Claim: If $\Delta(\mathcal{T})$ denotes the maximum degree of $\mathcal{T}$, then $2 \leq \Delta(\mathcal{T}) \leq m-s+1$.

The lower bound is direct, as $m \geq 2$. By using Lemma 2.4, we have that the number of non-pendant vertices of $\mathcal{T}$ is at least s. Since the degree sum (sum of the degrees of all the vertices) of $\mathcal{T}$ is fixed for a given value of n and m, the degree sum of the non-pendant vertices of $\mathcal{T}$ is at least $(m+n-1) - (n-s) = m+s-1$. Therefore, the maximum degree of $\mathcal{T}$ is at most equal to $m+s-1-2(s-1) = m-s+1$.

Let $\mathcal{T}$ be an arbitrary hypertree containing p pendant vertices and $q = n - p$ non-pendant vertices with independence number $n - s$. Also, let $2 + \epsilon_i, 1 \leq i \leq q$ be the degrees of the q non-pendant vertices of $\mathcal{T}$, where $0 \leq \epsilon_1 \leq \ldots \leq \epsilon_q \leq \Delta(\mathcal{T}) - 2$.
Consider the case when at least one of $\epsilon_i, 1 \leq i \leq q-1$ is non-zero. From Lemma 2.5, for a strictly convex function φ, we have that

$$\sum_{i=1}^{q} \varphi(2+\epsilon_i) < \sum_{i=1}^{q-1} \varphi(2) + \varphi(2 + \sum_{i=1}^{q} \epsilon_i).$$

Now, if $q > s$, then from Lemma 2.5, for any $t > 2$, we have:

$$p\varphi(1) + (q-1)\varphi(2) + \varphi(t) < (p+q-s)\varphi(1) + (s-1)\varphi(2) + \varphi(t+q-s).$$

Therefore,

$$\begin{aligned}
\Phi(\mathcal{T}) &= p\varphi(1) + \sum_{i=1}^{q} \varphi(2+\epsilon_i) \\
&\leq p\varphi(1) + \sum_{i=1}^{q-1} \varphi(2) + \varphi(2 + \sum_{i=1}^{q} \epsilon_i) \\
&\leq (p+q-s)\varphi(1) + (s-1)\varphi(2) + \varphi(2 + \sum_{i=1}^{q} \epsilon_i + q - s) \\
&= (n-s)\varphi(1) + (s-1)\varphi(2) + \varphi(m-s+1) = \Phi(\mathcal{T}^*).
\end{aligned}$$

Now, it remains to show that $\Phi(\mathcal{T}^*)$ is maximized. By considering the expression of $\Phi(\mathcal{T}^*)$, this can be verified in three cases. For any positive integer t,

Case 1. $(n-s)\varphi(1) + (s-1)\varphi(2) < (n-s+t)\varphi(1) + (s-2t-1)\varphi(2) + t\varphi(3)$. This is not possible, as the number of pendant (resp. non-pendant) vertices is at most $n-s$ (resp. at least s).

Case 2. $(n-s)\varphi(1) + \varphi(m-s+1) < (n-s-t)\varphi(1) + t\varphi(0) + \varphi(m-s+t+1)$. This will never be an expression of the vertex-degree function index for any hypertree, since it is connected.

Case 3. $(s-1)\varphi(2) + \varphi(m-s+1) < \varphi(1) + (s-t-1)\varphi(2) + \varphi(m-s+t+1)$. This is also not possible in a hypertree with an independence number of $n-s$, since $\Delta(\mathcal{T}) \leq m-s+1$.

■

Definition 3.5. Given $1 \leq s \leq \lfloor \frac{m}{2} \rfloor$, let $\mathcal{P}^{(2s)} : e_1, v_1, e_2, \ldots, v_{2s-1}, e_{2s}$ be the hyperpath such that $v_i \in e_i \cap e_{i+1}$. Now, $\mathcal{T}^\dagger$ is a hypertree of size $m = ks + r, k \geq 2, 0 \leq r \leq s-1$ obtained from $\mathcal{P}^{(2s)}$ by attaching $k-1$ hyperedges to each of $v_{2j-1}, 1 \leq j \leq r$, and $k-2$ hyperedges to each of v_{2q-1} for $r+1 \leq q \leq s$.

It is very important to note that the above hypertree $\mathcal{T}^\dagger$ defined in Definition 3.5 has independence number $n-s$. Further, if $k \geq 3$, then the unique maximum independent set is given by $\mathcal{V}(\mathcal{T}^\dagger) \setminus \{v_1, v_3, \ldots, v_{2s-1}\}$.

Theorem 3.6. *Let $\mathcal{T}$ be a hypertree of order n and size $m = ks + r, k \geq 2, 0 \leq r \leq s-1$ having independence number $n-s$, $1 \leq s \leq \lfloor \frac{m}{2} \rfloor$. Then, for a strictly convex function φ,*

$$\Phi(\mathcal{T}) \geq (n-2s+1)\varphi(1) + (s-1)\varphi(2) + r\ \varphi(k+1) + (s-r)\ \varphi(k),$$

with equality if and only if $\mathcal{T}$ is isomorphic to $\mathcal{T}^\dagger$ with independence number $n-s$, defined in Definition 3.5.

Proof. From Lemma 2.4, we have that the number of pendant vertices in a hypertree $\mathcal{T}$ of order n having independence number $n-s$ is at least $n-2s+1$. In other words, the number of non-pendant vertices in $\mathcal{T}$ is at most $2s-1$. The construction of the minimal hypertree $\mathcal{T}^{\dagger}$ is as follows: Construct a hyperpath $\mathcal{P}^{(2s)}: e_1, v_1, e_2, v_2, \ldots, v_{2s-1}, e_{2s}$ of size $2s$. It is clear that the hyperpath $\mathcal{P}^{(2s)}$ has independence number $p-s$, where p is the order of the hyperpath. The addition of remaining hyperedges should be such that the independence number of the hypertree remains constant and the number of non-pendant vertices remains the same (as it has already attained its upper bound). These conditions are satisfied by attaching the remaining $(m-2s)$ hyperedges to the non-pendant vertices $v_1, v_3, \ldots, v_{2s-1}$. By using the property of strictly convex functions, it can be seen that the vertex-degree function index will be minimum if the addition of these pendant hyperedges is such that $|d_{2i-1} - d_{2j-1}| \leq 1$, for all $1 \leq i, j \leq s$.

Let $\mathcal{T}$ be an arbitrary hypertree of order n and size m with $p \geq n-2s+1$ pendant vertices and $q = n-p$ non-pendant vertices. Also, let the degrees of q non-pendant vertices be $2+\delta_i$, where each $\delta_i \geq 0, 1 \leq i \leq q$, and the degrees of $2s-1$ non-pendant vertices of $\mathcal{T}^{\dagger}$ be $2+\epsilon_j$, where each $\epsilon_j \geq 0, 1 \leq j \leq 2s-1$. Then, we have:

$$\begin{aligned}
\Phi(\mathcal{T}) &= p\varphi(1) + \sum_{i=1}^{q} \varphi(2+\delta_i) \\
&\geq (n-2s+1)\varphi(1) + (s-1)\ \varphi(2) + \sum_{i=1}^{s} \varphi(2+\epsilon_{2i}) \\
&\geq (n-2s+1)\varphi(1) + (s-1)\ \varphi(2) + r\ \varphi(3+\epsilon) + (s-r)\ \varphi(2+\epsilon) = \Phi(\mathcal{T}^{\dagger}).
\end{aligned}$$

As the degree sum of the vertices for a hypertree is $m+n-1$, the value of ϵ is $k-2$. Proving that the lower bound obtained is the minimum among all hypertrees having independence number $n-s$ is similar to the case of the upper bound. ■

3.2 Unicyclic hypergraphs

In the following definition, we define a unicyclic hypergraph $\mathcal{U}^*$, which attains the maximum value of the vertex-degree function index.

Definition 3.7. Let $\mathcal{T}_1$ be a hypertree obtained from a hyperstar $\mathcal{S}^{(m-s+1)}, 1 \leq s \leq \lfloor \frac{m}{2} \rfloor$, by attaching $s-2$ pendant hyperedges at any of the $m-s+1$ distinct pendant vertices of $\mathcal{S}^{(m-s+1)}$. Also, let $\mathcal{U}^*$ be a unicyclic hypergraph of size m having independence number $n-s$, obtained from $\mathcal{T}_1$ by attaching two vertices of a hyperedge $e, |e| \geq 3$, to any of the two pendant vertices of $\mathcal{T}_1$ (which results in a cycle of length 2).

Theorem 3.8. *Let $\mathcal{U}$ be a unicyclic hypergraph (with $rank(\mathcal{U}) \geq 3$) of order n and size m. If $\alpha(\mathcal{U}) = n-s,\ 1 \leq s \leq \lfloor \frac{m}{2} \rfloor$, then for a strictly convex function φ,*

$$\Phi(\mathcal{U}) \leq (n-s-1)\varphi(1) + s\ \varphi(2) + \varphi(m-s+1).$$

Equality in the above holds if and only if $\mathcal{U}$ is a unicyclic hypergraph with independence number $n-s$, defined in Definition 3.7.

Proof. As we have a similar result for graphs (2-uniform hypergraphs), we assume that $rank(\mathcal{U}) \geq 3$, where we allow cycles of length 2. Let $\mathcal{U}$ be a unicyclic hypergraph of order n and size $m \geq 2$ with independence number $n-s$.

Claim: If $\Delta(\mathcal{U})$ denotes the maximum degree of $\mathcal{U}$, then $2 \leq \Delta(\mathcal{U}) \leq m-s+1$.

The proof of the claim is similar to that of the hypertree. That is, from Lemma 2.4, we have that the number of non-pendant vertices in $\mathcal{U}$ is at least $s+1$. Also, the sum of the degrees of the non-pendant vertices of any unicyclic hypergraph of order n and size m with independence number $n-s$ is at most $(m+n)-(n-s-1) = m+s+1$. Hence, the maximum degree possible for a vertex in $\mathcal{U}$ with independence number equal to $n-s$ is $m+s+1-2s = m-s+1$.

Again, we repeat the same construction for unicyclic hypergraphs. Let $\mathcal{U}$ be an arbitrary unicyclic hypergraph containing p pendant vertices and $q = n-p$ non-pendant vertices with independence number $n-s$. Also, let $2+\epsilon_i, 1 \leq i \leq q$ be the degrees of the q non-pendant vertices of $\mathcal{U}$, where $0 \leq \epsilon_1 \leq \ldots \leq \epsilon_q \leq \Delta(\mathcal{U}) - 2$.
Consider the case when at least one of $\epsilon_i, 1 \leq i \leq q-1$ is non-zero. From Lemma 2.5, for a strictly convex function φ, we have that

$$\sum_{i=1}^{q} \varphi(2+\epsilon_i) < \sum_{i=1}^{q-1} \varphi(2) + \varphi(2+\sum_{i=1}^{q} \epsilon_i).$$

Now, if $q > s+1$, then from Lemma 2.5, for any $t > 2$, we have that

$$p\varphi(1) + (q-1)\varphi(2) + \varphi(t) < (p+q-s)\varphi(1) + (s-1)\varphi(2) + \varphi(t+q-s).$$

Therefore,

$$\begin{aligned}
\Phi(\mathcal{U}) &= p\varphi(1) + \sum_{i=1}^{q} \varphi(2+\epsilon_i) \\
&\leq p\varphi(1) + \sum_{i=1}^{q-1} \varphi(2) + \varphi(2+\sum_{i=1}^{q} \epsilon_i) \\
&\leq (p+q-s-1)\varphi(1) + s\varphi(2) + \varphi(2+\sum_{i=1}^{q} \epsilon_i + q - 1 - s) \\
&= (n-s-1)\varphi(1) + s\varphi(2) + \varphi(m-s+1) = \Phi(\mathcal{U}^*).
\end{aligned}$$

The argument is similar to the case of hypertree for proving that $\Phi(\mathcal{U}^*)$ is maximum. ■

Definition 3.9. Given $1 \leq s \leq \lfloor \frac{m}{2} \rfloor$, let $\mathcal{C}^{(2s)} : e_1, v_1, e_2, v_2, \ldots, e_{2s}, v_{2s}, e_1$ be a hypercycle of size $2s$. Now, $\mathcal{U}^\dagger$ is the unicyclic hypergraph of size $m = ks + r, k \geq 2, 0 \leq r \leq s-1$ obtained from $\mathcal{C}^{(2s)}$, by attaching (to non-pendant vertices only) $k-1$ hyperedges to each of $v_{2j}, 1 \leq j \leq r$ and $k-2$ hyperedges to each of $v_{2q}, q \geq r+1$.

Theorem 3.10. *Let $\mathcal{U}$ be a unicyclic hypergraph of order n and size m. If $\alpha(\mathcal{U}) = n-s$, $1 \leq s \leq \lfloor \frac{m}{2} \rfloor$, then for a strictly convex function φ,*

$$\Phi(\mathcal{U}) \geq (n-2s)\varphi(1) + s\ \varphi(2) + r\ \varphi(k+1) + (s-r)\ \varphi(k).$$

Equality in the above holds if and only if $\mathcal{U}$ is isomorphic to $\mathcal{U}^\dagger$ having independence number $n-s$ defined in Definition 3.9.

Proof. From Lemma 2.4, we have that the number of pendant vertices in a unicyclic hypergraph $\mathcal{U}$ of order n having independence number $n-s$ is at least $n-2s$. In other words, the number of non-pendant vertices in $\mathcal{U}$ is at most $2s$. The construction of the minimal hypergraph $\mathcal{U}^\dagger$ is as follows: As the sum of the degrees of the vertices is fixed, construct a hypercycle $\mathcal{C}^{(2s)}$ of size $2s$. We know that the hypercycle $\mathcal{C}^{(2s)}$ has independence number $p-s$, where p is the

order of the hypercycle. Now, the addition (and also the vertices) of the hyperedges to the hypercycle should be such that the independence number of the hypergraph remains constant and no additional cycle should be formed. These conditions can be satisfied by adding the pendant hyperedges $(m-2s)$ on the alternate non-pendant vertices, say $v_2, v_4, \ldots, v_{2s}$ of the hypercycle $\mathcal{C}^{(2s)}$. By using the property of strictly convex functions, it can be seen that the vertex-degree function index will be minimum if the addition of these pendant hyperedges is such that $|d_{2i} - d_{2j}| \leq 1$, for all $1 \leq i, j \leq s$.

Let $\mathcal{U}$ be an arbitrary unicyclic hypergraph of order n and size m with $p \geq n-2s$ pendant vertices and $q = n-p$ non-pendant vertices. Also, let the degrees of q non-pendant vertices be $2+\delta_i$, where each $\delta_i \geq 0, 1 \leq i \leq q$, and the degrees of $2s$ non-pendant vertices of $\mathcal{U}^\dagger$ be $2+\epsilon_j$, where each $\epsilon_j \geq 0, 1 \leq j \leq 2s$. Then, by using the Lemma 2.5, we have

$$\begin{aligned}
\Phi(\mathcal{U}) = &\ p\varphi(1) + \sum_{i=1}^{q} \varphi(2+\delta_i) \\
\geq &\ (n-2s)\varphi(1) + s\ \varphi(2) + \sum_{j=1}^{s} \varphi(2+\epsilon_{2j}) \\
\geq &\ (n-2s)\varphi(1) + s\ \varphi(2) + r\ \varphi(2+(\epsilon+1)) + (s-r)\ \varphi(2+\epsilon) = \Phi(\mathcal{U}^\dagger).
\end{aligned}$$

As the degree sum of the vertices for a unicyclic hypergraph is $m+n$, the value of ϵ is $k-2$. Also, showing that the lower bound obtained is the minimum among all unicyclic hypergraphs having independence number $n-s$ is similar. ■

4 Conclusion

This study broadens the understanding of hypergraph structures by examining the vertex-degree function index with a fixed independence number. We established bounds for this index in hypertrees and unicyclic hypergraphs, extending classical graph theory results. These findings provide useful tools for analyzing complex networks across various fields. Future research may investigate similar indices under different conditions, and also for other hypergraph classes.

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