Iranian Journal of Mathematical Chemistry

IJMC

DOI: 10.22052/IJMC.2024.254494.1840
Vol. 16, No. 2, 2025, pp.109-126
Research Paper

Topological Properties of Commuting Graphs over Semi-Dihedral Groups

Fawad Ali[1,2*], Kumail Raza[3] and Tanzeela Rubab[3]

[1]College of Mathematical Sciences and Statistics, Baise University, Guangxi, Baise 533000, China
[2]Department of Mathematics, COMSATS University Islamabad, Park Road, Tarlai Kalan, Islamabad 455505 Pakistan
[3]Institute of Numerical Sciences, Kohat University of Science & Technology, Kohat 26000, KPK, Pakistan

Keywords:

Distance property,
Chemical graph,
Molecular structure,
Commuting graph,
Finite group

AMS Subject Classification (2020):

05C09; 05C92; 05C31

Article History:

Received: 1 March 2024
Accepted: 15 December 2024

Abstract

The algebraic value of a chemical composition plays a crucial role in determining its physical properties, chemical reactivity, and biological activity. Algebraic graph theory investigates the connection between abstract algebra and graph theory. Focusing on the commuting graphs of semi-dihedral groups, we examine their various topological properties. Furthermore, we provide a detailed exploration of key topological graph indices, including the general Randić index, the Wiener index, the atom-bond connectivity index (and its fourth variation), the Schultz molecular topological index, the geometric-arithmetic index, the harmonic index, and the Harary index. This research reveals the structural as well as mathematical features of these graphs, providing valuable insights into their possible applications.

© 2025 University of Kashan Press. All rights reserved.

1 Introduction

In studies of quantitative structure-property relationships (QSPR), physical properties such as melting point, strain energy, stability and certain topological indices are used to determine the bioactivity of chemical compounds. The main objective of topological indices is the quantification of molecular structures and values unaffected by any action that maintains the structural integrity of a molecular structure, providing unique metrics for assessing symmetry and characterizing its topology [1]. Various indices can be used to examine specific chemically structured material properties. A prime instance was derived in 1947 when Wiener came up with the idea of a topological description while investigating the paraffin melting temperature [2] and it is

*Corresponding author
E-mail addresses: fawad.ali@comsats.edu.pk (F. Ali), kumailraza050@gmail.com (K. Raza), tanzeelarubab28@gmail.com(T. Rubab)
Academic Editor: Gholam Hossein Fath-Tabar

known as the beginning of a path number. Since then, some other degree and distance-based topological descriptors have been created and examined; see, for example, [3, 4] and the references therein. Pólya [5] investigated the polynomial, which several researchers later used to figure out the polyunsaturated hydrocarbon orbital. Additionally, they extended this concept and developed the spectral theory of graphs. In 1988, Hosoya [6] came up with this idea, and several chemical structures of graphs were explored using certain polynomial computations. After that, in 1996, Sagan et al. [7] subsequently named it the Wiener polynomial. Estrada et al. [8] examined numerous applications of the generalized Wiener index. Several essential indices are based on the degrees and distances of vertices of a graph. The Randić connectivity index is a prominent molecular index introduced in 1975; see [9] for further details. In 1993, Ivanciuc et al. [10] defined the Harary index as the sum of the reciprocal distances between all pairs of nonadjacent vertices in a graph. Furthermore, Estrada et al. [11] deliberated the atomic-bond connectivity (ABC) index as a modification of the Randić index. This concept is strongly related to physicochemical parameters, including the formation enthalpy and stability of alkanes, as well as the strain energy of cycloalkanes.

Graph theory is an essential branch of mathematics with applications in various fields including but not limited to computer science, chemistry and statistics. It is beneficial to use graphs and understand their properties to understand a wide variety of mathematical problems or to explain real-world situations. In this paper, we are concerned with the so-called *commuting graphs*, which are defined to reflect the commuting structure of a given (in general, non-abelian) group. More precisely, for a given group H with center $Z(H)$, one can consider the notion of a classical commuting graph as $G = \mathcal{C}(H, \Gamma)$ where Γ is a non-empty subset of $H \backslash Z(H)$, for which Γ is the vertex set and any $x, y \in \Gamma$ are joined by an edge if and only if x and y commute in H. Clearly, the concept of commuting graphs has been analyzed in [12] in several contexts. The connectivity and spectral radius of the commuting graphs of dihedral and dicyclic groups were investigated in [13]. Tolue [14] studied the graph-theoretic properties of the non-commuting graphs over the AC-group and dihedral groups. Moreover, in [15], the authors examined the structure of the commuting conjugacy class graph of a group G having $\frac{G}{Z(G)} \cong D_{2n}$ property.

The relationship between chemical characteristics and algebraic graphs constitutes a fascinating area of research, especially when examined from the perspective of group theory. Symmetry groups, semi-dihedral, and generalized quaternion groups have significantly enhanced the understanding of molecular structures and electron configurations. These algebraic tools are particularly effective in studying structural isomerism and conformational analysis. By representing atoms as vertices and bonds as edges, the algebraic graphs offer a precise visualization of molecular structures, enabling the investigation of all possible conformations and isomers. In [16–19], the authors found several Zagreb indices, the degree-based, and distance-based topological indices of power and commuting graphs over certain finite groups. Inspired by their work, we focus on the commuting graphs of semi-dihedral groups. However, computing topological indices for an algebraic graph is a challenging effort. This study tackles this complexity by analyzing degree- and distance-based topological descriptors (summarized in Table 1) for commuting graphs of semi-dihedral groups.

The present literature study still contains plenty of gaps regarding the computations of commuting graphs. Also, the Hosoya polynomials have been explored in various contexts. Determining certain topological invariants for such nontrivial structures is very challenging. This article examines several well-known topological properties of the commuting graphs of semi-dihedral groups. Additionally, we deliberate on resolving and Hosoya polynomials for the same algebraic graphs.

This paper is structured as follows: Section 2 summarizes significant findings and fundamental terminology that serve as the foundation for the rest of this paper. Moving to Section 3, we analyze the degree and distance-based topological indices of the commuting graphs over semi-dihedral groups. The concluding remarks are given in Section 5.

2 Preliminaries

This part is a concise overview of the basic concepts and significant discoveries in graph theory. It will serve as a useful reference point as we dive deeper into the topic throughout the article.

All graphs studied in this paper are simple graphs, that is unweighted, undirected graphs with no loops or multiple edges. Let a and b be two vertices in a graph Γ. The number of edges incident to a vertex a is known as its *degree* (or *valency*) and is denoted by d_a. The *neighborhood* of a vertex a, denoted by $N(a)$ is the set of all connected vertices to a in Γ. The sum of all distinct edges connecting every vertex in $N(a)$ is known as the *ve-degree* of a, and it is represented by $d_{ve}(a)$. A *complete graph* K_n is a graph in which every pair of distinct vertices is connected by a unique edge. The *distance* from a to b in Γ, denoted by $dis(a, b)$, is the length of the shortest path between a and b, while the longest path from a to b in Γ is denoted by $dis_D(a, b)$. The largest distance between a vertex a and any other vertex of Γ is called its *eccentricity* and denoted by $e(a)$. The *diameter* of Γ ($diam(\Gamma)$) is the largest among all the vertices of a graph Γ. Also, the *radius* $r(\Gamma)$ is the smallest eccentricity among all the vertices of Γ.

Table 1: A list of potential topological indices.

The index's name	Symbol	Formula
General Randić index [20]	$R_\alpha(\Gamma)$	$\sum_{v\sim w} (d_v \times d_w)^\alpha$
Schultz molecular topological index [21]	$MTI(\Gamma)$	$\sum_{\{v,w\}\in V(\Gamma)} (d_v + d_w)\,\mathrm{dis}(v, w) + \sum_{w\in V(\Gamma)} d_w^2$
Atomic-bond connectivity (ABC) index [22]	$ABC(\Gamma)$	$\sum_{v\sim w} \sqrt{\frac{d_v+d_w-2}{d_v\times d_w}}$
Fourth version of ABC index [23]	$ABC_4(\Gamma)$	$\sum_{v\sim w} \sqrt{\frac{S_v+S_w-2}{S_v\times S_w}}$
Geometric-arithmetic (GA) index [22]	$GA(\Gamma)$	$\sum_{v\sim w} \frac{2\sqrt{d_v\times d_w}}{d_v+d_w}$
Fifth version of GA index [23]	$GA_5(\Gamma)$	$\sum_{v\sim w} \frac{2\sqrt{S_v\times S_w}}{S_v+S_w}$
Sankruti index [24]	$S(\Gamma)$	$\sum_{uv\in E(\Gamma)} \left(\frac{S_u S_v}{S_u+S_v-2}\right)^3$
Hosoya polynomial [3]	$\mathbb{H}(\Gamma, x)$	$\sum_{i\geq 0} \mathrm{dis}(\Gamma, i)x^i$
Reciprocal status Hosoya polynomial [3]	$\mathbb{H}_{rs}(\Gamma, x)$	$\sum_{vw\in E(\Gamma)} x^{rs(v)+rs(w)}$, where $rs(w) = \sum_{v\in V(\Gamma), w\neq v} \frac{1}{\mathrm{dis}(w,v)}$

Definition 2.1. Let G be a group and Ω be a nonempty subset of G. The commuting graph $\Gamma(G, \Omega)$ has vertex set Ω, where $x_1, x_2 \in \Omega$ are adjacent whenever $x_1x_2 = x_2x_1$ in G.

Furthermore, for $m \geq 2$, the presentation of semi-dihedral group is given as:

$$SD_{8m} = \langle a, b \mid a^{4m} = b^2 = e, bab = a^{2m-1}\rangle.$$

Given the structure of SD_{8m} when m is odd, it is clear that $Z(SD_{8m}) = \{e, a^m, a^{2m}, a^{3m}\}$ is the center of SD_{8m}. Now, we divide SD_{8m} into $\phi_1 = \{e, a, a^2, \ldots, a^{4m-1}\}$ and $\phi_2 = \{b, ba, \ldots, ba^{4m-1}\} = \bigcup_{i=0}^{m-1} \phi_2^i$, where $\phi_2^i = \{ba^i, ba^{m+i}, ba^{2m+i}, ba^{3m+i}\}$ and $\phi_3 = \phi_1 \backslash Z(SD_{8m})$ are the subsets of SD_{8m}.

If m is even then $Z(SD_{8m}) = \{e, a^{2m}\}$. Moreover, $\psi_1 = \{e, a, \ldots, a^{4m-1}\}$, $\psi_2 = \{b, ab, \ldots, a^{4m-1}b\}$ $= \bigcup_{i=0}^{2m-1} \psi_2^i$, where $\psi_2^i = \{ba^i, ba^{2m+i}\}$ and $\psi_3 = \psi_1 \setminus Z(SD_{8m})$ are the subsets of SD_{8m}. Therefore, the structure of the commuting graph of SD_{8m} is given in the following lemma.

Lemma 2.2. *([25]). For $m \geq 2$, we have*

$$\Gamma(SD_{8m}) = \begin{cases} K_4 \vee (mK_4 \cup K_{4m-4}), & \text{if } m \text{ is odd,} \\ K_2 \vee (2mK_2 \cup K_{4m-2}), & \text{if } m \text{ is even.} \end{cases}$$

3 Topological properties

Topological indices have a vital role in graph theory and have a lot of applications in physical chemistry. In the existing literature, various indices have been discussed for different algebraic graphs, see for example [26, 27]. In the present section, we extend this concept and discuss certain topological properties of commuting graphs over finite semi-dihedral groups.

Proposition 3.1. *The all possible distance of $\Gamma(SD_{8m}) = \Gamma(SD_{8m}, SD_{8m})$ is given as follows:*
When m is odd,

$$dis(\Gamma(SD_{8m}), k) = \begin{cases} 4m(2m+5), & \text{if } k = 1, \\ 48m(m-1), & \text{if } k = 2. \end{cases}$$

When m is even,

$$dis(\Gamma(SD_{8m}), k) = \begin{cases} 8m(m+1), & \text{if } k = 1, \\ 8m(4m-2), & \text{if } k = 2. \end{cases}$$

Proposition 3.2. *The degree of $x \in \Gamma(SD_{8m})$ can be expressed as follows:*
When m is odd,

$$d_x = \begin{cases} 8m-1, & \text{if } x \in Z(SD_{8m}), \\ 7, & \text{if } x \in \phi_2, \\ 4m-1, & \text{if } x \in \phi_3. \end{cases}$$

When m is even,

$$d_x = \begin{cases} 8m-1, & \text{if } x \in Z(SD_{8m}), \\ 3, & \text{if } x \in \psi_2, \\ 4m-1, & \text{if } x \in \psi_3. \end{cases}$$

Now, using the above proposition, we have the following results.

Theorem 3.3. *The atomic-bond connectivity index of the commuting graph of the semi-dihedral group is given as:*
When m is odd,

$$ABC(\Gamma(SD_{8m})) = \frac{4}{7}\Bigg(3\sqrt{3}m + \sqrt{\frac{2m-1}{(4m-1)^2}}(28m^2 - 63m + 35) + 8\sqrt{7}m\sqrt{\frac{2m+1}{8m-1}} + 21\sqrt{\frac{4m-1}{(8m-1)^2}} + 56\sqrt{\frac{3m-1}{32m^2-12m+1}}(m-1)\Bigg).$$

When m is even, then

$$ABC(\Gamma(SD_{8m})) = \frac{2}{3}\Bigg(2m + 3\sqrt{\frac{4m-1}{(8m-1)^2}} + \sqrt{\frac{2m-1}{(4m-1)^2}}(24m^2 - 30m + 9)$$

$$+8\sqrt{6}m\sqrt{\frac{m}{8m-1}}+\sqrt{\frac{3m-1}{32m^2-12m+1}}(24m-12)\Bigg).$$

Proof. Using the structure of $\Gamma(SD_{8m})$ Proposition 3.2, and the atomic-bond connectivity index, we can write:

If m is odd then

$$\begin{aligned}
ABC(\Gamma(SD_{8m})) &= 6\sqrt{\frac{(8m-1+8m-3)}{(8m-1)(8m-1)}}+6m\sqrt{\frac{12}{49}}+\binom{4m-4}{2}\sqrt{\frac{4m-1+4m-3}{(4m-1)^2}}\\
&\quad+16m\sqrt{\frac{8m+7-3}{(7)(8m-1)}}+16(m-1)\sqrt{\frac{8m-1+4m-3}{(4m-1)(8m-1)}}\\
&= 6\sqrt{\frac{(16m-4)}{(8m-1)^2}}+\frac{6\sqrt{12}}{7}m+(8m^2-18m+10)\sqrt{\frac{8m-4}{(4m-1)^2}}\\
&\quad+16m\sqrt{\frac{8m+4}{(56m-7)}}+16(m-1)\sqrt{\frac{12m-4}{(32m^2-12m+1)}}\\
&= \frac{4}{7}\Bigg(3\sqrt{3}m+\sqrt{\frac{2m-1}{(4m-1)^2}}(28m^2-63m+35)+21\sqrt{\frac{4m-1}{(8m-1)^2}}\\
&\quad+8\sqrt{7}m\sqrt{\frac{2m+1}{8m-1}}+56\sqrt{\frac{3m-1}{32m^2-12m+1}}(m-1)\Bigg).
\end{aligned}$$

Similarly, if m is even then

$$\begin{aligned}
ABC(\Gamma(SD_{8m})) &= \sqrt{\frac{(8m-1+8m-3)}{(8m-1)(8m-1)}}+2m\sqrt{\frac{4}{9}}+\binom{4m-2}{2}\sqrt{\frac{4m-1+4m-3}{(4m-1)^2}}\\
&\quad+8m\sqrt{\frac{8m+3-3}{3(8m-1)}}+4(2m-1)\sqrt{\frac{8m+4m-4}{(4m-1)(8m-1)}}\\
&= \sqrt{\frac{(16m-4)}{(8m-1)^2}}+2m\sqrt{\frac{4}{9}}+(8m^2-10m+3)\sqrt{\frac{8m-4}{(4m-1)^2}}\\
&\quad+8m\sqrt{\frac{8m}{(24m-3)}}+4(2m-1)\sqrt{\frac{12m-4}{(32m^2-12m+1)}}\\
&= \frac{2}{3}\Bigg(2m+\sqrt{\frac{2m-1}{(4m-1)^2}}(24m^2-30m+9)+3\sqrt{\frac{4m-1}{(8m-1)^2}}\\
&\quad+8m\sqrt{6}\sqrt{\frac{m}{8m-1}}+\sqrt{\frac{3m-1}{32m^2-12m+1}}(24m-12)\Bigg).
\end{aligned}$$

We obtain our desired results after simplifying the calculations. ■

Theorem 3.4. *The general Randić index of* $\Gamma(SD_{8m})$ *is given below:*
When m *is odd,*

$$R_\alpha(\Gamma(SD_{8m}))=6(8m-1)^\alpha\big((8m-1)^\alpha+\frac{8(7)^\alpha m}{3}+\frac{8}{3}(m-1)(4m-1)^\alpha\big)$$

$$+\,2\big(3m(7)^{2\alpha} + (4m^2-9m+5)(4m-1)^{2\alpha}\big).$$

When m is even,

$$R_\alpha(\Gamma(SD_{8m})) = (8m-1)^\alpha\big((8m-1)^\alpha + 8(3)^\alpha m + (8m-4)(4m-1)^\alpha\big) + 2m\big((3)^{2\alpha} + (8m^2-10m+3)(4m-1)^{2\alpha}\big).$$

Proof. Based on Lemma 2.2 and Proposition 3.2, the general Randić index can be expressed as follows:

if m is odd then

$$\begin{aligned}
R_\alpha(\Gamma(SD_{8m})) &= 6((8m-1)^2)^\alpha + 6m\left((7)^2\right)^\alpha + \binom{4m-4}{2}\left((4m-1)^2\right)^\alpha \\
&\quad + 16m\left((8m-1)(7)\right)^\alpha + 4(4m-4)\left((8m-1)(4m-1)\right)^\alpha \\
&= 6(8m-1)^{2\alpha} + 6m(49)^\alpha + (8m^2-18m+10)(4m-1)^{2\alpha} \\
&\quad + 16m(56m-7)^\alpha + 16(m-1)(32m^2-12m+1)^\alpha \\
&= 6(8m-1)^\alpha\Big((8m-1)^\alpha + \frac{8}{3}m(7)^\alpha + \frac{8}{3}(m-1)(4m-1)^\alpha\Big) + 2\Big(3m(7)^{2\alpha} \\
&\quad + (4m^2-9m+5)(4m-1)^{2\alpha}\Big).
\end{aligned}$$

After simplifications, we get

$$\begin{aligned}
R_\alpha(\Gamma(SD_{8m})) &= 6(8m-1)^\alpha\big((8m-1)^\alpha + \frac{8(7)^\alpha m}{3} + \frac{8}{3}(m-1)(4m-1)^\alpha\big) \\
&\quad + 2\big(3m(7)^{2\alpha} + (4m^2-9m+5)(4m-1)^{2\alpha}\big).
\end{aligned}$$

Similarly, if m is even then

$$\begin{aligned}
R_\alpha(\Gamma(SD_{8m})) &= \left((8m-1)^2\right)^\alpha + 2m\left((3)^2\right)^\alpha + \binom{4m-2}{2}\left((4m-1)^2\right)^\alpha \\
&\quad + 8m\left(3(8m-1)\right)^\alpha + 2(4m-2)\left((8m-1)(4m-1)\right)^\alpha \\
&= (8m-1)^{2\alpha} + 2m(9)^\alpha + (8m^2-10m+3)(4m-1)^{2\alpha} + 8m(24m-3)^\alpha \\
&\quad + 4(2m-1)(32m^2-12m+1) \\
&= (8m-1)^\alpha\Big((8m-1)^\alpha + 8m(3)^\alpha + (8m-4)(4m-1)^\alpha\Big) + 2m\Big((3)^{2\alpha} \\
&\quad + (8m^2-10m+3)(4m-1)^{2\alpha}\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
R_\alpha(\Gamma(SD_{8m})) &= (8m-1)^\alpha\big((8m-1)^\alpha + 8(3)^\alpha m + (8m-4)(4m-1)^\alpha\big) \\
&\quad + 2m\big((3)^{2\alpha} + (8m^2-10m+3)(4m-1)^{2\alpha}\big),
\end{aligned}$$

and the proof is completed. ■

Theorem 3.5. *The Schultz molecular topological index of $\Gamma(SD_{8m})$, is given as:*

$$MTI(\Gamma(SD_{8m})) = \begin{cases} (8m-1)(40m^2+47m+4), & \text{if } m \text{ is odd.} \\ (8m-1)(40m^2+15m+2), & \text{if } m \text{ is even.} \end{cases}$$

Proof. Using the Schultz molecular topological index formula, Lemma 2.2, and Proposition 3.2, we may write that if m is odd then

$$\begin{aligned}
MTI(\Gamma(SD_{8m})) &= 12(8m-1) + \binom{4m-4}{2}(8m-2) + 84m + 16m(8m+6) \\
&\quad + 4(4m-4)(12m-2) + 8m(4m-4)(4m-6) + 224m(m-1) \\
&\quad + m(8m-1)^2 + 4m(7)^2 + (4m-4)(4m-1)^2 \\
&= 6(16m-2) + 6m(14) + (8m^2-18m+10)(8m-2) + 16m(8m+6) \\
&\quad + m(8m-1)^2 + 16(m-1)(12m-2) + 32m(m-1)(4m+6) \\
&\quad + 224m(m-1) + 196m + (4m-4)(4m-1)^2 \\
&= (8m-1)(40m^2+47m+4).
\end{aligned}$$

Similarly, if m is even then

$$\begin{aligned}
MTI(\Gamma(SD_{8m})) &= 2m(3+3) + 2(8m-1) + \binom{4m-2}{2}(8m-2) + 8m(8m+2) \\
&\quad + 4(2m-1)(12m-2) + 8m(4m-2)(4m+2) + 48m(2m-1) \\
&\quad + m(8m-1)^2 + 4m(3)^2 + (4m-2)(4m-1)^2 \\
&= (16m-2) + 12m + (8m^2-10m+3)(8m-2) + 8m(8m+2) \\
&\quad + (4m-2)(4m-1)^2 + 4(2m-1)(12m-2) + 16m(m-1)(4m+2) \\
&\quad + 48m(2m-1) + m(8m-1)^2 + 4m(3)^2 \\
&= (8m-1)(40m^2+15m+2).
\end{aligned}$$

Which is the required proof. ■

Theorem 3.6. *The fourth atomic-bond connectivity index of $\Gamma(SD_{8m})$ is given below: if m is odd then*

$$\begin{aligned}
ABC_4(\Gamma(SD_{8m})) &= \sqrt{\frac{m(2m+1)}{(4m+1)^4}}(32m^2-72m+40) \\
&\quad + 24\sqrt{2}\sqrt{\frac{m(m+2)}{(16m^2+32m+1)^2}}\left(24m\sqrt{\frac{4m+2}{(32m+17)^2}}\right) \\
&\quad + 32\sqrt{2}\,(m-1)\sqrt{\frac{m(4m+5)}{(4m+1)^2(16m^2+32m+1)}} \\
&\quad + 64m\sqrt{\frac{m^2+4m+1}{512m^3+1296m^2+576m+17}}.
\end{aligned}$$

If m is even then

$$ABC_4(\Gamma(SD_{8m})) = \frac{m\sqrt{2}(32m^2-40m+12)}{(16m^2+1)} + 4\sqrt{\frac{m(2m+1)}{(4m+1)^4}}$$

$$+(16m\sqrt{2}-8\sqrt{2})\sqrt{\frac{m}{64m^3+16m^2+4m+1}}+16m\sqrt{2}\sqrt{\frac{m(2m+3)}{(4m+1)^2(16m+1)}}+8m\sqrt{2}\sqrt{\frac{m}{(16m+1)^2}}.$$

Proof. Using the fourth atomic-bond connectivity index formula, for both odd and even m, we can write

$$ABC_4(\Gamma(SD_{8m})) = 6\left(\frac{\sqrt{2\,(3(8m-1)+28m+(4m-4)(4m-1))-2}}{(3(8m-1)+28m+(4m-4)(4m-1))^2}\right)$$
$$+\quad 6m\left(\frac{\sqrt{(8(8m-1)+21)-2}}{(4(8m-1)+21)^2}\right)$$
$$+\quad \binom{4m-4}{2}\left(\frac{\sqrt{2\,(4(8m-1)+(4m-5)(4m-1))-2}}{(4(8m-1)+(4m-5)(4m-1))^2}\right)$$
$$+\quad 16m\left(\frac{\sqrt{3(8m-1)+28m+(4m-4)(4m-1)+4(8m-1)+19}}{(3(8m-1)+28m+(4m-4)(4m-1))(4(8m-1)+21)}\right)$$
$$+\quad 4(4m-4)\left(\frac{\sqrt{(3(8m-1)+28m+(4m-4)(4m-1))+(4(8m-1)+(4m-5)(4m-1))-2}}{(3(8m-1)+28m+(4m-4)(4m-1))(4(8m-1)+(4m-5)(4m-1))}\right)$$
$$=\quad 8\left(5\sqrt{\frac{m(2m+1)}{(4m+1)^4}}+4m^2\sqrt{\frac{m(2m+1)}{(4m+1)^4}}+3\sqrt{2}\sqrt{\frac{m(m+2)}{(16m^2+32m+1)^2}}\right.$$
$$-\quad 4\sqrt{2}\sqrt{\frac{m(4m+5)}{(4m+1)^2(16m^2+32m+1)}}+m\left(-9\sqrt{\frac{m(2m+1)}{(4m+1)^4}}+3\sqrt{\frac{4m+2}{(32m+17)^2}}\right.$$
$$+\quad \left.\left.4\sqrt{2}\sqrt{\frac{m(4m+5)}{(4m+1)^2(16m^2+32m+1)}}+8\sqrt{\frac{m^2+4m+1}{512m^3+1296m^2+576m+17}}\right)\right)$$
$$=\quad \sqrt{\frac{m(2m+1)}{(4m+1)^4}}(32m^2-72m+40)+(24)^2\sqrt{2}m\sqrt{\frac{m(m+2)}{(16m^2+32m+1)^2}}\sqrt{\frac{4m+2}{(32m+17)^2}}$$
$$+\quad 32\sqrt{2}(m-1)\sqrt{\frac{m(4m+5)}{(4m+1)^2(16m^2+32m+1)}}+64m\sqrt{\frac{m^2+4m+1}{512m^3+1296m^2+576m+17}}.$$

and,

$$ABC_4(\Gamma(SD_{8m}))$$
$$=\quad \frac{\sqrt{2((8m-1)+12m+(4m-2)(4m-1))-2}}{((8m-1)+12m+(4m-2)(4m-1))^2}+2m\left(\frac{\sqrt{2(2(8m-1)+3)-2}}{(2(8m-1)+3)^2}\right)$$
$$+\quad \binom{4m-2}{2}\left(\frac{\sqrt{2(2(8m-1)+(4m-3)(4m-1))-2}}{(2(8m-1)+(4m-3)(4m-1))^2}\right)$$
$$+\quad 8m\left(\frac{\sqrt{((20m-1+(4m-2)(4m-1))+(16m+1)-2}}{((8m-1)+12m+(4m-2)(4m-1))(2(8m-1)+3)}\right)$$
$$+\quad 2(4m-2)\left(\frac{\sqrt{((20m-1+(4m-2)(4m-1))+(16m-2+(4m-3)(4m-1))-2}}{(20m-1+(4m-2)(4m-1))(2(8m-1)+(4m-3)(4m-1))}\right)$$
$$=\quad 4\left(\sqrt{\frac{m(2m+1)}{(4m+1)^4}}+\frac{48m^4}{(16m^2+1)^2}-2\sqrt{2}\sqrt{\frac{m}{64m^3+16m^2+4m+1}}\right.$$
$$+\quad 2m\sqrt{2}\left(\sqrt{\frac{m}{(16m^2+1)^2}}+2\sqrt{\frac{m(2m+3)}{(4m+1)^2(16m+1)}}-5\sqrt{\frac{m^2}{(16m^2+1)^2}}\right.$$

$$
\begin{aligned}
&+\quad 2\sqrt{\frac{m}{64m^3+16m^2+4m+1}}\Bigg)\Bigg)\\
&=\quad \sqrt{\frac{2m^2}{(16m^2+1)^2}}(32m^2-40m+12)+\sqrt{\frac{2m}{64m^3+16m^2+4m+1}}(16m-8)\\
&+\quad 4\sqrt{\frac{m(2m+1)}{(4m+1)^4}}+8m\sqrt{\frac{2m}{(16m+1)^2}}+16m\sqrt{\frac{2m(2m+3)}{(4m+1)^2(16m+1)}}.
\end{aligned}
$$

As a result of the simplification, we get the desired results for both cases. ■

Theorem 3.7. *The geometric arithmetic index of* $\Gamma(SD_{8m})$ *is given as:*

$$
GA(\Gamma(SD_{8m}))=\begin{cases}6(m+1)+\frac{16m\sqrt{56m-7}}{4m+3}+\frac{2(1-4m)(4m^2-9m+5)}{4m-1}\\ \quad +\frac{16(m-1)\sqrt{32m^2-12m+1}}{6m-1}, & \textit{if } m \textit{ is odd},\\ m+\frac{1}{2}+\frac{4m\sqrt{24m-3}}{4m+1}+\frac{(1-4m)(8m^2-10m+3)}{8m-2}\\ \quad +\frac{2(2m-1)\sqrt{32m^2-12m+1}}{6m-1}, & \textit{if } m \textit{ is even}.\end{cases}
$$

Proof. By using the structure of $\Gamma(SD_{8m})$, if m is odd, then

$$
\begin{aligned}
GA(\Gamma(SD_{8m})) &= 6\left(\frac{2\sqrt{(8m-1)^2}}{8m-1}\right)+6m+\binom{4m-4}{2}\left(\frac{2\sqrt{(4m-1)^2}}{2(4m-1)}\right)\\
&\quad +32m\left(\frac{2\sqrt{7(8m-1)}}{8m+7-1}\right)+4(4m-4)\left(\frac{2\sqrt{(4m-1)(8m-1)}}{12m-2}\right)\\
&= 6+6m+\frac{(16m^2-36m+20)}{2}+32m\left(\frac{2\sqrt{(56m-7)}}{8m+6}\right)\\
&\quad +16(m-1)\left(\frac{\sqrt{32m^2-12m+1}}{6m-1}\right)\\
&= 6(m+1)+\frac{16m\sqrt{56m-7}}{4m+3}+\frac{2(1-4m)(4m^2-9m+5)}{4m-1}\\
&\quad +\frac{16(m-1)\sqrt{32m^2-12m+1}}{6m-1}.
\end{aligned}
$$

Similarly, if m is even then

$$
\begin{aligned}
GA(\Gamma(SD_{8m})) &= \frac{2\sqrt{(8m-1)^2}}{2(8m-1)}+2m\left(\frac{2\sqrt{3\times 3}}{3+3}\right)+\binom{4m-2}{2}\left(\frac{2\sqrt{(4m-1)^2}}{2(4m-1)}\right)\\
&\quad +16m\left(\frac{2\sqrt{3(8m-1)}}{8m+2}\right)+2(4m-2)\left(\frac{2\sqrt{(4m-1)(8m-1)}}{12m-2}\right)\\
&= 1+2m+(16m^2-20m+6)+4(2m-1)\left(\frac{\sqrt{(32m^2-12m+1)}}{(6m-1)}\right)\\
&\quad +16m\left(\frac{\sqrt{(24m-3)}}{4m+1}\right)\\
&= m+\frac{1}{2}+\frac{4m\sqrt{24m-3}}{4m+1}+\frac{(1-4m)(8m^2-10m+3)}{8m-2}
\end{aligned}
$$

$$+\frac{2(2m-1)\sqrt{32m^2-12m+1}}{6m-1},$$

which is the required result. ■

Next, we are going to compute the fifth version of geometric arithmetic index.

Theorem 3.8. *The fifth version of GA of* $\Gamma(SD_{8m})$ *is computed as:*
if m *is odd then*

$$GA_5(\Gamma(SD_{8m})) = 2(m-1)(4m-5) + 16m\frac{\sqrt{(32m+17)(16m^2+32m+1)}}{(8m^2+32m+9)} + 6(m+1) + 16(m-1)(4m+1)\frac{\sqrt{16m^2+32m+1}}{(16m^2+20m+1)}.$$

If m *is even then*

$$GA_5(\Gamma(SD_{8m})) = 1 + 2m + (2m-1)(4m-3) + 8m(4m+1)\frac{\sqrt{16m+1}}{(8m^2+12m+1)} + 4(2m-1)(4m+1)\frac{\sqrt{16m^2+1}}{(16m^2+4m+1)}.$$

Proof. Using the structure of $\Gamma(SD_{8m})$, if m is odd, then

$$\begin{aligned}
GA_5(\Gamma(SD_{8m})) &= 6\left(\frac{2\sqrt{\big(3(8m-1)+28m+(4m-4)(4m-1)\big)^2}}{2\big(3(8m-1)+28m+(4m-4)(4m-1)\big)}\right) \\
&+ \binom{4m-4}{2}\left(\frac{2\sqrt{(4(8m-1)+(4m-5)(4m-1))^2}}{2(4(8m-1)+(4m-5)(4m-1))}\right) \\
&+ 6m\left(\frac{2\sqrt{(4(8m-1)+21)^2}}{2(4(8m-1)+21)}\right) \\
&+ 16m\left(\frac{2\sqrt{\big(52m-3+(4m-4)(4m-1)(32m+17)\big)}}{\big(3(8m-1)+28m+(4m-4)(4m-1)\big)+328m+17}\right) \\
&+16(m-1)\left(\frac{2\sqrt{\big(52m-3+(4m-4)(4m-1)\big)\big(4(8m-1)+(4m-5)(4m-1)\big)}}{\big(52m-3+(4m-4)(4m-1)\big)+4(8m-1)+(4m-5)(4m-1)}\right) \\
&= 2\Bigg((m-1)(4m-5) + 3(m+1) + \frac{8(m-1)(4m+1)}{\sqrt{16m^2+20m+1}} \\
&+ \frac{8m\sqrt{512m^3+1296m^2+576m+17}}{(8m^2+32m+9)}\Bigg) \\
&= 6(m+1) + 2(m-1)(4m-5) + 16m\frac{\sqrt{(32m+17)(16m^2+32m+1)}}{(8m^2+32m+9)} \\
&+ 16(m-1)(4m+1)\frac{\sqrt{16m^2+32m+1}}{(16m^2+20m+1)}.
\end{aligned}$$

Similarly if m is even then

$$\begin{aligned} GA_5(\Gamma(SD_{8m})) &= \frac{2\sqrt{\big((8m-1)+4m(3)+(4m-2)(4m-1)\big)^2}}{2\big((8m-1)+12m+(4m-2)(4m-1)\big)} \\ &\quad + \binom{4m-2}{2}\left(\frac{2\sqrt{\big(2(8m-1)+(4m-3)(4m-1)\big)^2}}{2\big(2(8m-1)+(4m-3)(4m-1)\big)}\right) \\ &\quad + 2m\left(\frac{2\sqrt{(2(8m-1)+3)^2}}{2(2(8m-1)+3)}\right) \\ &\quad + 8m\left(\frac{2\sqrt{(8m-1)+12m+(4m-2)(4m-1)(2(8m-1)+3)}}{((8m-1)+12m+(4m-2)(4m-1))+2(8m-1)+3}\right) \\ &\quad +4(2m-1)\left(\frac{2\sqrt{(20m-1+(4m-2)(4m-1))(2(8m-1)+(4m-3)(4m-1))}}{(20m-1+(4m-2)(4m-1))+(2(8m-1)+(4m-3)(4m-1))}\right) \\ &= 1+2m+(2m-1)(4m-3)+8m(4m+1)\frac{\sqrt{16m+1}}{(8m^2+12m+1)} \\ &\quad + 4(2m-1)(4m+1)\frac{\sqrt{16m^2+1}}{(16m^2+4m+1)}. \end{aligned}$$

After a few steps of simplification, we get our desired results. ■

Theorem 3.9. *The Sanskruti index of* $\Gamma(SD_{8m})$ *is given as:*

$$S(\Gamma(SD_{8m})) = \begin{cases} \frac{3}{(16384)}\left(\frac{(16m^2+32m+1)^6}{m(m+2)^3}+\frac{m(32m+17)^6}{(2m+1)^3}\right)+\frac{(m-1)(4m-5)}{2048}\frac{(4m+1)^{12}}{m(2m+1)^3} \\ \quad +\frac{m}{256}\left(\frac{(32m+17)(16m^2+32m+1)}{(m^2+4m+1)}\right)^3+\frac{(m-1)}{32}\frac{(4m+1)^6(16m^2+32m+1)^3}{(m(4m+5))^3}, & \text{if } m \text{ is odd,} \\ \frac{(16m^2+8m+1)^6}{4096(m(2m+1))^3}+\frac{(16m+1)^6}{(128m)^2}+\frac{(2m-1)(4m-3)}{32768}\left(\frac{16m^2+1}{m}\right)^6 \\ \quad +\frac{1}{64m^2}\left(\frac{(4m+1)^2(16m+1)}{(2m+3)}\right)^3+\frac{2m-1}{128}\left(\frac{(4m+1)(16m^2+1)}{m}\right)^3, & \text{if } m \text{ is even.} \end{cases}$$

Proof. To prove the given result, we use the Sanskruti index formula, Lemma 2.2, and Proposition 3.2. Now if m is odd then

$$\begin{aligned} S(\Gamma(SD_{8m})) &= 6m\left(\frac{(32m-17)^2}{64m-36}\right)^3+6\left(\frac{\big(52m-1+(4m-4)(4m-1)\big)^2}{2\big(52m-1+(4m-4)(4m-1)\big)-2}\right)^3 \\ &\quad +\binom{4m-4}{2}\left(\frac{\big(4(8m-1)+(4m-5)(4m-1)\big)^2}{\big(4(8m-1)+(4m-5)(4m-1)\big)-2}\right)^3 \\ &\quad +16m\left(\frac{(52m-1+(4m-4))((4m-1)(4(8m-1)+21))}{(52m-1+(4m-4)(4m-1))+(4(8m-1)+19)}\right)^3 \\ &\quad +16(m-1)\left(\frac{\big(52m-1+(4m-4)(4m-1)\big)\big(4(8m-1)+(4m-5)(4m-1)\big)}{52m+(4m-4)(4m-1)+4(8m-1)+(4m-5)(4m-1)-3}\right)^3 \\ &= \frac{1}{16384m^3}\left(\frac{8(m-1)(4m-5)(4m+1)^{12}}{(2m+1)^3}+\frac{3m^4(32m+17)^6}{(2m+1)^3}\right. \end{aligned}$$

$$+\frac{512(m-1)(4m+1)^6(16m^2+32m+1)}{(4m+5)^3}$$
$$+\frac{64m^4(32m+17)^3(16m^2+32m+1)^3}{(m^2+4m+1)^3}+\frac{3(16m^2+32m+1)^6}{(m+2)^3}\Bigg)$$
$$= \frac{1}{(2048)}\left(\frac{3(16m^2+32m+1)^6}{8m(m+2)^3}+\frac{3m(32m+17)^6}{8(2m+1)^3}\right)+\frac{(m-1)(4m-5)(4m+1)^{12}}{m(2m+1)^3}$$
$$+\frac{m}{256}\left(\frac{(32m+17)(16m^2+32m+1)}{(m^2+4m+1)}\right)^3+\frac{(m-1)}{32}\frac{(4m+1)^6(16m^2+32m+1)^3}{(m(4m+5))^3}.$$

In the same manner, if m is even then

$$S(\Gamma(SD_{8m}))=\left(\frac{\big((8m-1)+12m+(4m-2)(4m-1)\big)^2}{28m+2(4m-2)(4m-1)-4}\right)^3+2m\Big(\frac{\big(16m+1\big)^2}{32m}\Big)^3$$
$$+\binom{4m-2}{2}\left(\frac{\big(2(8m-1)+(4m-3)(4m-1)\big)^2}{16m+(4m-3)(4m-1)-4}\right)^3$$
$$+8m\left(\frac{\big((8m-1)+12m+(4m-2)\big)\big((4m-1)(2(8m-1)+3)\big)}{(8m-1)+12m+(4m-2)(4m-1)+2(8m-1)+1}\right)^3$$
$$+2(4m-2)\left(\frac{\big((8m-1)+12m+(4m-2)(4m-1)\big)\big(2(8m-1)+(4m-3)(4m-1)\big)}{\big((8m-1)+12m+(4m-2)(4m-1)\big)+2(8m-1)+(4m-3)(4m-1)-2}\right)^3.$$

After simplification, we get

$$S(\Gamma(SD_{8m}))=\frac{(16m^2+8m+1)^6}{4096(m(2m+1))^3}+\frac{(16m+1)^6}{(128m)^2}+\frac{(2m-1)(4m-3)}{32768}\left(\frac{16m^2+1}{m}\right)^6$$
$$+\frac{1}{64m^2}\Big(\frac{(4m+1)^2(16m+1)}{(2m+3)}\Big)^3+\frac{2m-1}{128}\Big(\frac{(4m+1)(16m^2+1)}{m}\Big)^3.$$

This gives the required result. ■

4 Hosoya properties

Several chemists have used Polya's concept of counting polynomials to determine the molecular orbitals of unsaturated hydrocarbons [5]. In the concept of graph spectra, the characteristic polynomials have been intensively investigated. In 1988, Hosoya used such a concept to introduce polynomials of numerous chemical structures, which are now commonly known as Hosoya polynomials and have received a lot of attention in the past years, see for example [28]. The Hosoya polynomial provides a wide range of knowledge regarding distance-based graph invariants.

The Hosoya polynomial of a finite graph Γ of order m is

$$\mathbb{H}(\Gamma,y)=\sum_{i\geq 0} dis(\Gamma,i)y^i. \tag{1}$$

For every $i \leq diam(\Gamma)$, the coefficient $dis(\Gamma, i)$ indicates certain pair of vertices (v, u) that satisfy $dis(v, u) = i$. As stated by Ramane and Talwar [29], the reciprocal status Hosoya polynomial of Γ is

$$\mathbb{H}_{rs}(\Gamma, y) = \sum_{vu \in E(\Gamma)} y^{rs(v)+rs(u)}, \tag{2}$$

where $rs(u) = \sum_{v \in V(\Gamma), u \neq v} \frac{1}{dis(v,u)}$ is known as the reciprocal status of u.

Therefore, we have extended this concept on commuting graphs of semi-dihedral groups.

Theorem 4.1. *The Hosoya polynomial of* $\Gamma(SD_{8m})$ *is*

$$\mathbb{H}(\Gamma(SD_{8m}), y) = \begin{cases} \big(24m(m-1)\big)y^2 + 4m(2m+5)y + 8m, & \text{if } m \text{ is odd,} \\ \big(12m(2m-1)\big)y^2 + 8m(m+1)y + 8m, & \text{if } m \text{ is even.} \end{cases}$$

Proof. Since, $diam(\Gamma(SD_{8m})) = 2$. We need to examine $\mathrm{dis}(\Gamma(SD_{8m}), 0)$, $dis(\Gamma(SD_{8m}), 1)$ and $dis(\Gamma(SD_{8m}), 2)$. Next, for an odd m, we consider a vertex set V_k for any pair of vertices of $\Gamma(SD_{8m})$ and we get:

$$|V_k| = \binom{8m}{2} + 8m = \frac{8m(8m+1)}{2}.$$

Let

$$\mathcal{C}(\Gamma(SD_{8m}), k) = \{(v_1, v_2); v_1, v_2 \in V(\Gamma(SD_{8m})) \mid \mathrm{dis}(v_1, v_2) = k\},$$

and $dis(\Gamma(SD_{8m}), k) = |\mathcal{C}(\Gamma(SD_{8m}), k)|$. Then,

$$V_k = \mathcal{C}(\Gamma(SD_{8m}), 0) \cup \mathcal{C}(\Gamma(SD_{8m}), 1) \cup \mathcal{C}(\Gamma(SD_{8m}), 2).$$

Since, for each $v_1 \in V(\Gamma(SD_{8m})), dis(v_1, v_1) = 0$ Thus, $\mathcal{C}(\Gamma(SD_{8m}), 0) = 8m$ and

$$\begin{aligned} \mathcal{C}(\Gamma(SD_{8m}), 1) &= 4(4m) + 4(4m-4) + \binom{4}{2} + m\binom{4}{2} + \binom{4m-4}{2} \\ &= 16m + 16m - 16 + 6 + 6m + 8m^2 - 18m + 10 \\ &= 8m^2 + 20m. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{C}(\Gamma(SD_{8m}), 1) &= |V_k| - dis(\Gamma(SD_{8m}), 0) - \mathrm{dis}(\Gamma(SD_{8m}), 1) \\ &= \frac{8m(8m+1)}{2} - 8m - (8m^2 + 20m) \\ &= \frac{48m^2 - 48m}{2} = 24m(m-1), \end{aligned}$$

and

$$\mathbb{H}(\Gamma(SD_{8m}), y) = (24m(m-1))y^2 + (8m^2 + 20m)y + 8m.$$

Now, if m is even then we consider the vertex set V_k and we have:

$$|V_k| = \binom{8m}{2} + 8m = \frac{8m(8m+1)}{2}.$$

Suppose

$$\mathcal{C}(\Gamma(SD_{8m}), k) = \{(v_1, v_2); v_1, v_2 \in V(\Gamma(SD_{8m})) \mid \mathrm{dis}(v_1, v_2) = k\},$$

and $\text{dis}(\Gamma(SD_{8m}), k) = |\mathcal{C}(\Gamma(SD_{8m}), k)|$. Then,

$$V_k = \mathcal{C}(\Gamma(SD_{8m}), 0) \cup \mathcal{C}(\Gamma(SD_{8m}), 1) \cup \mathcal{C}(\Gamma(SD_{8m}), 2).$$

Since $\text{dis}(v_1, v_1) = 0$, for each $v_1 \in V(\Gamma(SD_{8m}))$ $\mathcal{C}(\Gamma(SD_{8m}), 0) = 8m$. Next,

$$\begin{aligned}\mathcal{C}(\Gamma(SD_{8m}), 1) &= 8m + 2(4m-2) + \binom{2}{2} + 2m\binom{2}{2} + \binom{4m-2}{2} \\ &= 8m + 8m - 4 + 1 + 2m + 8m^2 - 10m + 3 \\ &= 8m^2 + 8m,\end{aligned}$$

and,

$$\begin{aligned}\mathcal{C}(\Gamma(SD_{8m}), 2) &= |V_k| - \text{dis}(\Gamma(SD_{8m}), 0) - \text{dis}(\Gamma(SD_{8m}), 1) \\ &= \frac{8m(8m+1)}{2} - 8m - (8m^2 + 8m) \\ &= \frac{48m^2 - 24m}{2} = 12m(2m-1).\end{aligned}$$

Hence,

$$\mathbb{H}(\Gamma(SD_{8m}), y) = (12m(2m-1))y^2 + (8m^2 + 8m)y + 8m.$$

This is the required proof. ■

Proposition 4.2. *If y is a vertex of $\Gamma(SD_{8m})$, where m is odd, then*

$$rs(y) = \begin{cases} 4m+4, & y \in Z(SD_{8m}), \\ 4m, & y \in \phi_2, \\ 2(2m-1), & y \in \phi_3. \end{cases} \tag{3}$$

Proof. For odd m, we know that $\Gamma(SD_{8m}) = K_4 \vee (mK_4 \cup K_{4m-4})$ whose vertex set is $Z(SD_{8m}) \cup \phi_2 \cup \phi_3$. Therefore, if $y \in Z(SD_{8m})$ then $ec(y) = 1$, and using the reciprocal status concept, we get

$$rs(y) = \left(\frac{1}{1}\right)(4m+4) = 4m+4.$$

Again using the reciprocal status concept and $ec(y) = 1$ for $y \in \phi_2$, we obtain

$$rs(y) = \left(\frac{1}{1}\right)(4 + 4m - 4) = 4m.$$

Moreover, for any $y \in \phi_3$, we have $ec(y) = 2$. Also, apply the reciprocal status concept, we get

$$\begin{aligned} rs(y) &= \left(\frac{1}{2}\right)(4m + 4m - 4) \\ &= \frac{8m-4}{2} = 2(2m-1).\end{aligned}$$

Combining these, we obtain the required result. ■

Proposition 4.3. *If m is even and y is a vertex of $\Gamma(SD_{8m})$ then*

$$rs(y) = \begin{cases} 4m+2, & y \in Z(SD_{8m}), \\ 4m, & y \in \psi_2, \\ 4m-1, & y \in \psi_3. \end{cases} \tag{4}$$

Proof. Since the structure of $\Gamma(SD_{8m})$ is $K_2 \vee (2mK_2 \cup K_{4m-2})$, where m is even, its vertex set is $Z(SD_{8m}) \cup \psi_2 \cup \psi_3$. Therefore, by using the reciprocal status formula, we get $ec(y) = 1$ for $y = Z(SD_{8m})$ and so,

$$rs(y) = \left(\frac{1}{1}\right)(2 + 2(2m)) = 4m + 2.$$

If $y \in \psi_2$ then $ec(y) = 1$. So

$$rs(y) = \left(\frac{1}{1}\right)(2 + 4m - 2) = 4m.$$

Likewise, if $y \in \psi_3$ then $ec(y) = 2$ and we have:

$$\begin{aligned} rs(y) &= \left(\frac{1}{2}\right)(2(2m) + 4m - 2) \\ &= \frac{8m-2}{2} = (4m-1). \end{aligned}$$

Combining all cases, we get the required result. ■

Theorem 4.4. *The reciprocal status Hosoya polynomial of $\Gamma(SD_{8m})$ is*

$$\mathbb{H}_{rs}(\Gamma(SD_{8m}), y) = \begin{cases} 4(4m)y^{4(m+1)} + 4(4m-4)y^{4m} + 4m(4m-4)y^{2(2m-1)}, & \text{if } m \text{ is odd,} \\ (4m)y^{2(2m+1)} + (8m-4)y^{4m} + (16m^2-8)y^{4m-1}, & \text{if } m \text{ is even.} \end{cases}$$

Proof. By using Proposition 4.2, there are three types of edges $(v_1 \sim v_2, v_1 \sim v_3, v_2 \sim v_3)$ in $\Gamma(SD_{8m})$, and according to the end vertices reciprocal status, we have $v_1 + v_2 = 4(m+1)$, $v_1 + v_3 = 4m$ and $v_2 + v_3 = 2(2m-1)$ edges. By incorporating the edge set's and the formula for the reciprocal status Hosoya polynomial, we obtain

$$\begin{aligned} \mathbb{H}_{rs}(\Gamma(SD_{8m}), y) &= \sum_{v_1 \sim v_2} y^{v_1+v_2} + \sum_{v_1 \sim v_3} y^{v_1+v_3} + \sum_{v_2 \sim v_3} y^{v_2+v_3} \\ &= 4(4m)y^{4(m+1)} + 4(4m-4)y^{4m} + 4m(4m-4)y^{2(2m-1)}. \end{aligned}$$

Similarly, if m is even, then three distinct types of edges exist, that is $(v_1 \sim v_2, v_1 \sim v_3, v_2 \sim v_3)$. Therefore, using Proposition 4.3, we get

$$v_1 + v_2 = 2(2m+1), \quad v_1 + v_3 = 4m, \quad v_2 + v_3 = (4m-1).$$

By incorporating the edge sets and formula, we get the reciprocal status Hosoya polynomial in the following manner:

$$\begin{aligned} \mathbb{H}_{rs}(\Gamma(SD_{8m}), y) &= \sum_{v_1 \sim v_2} y^{v_1+v_2} + \sum_{v_1 \sim v_3} y^{v_1+v_3} + \sum_{v_2 \sim v_3} y^{v_2+v_3} \\ &= (4m)y^{2(2m+1)} + (8m-4)y^{4m} + (16m^2-8)y^{4m-1}. \end{aligned}$$

Which is the required proof. ■

5 Concluding remarks

This work aimed to explore the structural characteristics of the commuting graphs over certain non-abelian groups. The group of symmetries, semi-dihedral and dicyclic groups have familiar algebraic structures that have substantially enhanced our understanding of molecules and electron configurations. In this paper, we examined several (detour) distance properties, deliberated the metric dimension, and resolving polynomial of the commuting graphs of dicyclic groups. Moreover, this approach has also allowed us to examine numerous essential degree-based topological invariants of the commuting graphs of semi-dihedral groups.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

Acknowledgments. The authors gratefully acknowledge the constructive comments and suggestions of the anonymous referee, which significantly improved the readability of the manuscript.

References

[1] M. Ghorbani and M. A. Hosseinzadeh, Computing abc4 index of nanostar dendrimers, *Optoelectron. Adv. Mater-Rapid Commun.* **4** (2010) 1419–1422.

[2] H. Wiener, Structural determination of paraffin boiling points, *J. Am. Chem. Soc.* **69** (1947) 17–20.

[3] F. Ali, B. A. Rather, A. Din, T. Saeed and A. Ullah, Power graphs of finite groups determined by hosoya properties, *Entropy* **24** (2022) #213, https://doi.org/10.3390/e24020213.

[4] S. Hafeez and R. Farooq, Inverse sum indeg energy of graphs, *IEEE Access* **7** (2019) 100860–100866.

[5] G. Pólya, Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen, *Acta Math.* **68** (1937) 145–254.

[6] H. Hosoya, On some counting polynomials in chemistry, *Discrete Appl. Math.* **19** (1988) 239–257, https://doi.org/10.1016/0166-218X(88)90017-0.

[7] B. E. Sagan, Y. N. Yeh and P. Zhang, The wiener polynomial of a graph, *Int. J. Quantum. Chem.* **60** (1996) 959–969.

[8] E. Estrada, O. Ivanciuc, I. Gutman, A. Gutierrez and L. Rodríguez, Extended wiener indices. A new set of descriptors for quantitative structure-property studies, *New J. Chem.* **22** (1998) 819–822, https://doi.org/10.1039/A709255E.

[9] M. Randic, Characterization of molecular branching, *J. Am. Chem. Soc.* **97** (1975) 6609–6615, https://doi.org/10.1021/ja00856a001.

[10] O. Ivanciuc, T. S. Balaban and A. T. Balaban, Design of topological indices. Part 4. Reciprocal distance matrix, related local vertex invariants and topological indices, *J. Math. Chem.* **12** (1993) 309–318, https://doi.org/10.1007/BF01164642.

[11] E. Estrada, L. Torres, L. Rodriguez and I. Gutman, An atom-bond connectivity index: modelling the enthalpy of formation of alkanes, *Indian J. Chem. A* **37** (1998) 849–855.

[12] R. Brauer and K. A. Fowler, On groups of even order, *Ann. of Math.* **62** (1955) 565–583, https://doi.org/10.2307/1970080.

[13] F. Ali and Y. Li, The connectivity and the spectral radius of commuting graphs on certain finite groups, *Linear Multilinear Algebra* **69** (2021) 2945–2958, https://doi.org/10.1080/03081087.2019.1700893.

[14] B. Tolue, The twin non-commuting graph of a group, *Rend. Circ. Mat. Palermo, II. Ser* **69** (2020) 591–599, https://doi.org/10.1007/s12215-019-00421-4.

[15] M. A. Salahshour, Commuting conjugacy class graph of G when $\frac{G}{Z(G)} \cong D_{2n}$, *Math. Interdisc. Res.* **5** (2020) 379–385, https://doi.org/10.22052/MIR.2021.240226.1232.

[16] F. Ali, B. A. Rather, M. Naeem and W. Wang, Degree and distance based topological descriptors of power graphs of finite non-abelian groups, *Discrete Appl. Math.* **345** (2024) 62–76, https://doi.org/10.1016/j.dam.2023.11.038.

[17] R. Ismail, F. Ali, R. Qasim, M. Naeem, W. K. Mashwani and S. Khan, Several zagreb indices of power graphs of finite non-abelian groups, *Heliyon* **9** (2023) #e19560, https://doi.org/10.1016/j.heliyon.2023.e19560.

[18] S. Hayat, S. Wang and J. B. Liu, Valency-based topological descriptors of chemical networks and their applications, *Appl. Math. Model.* **60** (2018) 164–178, https://doi.org/10.1016/j.apm.2018.03.016.

[19] H. P. Schultz, E. B. Schultz and T. P. Schultz, Topological organic chemistry. 8. graph theory and topological indexes of heteronuclear systems, *J. Chem. Inf. Comput. Sci.* **34** (1994) 1151–1157, https://doi.org/10.1021/ci00021a022.

[20] O. Ivanciuc, Qsar comparative study of wiener descriptors for weighted molecular graphs, *J. Chem. Inf. Comput. Sci.* **40** (2000) 1412–1422, https://doi.org/10.1021/ci000068y.

[21] M. R. Rajesh Kanna, R. Pradeep Kumar and R. Jagadeesh, Computation of topological indices of dutch windmill graph, *Open J. Discrete Math.* **6** (2016) 74–81, https://doi.org/10.4236/ojdm.2016.62007 .

[22] J.-B. Liu, C. Wang, S. Wang and B. Wei, Zagreb indices and multiplicative zagreb indices of eulerian graphs, *Bull. Malays. Math. Sci. Soc.* **42** (2019) 67–78, https://doi.org/10.1007/s40840-017-0463-2.

[23] A. Graovac, M. Ghorbani and M. A. Hosseinzadeh, Computing fifth geometric-arithmetic index for nanostar dendrimers, *J. Math. Nanosci.* **1** (2011) 33–42.

[24] S. M. Hosamani, Computing sanskruti index of certain nanostructures, *J. Appl. Math. Comput.* **54** (2017) 425–433, https://doi.org/10.1007/s12190-016-1016-9.

[25] T. Cheng, M. Dehmer, F. Emmert-Streib, Y. Li and W. Liu, Properties of commuting graphs over semidihedral groups, *Symmetry* **13** (2021) #103, https://doi.org/10.3390/sym13010103.

[26] F. Ali, B. A. Rather, N. Fatima, M. Sarfraz, A. Ullah, K. A. M. Alharbi and R. Dad, On the topological indices of commuting graphs for finite non-abelian groups, *Symmetry* **14** (2022) #1266, https://doi.org/10.3390/sym14061266.

[27] F. Ali, B. A. Rather, M. Sarfraz, A. Ullah, N. Fatima and W. K. Mashwani, Certain topological indices of non-commuting graphs for finite non-abelian groups, *Molecules* **27** (2022) #6053, https://doi.org/10.3390/molecules27186053.

[28] B. A. Rather, F. Ali, S. Alsaeed and M. Naeem, Hosoya polynomials of power graphs of certain finite groups, *Molecules* **27** (2022) #6081, https://doi.org/10.3390/molecules27186081.

[29] H. S. Ramane and S. Y. Talwar, Reciprocal transmission Hosoya polynomial of graphs, *International J. Math. Combin.* **3** (2019) 70–79.