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Research Paper

Construction of Zero-Divisor Graph of a Hyperlattice with Respect to Hyperideals

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Abstract

In this paper, we define the zero-divisor graph of a meet-hyperlattice with respect to a hyperideal. We prove the diameter of a $P-$hyperlattice and Nakano hyperlattice are at most 3 and 4 respectively. We obtain that the zero-divisor graph with respect to the intersection of two prime hyperideals is complete bipartite. We prove certain properties of these zero-divisor graphs with suitable examples.

1 Introduction

In recent years, researchers [1–3] have explored the construction of graphs from algebraic structures such as rings, groups, and modules. Ehsan and Khashyarmanesh [4] have studied the zero-divisor graphs of lattices and characterized them in terms of atoms in a lattice. Joshi et al. [5] described zero-divisor graphs of lattices with respect to an ideal, and computed their diameter, girth, and characterized bipartite zero-divisor graphs. Joshi and Khishte [6] examined the zero-divisor graph of lattices using the spectrum of a lattice and provided conditions for adjacency. Domination in lattices using atoms in lattices was studied by Chelvam and Nithya [7]. Tapatee et al. [8, 9] studied the graph of a lattice with respect to superfluous elements and essential elements and established related properties. In [10], the authors studied the zero-divisor graphs of posets.

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The idea of hyperoperation, $\circ : \mathbb{H}^2 \to \mathcal{P}^*(\mathbb{H})$, where $\mathbb{H}$ is a non-empty set and $\mathcal{P}^*(\mathbb{H})$ is the set of non-empty subsets of $\mathbb{H}$ extends the concept of binary operations in a classical algebraic system. A binary operation deals with cases where the combination of two elements yields one outcome. However, this is a limitation, as in most instances found in natural phenomena, the combination of two elements can yield multiple outcomes. Konstantinidou [11] generated hyperlattices from lattices and investigated their distributivity. The notion of complete join hyperlattices was studied by Lashkenari and Davvaz [12]. Ameri et al. [13] investigated join hyperlattices and established the relationship between prime hyperideals and prime hyperfilters. Bideshki et al. [14] analyzed the properties of hyperideals and hyperfilters in a meet-hyperlattice. In [15] Pallavi et al. defined various generalizations of prime hyperideals in a meet-hyperlattice. The idea of a fundamental relation on a hyperlattice was introduced by Rasouli and Davvaz [16].

In Section 2 of the paper, we give necessary preliminaries on hyperlattices from [11–13, 17] and we refer to [18] for preliminaries in graph theory. In Section 3, we define the notion of an element prime to a hyperideal in a meet-hyperlattice. Using these elements, we establish the definition of a zero-divisor graph of a meet-hyperlattice with respect to hyperideals. We prove that the diameter of a P-hyperlattice and Nakano hyperlattice are at most 3 and 4, respectively. Finally, we show that the zero-divisor graph with respect to the intersection of two prime hyperideals is complete bipartite. In Section 4, we provide examples of meet-hyperlattices of chemical compounds.

2 Preliminaries

We use the following notations in the paper: $\bigwedge$ denotes a meet-hyperoperation, $\wedge$ denotes a meet (binary) operation, $\vee$ denotes a join (binary) operation, $\sqcap$ denotes Nakano hyperoperation and $\bigwedge^P$ denotes P-hyperoperation.

Definition 2.1. ([11]). An algebraic system $(\mathbb{L}, \bigwedge, \vee)$ where $\vee$ is a binary operation and $\bigwedge$ is a hyperoperation, is called a meet-hyperlattice ($\bigwedge$) if it satisfies:

1. $l_1 \in l_1 \bigwedge l_1$ and $l_1 = l_1 \vee l_1$,
2. $l_1 \bigwedge (l_2 \bigwedge l_3) = (l_1 \bigwedge l_2) \bigwedge l_3$ and $l_1 \vee (l_2 \vee l_3) = (l_1 \vee l_2) \vee l_3$,
3. $l_1 \bigwedge l_2 = l_2 \bigwedge l_1$ and $l_1 \vee l_2 = l_2 \vee l_1$,
4. $l_2 \in l_2 \bigwedge (l_1 \vee l_2) \cap l_2 \vee (l_2 \bigwedge l_1)$,

for all $l_1, l_2, l_3 \in \mathbb{L}$.
Further, a meet-hyperlattice $\mathbb{L}$ is called a strong meet-hyperlattice if for all $l_1, b \in \mathbb{L}$ with $l_1 \in l_1 \bigwedge b$ implies $l_1 \vee b = b$.

Throughout, $\mathbb{L}$ denotes a strong meet-hyperlattice.
Remark 1. Let $(\mathbb{L}, \bigwedge, \vee)$ be a meet-hyperlattice. For $l_1, l_2 \in \mathbb{L}$, the relation

$$l_1 \leq l_2 \text{ if and only if } l_2 = l_1 \vee l_2,$$

is a partial order on $\mathbb{L}$.

Example 2.2. ([12]). Let L be a modular lattice. For all $a, b \in L$, we define

$$a \sqcap b = \{c \in L : c \wedge a = c \wedge b = a \wedge b\}.$$

Then $(L, \sqcap, \vee)$ is a strong meet-hyperlattice. This hyperlattice is called Nakano hyperlattice.

Example 2.3. ([15]). Let $\mathbb{L} = \{0, a_1, a_2, a_3, 1\}$. Let the hyperoperation $\bigwedge$ and the classical operation $\vee$ be defined as in the following tables:

$\bigwedge$	0	a_1	a_2	a_3	1
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a_1	$\{0\}$	$\{a_1\}$	$\{0\}$	$\{0\}$	$\{a_1\}$
a_2	$\{0\}$	$\{0\}$	$\{0, a_2\}$	$\{0\}$	$\{0, a_2\}$
a_3	$\{0\}$	$\{0\}$	$\{0\}$	$\{0, a_3\}$	$\{0, a_3\}$
1	$\{0\}$	$\{a_1\}$	$\{0, a_2\}$	$\{0, a_3\}$	$\{a_1, 1\}$

$\vee$	0	a_1	a_2	a_3	1
0	0	a_1	a_2	a_3	1
a_1	a_1	a_1	1	1	1
a_2	a_2	1	a_2	1	1
a_3	a_3	1	1	a_3	1
1	1	1	1	1	1

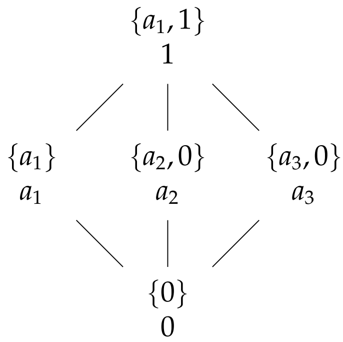

Figure 1: Lattice diagram of the hyperlattice $(\mathbb{L}, \bigwedge, \vee)$.

It can be seen that $(\mathbb{L}, \bigwedge, \vee)$ is a strong meet-hyperlattice (see Figure 1).

In [19, 20], the authors studied inheritance examples of algebraic hyperstuctures, particularly of hypergroups. Accordingly, in [15] the authors have constructed the following meet-hyperlattices.
Let "parents" be denoted by P, "filial generation" be denoted by f and mating by $\times$.

Example 2.4. ([15]). Consider the process of dihybrid crossing of pea plant. Let a_1 denote the phenotype tall and round, a_2 denote the phenotype short and round, a_3 denote the phenotype tall and wrinkled, and a_4 denote the phenotype short and wrinkled.

P :	Round, Tall $(RRTT)$	$\times$	Wrinkled, Short $(rrtt)$
		$\downarrow$	
f_1 :		Round, Tall $(RrTt)$	
$f_1 \times f_1$:	Round, Tall $(RrTt)$	$\times$	Round, Tall $(RrTt)$
		$\downarrow$	
f_2 :		a_1, a_2, a_3, a_4	

Take $\mathbb{K} = \{a_1, a_2, a_3, a_4\}$. Let $\bigwedge$ be the crossing between the phenotypes, and $\vee$ be the relation of dominance between the phenotypes, given by the following tables.

$\bigwedge$	a_1	a_2	a_3	a_4
a_1	$\mathbb{K}$	$\mathbb{K}$	$\mathbb{K}$	$\mathbb{K}$
a_2	$\mathbb{K}$	$\{a_2, a_4\}$	$\mathbb{K}$	$\{a_2, a_4\}$
a_3	$\mathbb{K}$	$\mathbb{K}$	$\{a_3, a_4\}$	$\{a_3, a_4\}$
a_4	$\mathbb{K}$	$\{a_2, a_4\}$	$\{a_3, a_4\}$	$\{a_4\}$

$\vee$	a_1	a_2	a_3	a_4
a_1	a_1	a_1	a_1	a_1
a_2	a_1	a_2	a_1	a_2
a_3	a_1	a_1	a_3	a_3
a_4	a_1	a_2	a_3	a_4

Then $(\mathbb{K}, \bigwedge, \vee)$ is a meet-hyperlattice.

Lemma 2.5. *([12]). For any $l_1, l_2 \in \mathbb{L}$, there exist $x, y \in l_1 \bigwedge l_2$ such that $x \leq l_1$ and $y \leq l_2$.*

3 Zero-divisor graph with respect to a hyperideal

Definition 3.1. 1. $\emptyset \neq J \subseteq \mathbb{L}$ is called a semi hyperideal if for $i_1 \in \mathbb{L}, i_2 \in J$ with $i_1 \leq i_2$ implies $i_1 \in J$.

2. A semi hyperideal J is called a hyperideal if $i_1, i_2 \in J$ implies $i_1 \vee i_2 \in J$.

3. A proper semi hyperideal (hyperideal) J is called prime if for $i_m \in \mathbb{L}, m = 1, 2$, $(i_1 \bigwedge i_2) \cap J \neq \emptyset$ implies either $i_1 \in J$ or $i_2 \in J$.

We denote the set of all hyperideals of $\mathbb{L}$ by $I(\mathbb{L})$.

Theorem 3.2. *([15]). Let $(L, \wedge, \vee)$ be a lattice and $\emptyset \neq P \subseteq L$ be such that for each $l_1 \in L$ there exists $p \in P$ be such that $l_1 \leq p$. We define a hyperoperation $\bigwedge^P$ by*

$$l_1 \overset{P}{\bigwedge} l_2 = l_1 \wedge l_2 \wedge P = \{l_1 \wedge l_2 \wedge p \ : \ p \in P\}.$$

Then $(L, \bigwedge^P, \vee)$ is a meet-hyperlattice.

For a lattice L and $P \subseteq L$, satisfying the conditions given in Theorem 3.2, we call the hyperlattice $(L, \bigwedge^P, \vee)$ as a P-hyperlattice.

Definition 3.3. Let $J \in I(\mathbb{L})$.

1. For $A \subseteq \mathbb{L}$, we define the subset $(J : A) = \{y \in \mathbb{L} : (y \bigwedge a) \cap J \neq \emptyset \text{ for all } a \in A\}$. If $A = \{a\}$, then we simply write $(J : A)$ as $(J : a)$. A is said to be prime to J if $(J : A) = J$.

2. $a \in \mathbb{L}$ is said to be prime to J if $(J : a) = J$. We denote by $\mathcal{S}(J)$, the set of all elements that are not prime to J.

3. J is called primal if $\mathcal{S}(J) \in I(\mathbb{L})$.

Lemma 3.4. *Let $I \in I(\mathbb{L})$ with $I \neq \mathbb{L}$. Then $I = \mathcal{S}(I)$ if and only if I is prime.*

Proof. Suppose that I is prime. Then clearly $I \subseteq \mathcal{S}(I)$. Now let $a \in \mathcal{S}(I)$. Then $(I : a) \neq I$, and so there exists $y \in (I : a) \setminus I$ such that $(a \bigwedge y) \cap I \neq \emptyset$. Since I is prime, we must have $a \in I$. Conversely suppose that $I = \mathcal{S}(I)$. Let $a_m \in \mathbb{L}, m = 1, 2$, with $(a_1 \bigwedge a_2) \cap I \neq \emptyset$ and $a_1 \notin I$. Then $a_2 \in \mathcal{S}(I) = I$. ∎

Definition 3.5. For $J \in I(\mathbb{L})$, we define

$$\mathcal{Z}(J) = \{r \in \mathbb{L} \setminus J : (r \bigwedge i) \cap J \neq \emptyset \text{ for some } i \notin J\}.$$

Remark 2. For $J \in I(\mathbb{L})$, $\mathcal{Z}(J) = \{r \notin J : (J : r) \neq J\}$.

Lemma 3.6. *For* $J \in I(\mathbb{L})$, $\mathcal{Z}(J) = \mathcal{S}(J) \setminus J$.

Proof. Let $l_1 \in \mathcal{Z}(J)$. Then there exists $y \in J^c$ such that $(l_1 \bigwedge y) \cap J \neq \emptyset$, and so $y \in (J : l_1)$ yielding $J \neq (J : l_1)$. So $l_1 \in \mathcal{S}(J) \setminus J$. Conversely, let $u \in \mathcal{S}(J) \setminus J$. Then $(J : u) \neq J$, and so there exists $w \in (J : u) \setminus J$ such that $(u \bigwedge w) \cap J \neq \emptyset$. This shows that $u \in \mathcal{Z}(J)$. ■

Remark 3. For $I \in I(\mathbb{L})$ $\mathcal{Z}(I) = \emptyset$ if and only if $I = \mathbb{L}$ or I is prime.

Definition 3.7. Let $J \in I(\mathbb{L})$. We define an undirected graph called the zero-divisor graph of $\mathbb{L}$ with respect to the hyperideal J, denoted by $G^J(\mathbb{L})$, whose vertex set is $V(G^J(\mathbb{L})) = \mathcal{Z}(J)$ and $x, y \in \mathcal{Z}(J)$ are adjacent if $x \neq y$ and $(x \bigwedge y) \cap J \neq \emptyset$.

For a connected graph G, we denote $d(x, y)$ as the distance between the vertices x and y in G.

Example 3.8. Let $\mathbb{L} = \{0, a_1, a_2, a_3, 1\}$. Let the hyperoperation $\bigwedge$ and the classical operation $\vee$ be defined by the following tables.

$\bigwedge$	0	a_1	a_2	a_3	1
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a_1	$\{0\}$	$\{0, a_1\}$	$\{0, a_1\}$	$\{0\}$	$\{0, a_1\}$
a_2	$\{0\}$	$\{0, a_1\}$	$\{0, a_2\}$	$\{0\}$	$\{0, a_2\}$
a_3	$\{0\}$	$\{0\}$	$\{0\}$	$\{a_3\}$	$\{a_3\}$
1	$\{0\}$	$\{0, a_1\}$	$\{0, a_2\}$	$\{a_3\}$	$\{a_3, 1\}$

$\vee$	0	a_1	a_2	a_3	1
0	0	a_1	a_2	a_3	1
a_1	a_1	a_1	a_2	1	1
a_2	a_2	a_2	a_2	1	1
a_3	a_3	1	1	a_3	1
1	1	1	1	1	1

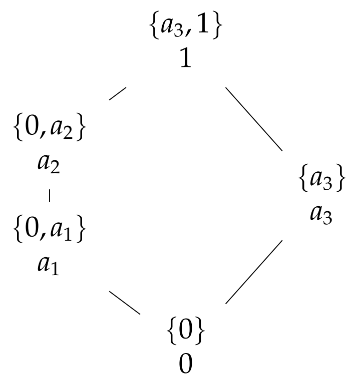

Figure 2: Lattice diagram of the hyperlattice $(\mathbb{L}, \bigwedge, \vee)$.

Then $(\mathbb{L}, \bigwedge, \vee)$ is a meet-hyperlattice as shown in Figure 2 and $I = \{0\}$ and $J = \{0, a_1\}$ (shown in Figure 5) are hyperideals of $\mathbb{L}$ whose zero-divisor graphs are given in Figure 3 and Figure 4, respectively.

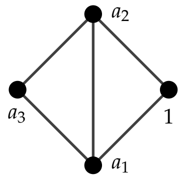

Figure 3: Zero-divisor graph with respect to the hyperideal $I = \{0\}$.

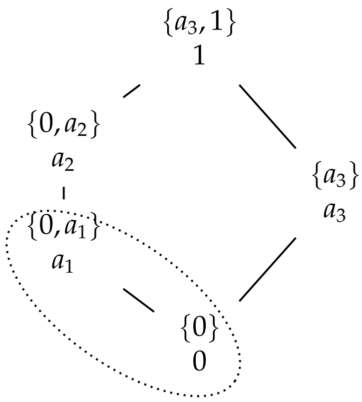

Figure 4: Hyperideal $J = \{0, a_1\}$ is represented by the dotted lines.

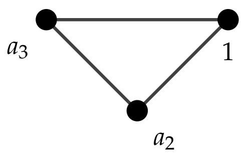

Figure 5: Graph with respect to the hyperideal $J = \{0, a_1\}$.

Remark 4. In Definition 3.7, if we drop the condition $x \neq y$, then we may have a graph with loops. For example, the zero-divisor graph of $\mathbb{L}$ with respect to the hyperideal $I = 0$ given in Example 3.8, by allowing the condition $x = y$ in the definition, is given in Figure 6.

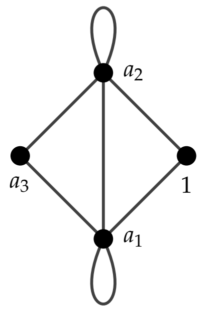

Figure 6: Zero-divisor graph with respect to the hyperideal $I = \{0\}$.

Theorem 3.9. *1. Let $(L, \wedge, \vee)$ be a lattice, and $P \subseteq L$ such that $(\mathbb{L}, \bigwedge^P, \vee)$ is a P-hyperlattice. Then for $J \in I(\mathbb{L})$, the graph $G^J(\mathbb{L})$ is connected and diam$(G^J(\mathbb{L})) \leq 3$.*

2. If L is modular, and $(\mathbb{L}, \sqcap, \vee)$ is the Nakano hyperlattice, then for $J \in I(\mathbb{L})$, the graph $G^J(\mathbb{L})$ is connected and diam$(G^J(\mathbb{L})) \leq 4$.

Proof. 1. Let $\mathbb{L}$ be a P-hyperlattice and I be hyperideal of $\mathbb{L}$. Let $l_1, l_2 \in \mathcal{Z}(I)$. Suppose $(l_1 \bigwedge^P l_2) \cap I \neq \emptyset$, then $l_1 l_2$ is an edge. Otherwise if $(l_1 \bigwedge^P l_2) \cap I = \emptyset$, then since $l_1, l_2 \in \mathcal{Z}(I)$, there exist $x, y \in \mathcal{Z}(I)$ such that $l_1 x$ and $l_2 y$ are edges.
Case 1: Suppose $x = y$. Then $l_1 x l_2$ constitutes a path.
Case 2: Suppose that $x \neq y$. If $l_1 y$ is an edge then $l_1 y l_2$ is a path. If $l_2 x$ is an edge, then $l_1 x l_2$ is a path, and hence $d(l_1, l_2) = 2$.
Case 3: Suppose that $x \neq y$ and $l_2 x$ and $l_1 y$ are not edges in $G^J(\mathbb{L})$. Since $\mathbb{L}$ is a P-hyperlattice, $(x \bigwedge^P l_1) \cap I \neq \emptyset$, so there is $p_1 \in P$ such that $x \wedge l_1 \wedge p_1 \in I$. Similarly, there is $p_2 \in P$ such that $y \wedge l_2 \wedge p_2 \in I$. Now take $u = l_2 \wedge x \wedge p_1$, $v = l_1 \wedge y \wedge p_2$. Since $(l_2 \bigwedge^P x) \cap I = \emptyset$, and $(l_1 \bigwedge^P y) \cap I = \emptyset$, it follows that $u, v \notin I$. Now $l_1 \wedge u = x \wedge l_1 \wedge l_2 \wedge p_1 \leq x \wedge l_1 \wedge p_1$, implies $l_1 u$ is an edge. Similarly, $l_2 v$ is an edge. As $u \wedge v = l_1 \wedge l_2 \wedge x \wedge y \wedge\ p_1 \wedge p_2 \leq l_1 \wedge p_1 \in I$. There exists $p' \in P$ such that $u \wedge v \leq p'$ and so $u \wedge u \in u \bigwedge^P v$, which shows that $(u \bigwedge^P v) \cap I \neq \emptyset$, whence uv is an edge. Thus $l_1 u v l_2$ is a path, and so $d(l_1, l_2) \leq 3$.

2. Let $\mathbb{L}$ be a Nakano hyperlattice and I be a hyperideal of $\mathbb{L}$. Let $l_1, l_2 \in \mathcal{Z}(I)$. Suppose $(l_1 \sqcap l_2) \cap I \neq \emptyset$. Then $l_1 l_2$ is an edge. Suppose that $(l_1 \sqcap l_2) \cap I = \emptyset$. Since $l_1, l_2 \in \mathcal{Z}(I)$, there exist $x, y \in \mathcal{Z}(I)$ and there exist $u, v \in \mathbb{L}$ such that

$$l_1 \wedge u = u \wedge x = l_1 \wedge x, u \in I, \text{ so that } l_1 x \text{ is an edge,}$$

and

$$l_2 \wedge v = y \wedge v = l_2 \wedge y, v \in I, \text{ so that } l_2 y \text{ is an edge.}$$

Suppose that $x = y$. Then $l_1 x y$ is a path. Suppose that $x \neq y$. Now $t = l_1 \wedge l_2 \notin I$. Then $x \wedge t = x \wedge l_1 \wedge l_2 \leq u \wedge l_2 \leq u \in I$, and so $(x \sqcap t) \cap I \neq \emptyset$. In a similar way, we can get $(y \sqcap t) \cap I \neq \emptyset$. Hence $l_1 x t y l_2$ is a path and $d(l_1, l_2) \leq 4$.

■

Example 3.10. Let $\mathbb{L} = \{0, a_1, a_2, \cdots, a_{10}, 1\}$. Define the hyperoperation $\bigwedge$ and the classical operation $\vee$ on $\mathbb{L}$ as given in the following tables:

$\bigwedge$	0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	1
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a_1	$\{0\}$	$\{a_1\}$	$\{0\}$	$\{0\}$	$\{a_1\}$	$\{a_1\}$	$\{0\}$	$\{0\}$	$\{a_1\}$	$\{a_1\}$	$\{0\}$	$\{a_1\}$
a_2	$\{0\}$	$\{0\}$	$\{a_2\}$	$\{0\}$	$\{a_2\}$	$\{0\}$	$\{a_2\}$	$\{0\}$	$\{a_2\}$	$\{0\}$	$\{a_2\}$	$\{a_2\}$
a_3	$\{0\}$	$\{0\}$	$\{0\}$	$\{a_3\}$	$\{0\}$	$\{a_3\}$	$\{a_3\}$	$\{a_3\}$	$\{a_3\}$	$\{a_3\}$	$\{a_3\}$	$\{a_3\}$
a_4	$\{0\}$	$\{a_1\}$	$\{a_2\}$	$\{0\}$	$\{a_4\}$	$\{a_1\}$	$\{a_2\}$	$\{0\}$	$\{a_4\}$	$\{a_1\}$	$\{a_2\}$	$\{a_4\}$
a_5	$\{0\}$	$\{a_1\}$	$\{0\}$	$\{a_3\}$	$\{a_1\}$	$\{a_5\}$	$\{a_3\}$	$\{a_3\}$	$\{a_5\}$	$\{a_5\}$	$\{a_3\}$	$\{a_5\}$
a_6	$\{0\}$	$\{0\}$	$\{a_2\}$	$\{a_3\}$	$\{a_2\}$	$\{a_3\}$	$\{a_6\}$	$\{a_3\}$	$\{a_6\}$	$\{a_3\}$	$\{a_6\}$	$\{a_6\}$
a_7	$\{0\}$	$\{0\}$	$\{0\}$	$\{a_3\}$	$\{0\}$	$\{a_3\}$	$\{a_3\}$	$\{a_3, a_7\}$	$\{a_3\}$	$\{a_3, a_7\}$	$\{a_3, a_7\}$	$\{a_3, a_7\}$
a_8	$\{0\}$	$\{a_1\}$	$\{a_2\}$	$\{a_3\}$	$\{a_4\}$	$\{a_5\}$	$\{a_6\}$	$\{a_3\}$	$\{a_8\}$	$\{a_5\}$	$\{a_6\}$	$\{a_8\}$
a_9	$\{0\}$	$\{a_1\}$	$\{0\}$	$\{a_3\}$	$\{a_1\}$	$\{a_5\}$	$\{a_3\}$	$\{a_3, a_7\}$	$\{a_5\}$	$\{a_5, a_9\}$	$\{a_3, a_7\}$	$\{a_5, a_9\}$
a_{10}	$\{0\}$	$\{0\}$	$\{a_2\}$	$\{a_3\}$	$\{a_2\}$	$\{a_3\}$	$\{a_6\}$	$\{a_3, a_7\}$	$\{a_6\}$	$\{a_3, a_7\}$	$\{a_6, a_{10}\}$	$\{a_6, a_{10}\}$
1	$\{0\}$	$\{a_1\}$	$\{a_2\}$	$\{a_3\}$	$\{a_4\}$	$\{a_5\}$	$\{a_6\}$	$\{a_3, a_7\}$	$\{a_8\}$	$\{a_5, a_9\}$	$\{a_6, a_{10}\}$	$\{a_8, 1\}$

$\vee$	0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	1
0	0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	1
a_1	a_1	a_1	a_4	a_5	a_4	a_5	a_8	a_9	a_8	a_9	1	1
a_2	a_2	a_4	a_2	a_6	a_4	a_8	a_6	a_{10}	a_8	1	a_{10}	1
a_3	a_3	a_5	a_6	a_3	a_8	a_5	a_6	a_7	a_8	a_9	a_{10}	1
a_4	a_4	a_4	a_4	a_8	a_4	a_8	a_8	1	a_8	1	1	1
a_5	a_5	a_5	a_8	a_5	a_8	a_5	a_8	a_9	a_8	a_9	1	1
a_6	a_6	a_8	a_6	a_6	a_8	a_8	a_6	a_{10}	a_8	1	a_{10}	1
a_7	a_7	a_9	a_{10}	a_7	1	a_9	a_{10}	a_7	1	a_9	a_{10}	1
a_8	a_8	a_8	a_8	a_8	a_8	a_8	a_8	1	a_8	1	1	1
a_9	a_9	a_9	1	a_9	1	a_9	1	a_9	1	a_9	1	1
a_{10}	a_{10}	1	a_{10}	a_{10}	1	1	a_{10}	a_{10}	1	1	a_{10}	1
1	1	1	1	1	1	1	1	1	1	1	1	1

Then $\mathbb{L}$ is a meet-hyperlattice (see Figure 7) and $I = \{0, a_2, a_3, a_6\}$ is a hyperideal of $\mathbb{L}$ and the zero-divisor graph of $\mathbb{L}$ with respect to the hyperideal I is given in Figure 8.

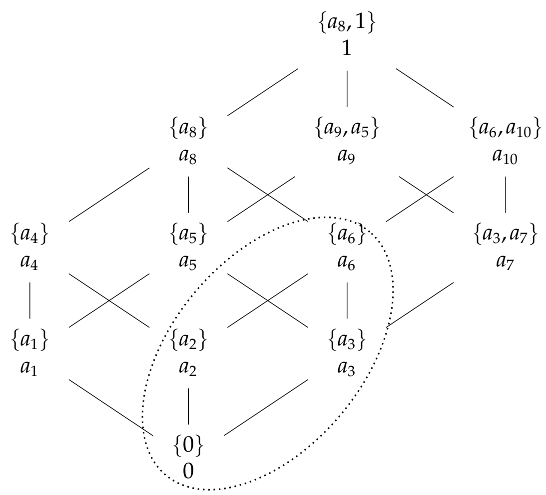

Figure 7: Hyperideal $\{0, a_2, a_3, a_6\}$ of $\mathbb{L}$ in dotted lines.

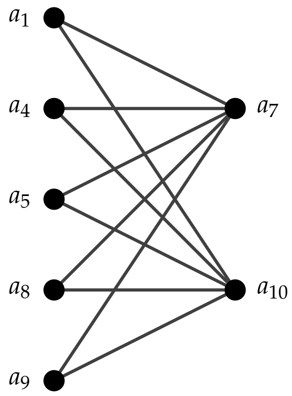

Figure 8: Zero-divisor graph with respect to the hyperideal $I = \{0, a_2, a_3, a_6\}$ is isomorphic to $K_{2,5}$.

Definition 3.11. $J \in I(\mathbb{L})$ is called semiprime, if $(l_1 \bigwedge l_2) \cap J \neq \emptyset$ and $(l_1 \bigwedge l_3) \cap J \neq \emptyset$ implies $(l_1 \bigwedge (l_2 \vee l_3)) \cap J \neq \emptyset$ for all $l_1, l_2, l_3 \in \mathbb{L}$.

Example 3.12. In Example 3.8, the hyperideal $J = \{0, a_1\}$ is a semiprime hyperideal. But J is not a prime hyperideal as $(1 \bigwedge 1) \cap J \neq \emptyset$, but $1 \notin J$.

Proposition 3.13. *Let $J \in I(\mathbb{L})$ be semiprime. Suppose $G^J(\mathbb{L})$ is a complete bipartite graph with J_1 and J_2 as partitions. If for $i = 1, 2$, $P_i = J \cup J_i$ are semi hyperideals, then $P_i = J \cup J_i \in I(\mathbb{L})$.*

Proof. Let $l_1, l_2 \in P_1$. If $l_1, l_2 \in J$, then clearly $l_1 \vee l_2 \in J \subseteq P_1$. If $l_1, l_2 \in J_1$, then for any $l' \in J_2$, we have $(l_i \bigwedge l') \cap J \neq \emptyset, i = 1, 2$. Since J is semiprime, we get $((l_1 \vee l_2) \bigwedge l') \cap J \neq \emptyset$. As $l_1 \vee l_2 \notin J$, we must have $l_1 \vee l_2 \in J_1 \subseteq P_1$. Suppose $l_1 \in J$ and $l_2 \in J_1$. Then for any $l' \in J_2$, $(l_2 \bigwedge l') \cap J \neq \emptyset$. Also $(l_1 \bigwedge l') \cap J \neq \emptyset$. Since J is semiprime, we get $((l_1 \vee l_2) \bigwedge l') \cap J \neq \emptyset$, and so $l_1 \vee l_2 \in P_1$. ■

Lower bound property (l.b. property): ([15]). We say that a strong meet-hyperlattice $\mathbb{L}$ satisfies l. b. property, if for all $x, y \in \mathbb{L}$, there exists $u \in x \bigwedge y$ with $u \leq x$ and $u \leq y$.
For the following result, we assume that $\mathbb{L}$ satisfies l.b. property.

Theorem 3.14. *Let $P_1, P_2 \in I(\mathbb{L})$ such that P_1 and P_2 are distinct Primes with $P_1 \setminus P_2 \neq \emptyset$ and $P_2 \setminus P_1 \neq \emptyset$. Let also $J = P_1 \cap P_2$. Then $G^J(\mathbb{L}) \simeq K_{|P_1 \setminus P_2|, |P_2 \setminus P_1|}$.*

Proof. Suppose that $l_1 l_2$ is an edge. Then $l_1, l_2 \in \mathcal{Z}(J)$ and $(l_1 \bigwedge l_2) \cap J \neq \emptyset$. Take $P_1 \setminus P_2 = X_1$, and $P_2 \setminus P_1 = X_2$. Since P_1 is prime, we get $l_1 \in P_1$ or $l_2 \in P_1$. Assume that $l_1 \in P_1$. Then as $l_1 \notin J$, $l_1 \in X_1$, and since P_2 is prime, we must have $l_2 \in P_2$. Further, as $l_2 \notin J$, we get $l_2 \in X_2$. Hence $\mathcal{Z}(I) = X_1 \cup X_2$. Now for any $l'_1 \in X_1$ and $l'_2 \in X_2$, we have $l'_i \in P_i (i = 1, 2)$. As $\mathbb{L}$ satisfies l.b. property, there exists $x \in l'_1 \bigwedge l'_2$ such that $x \leq l'_1, l'_2$, and so $(l'_1 \bigwedge l'_2) \cap J \neq \emptyset$. Hence $l'_1 l'_2$ is an edge, showing that $G^J(\mathbb{L})$ is a complete bipartite graph. ■

Example 3.15. Let $\mathbb{L} = \{1, 2, 3, 5, 6, 10, 15, 30\}$, (the positive divisors of 30) and let the hyperoperation and the binary operation be defined as follows:

$\sqcap$	1	2	3	5	6	10	15	30
0	$\mathbb{L}$	$\{1,5\}$	$\{1,2\}$	$\{1,3\}$	$\{1,5\}$	$\{1,3\}$	$\{1,2\}$	$\{1\}$
2	$\{1,5\}$	$\{2,6,10,30\}$	$\{1,5\}$	$\{1,3\}$	$\{2,10\}$	$\{6,2\}$	$\{1\}$	$\{2\}$
3	$\{1,2\}$	$\{1,5\}$	$\{3,6,15,30\}$	$\{1,2\}$	$\{15,3\}$	$\{1\}$	$\{6,3\}$	$\{3\}$
5	$\{1,3\}$	$\{1,3\}$	$\{1,2\}$	$\{5,10,15,30\}$	$\{1\}$	$\{15,5\}$	$\{10,5\}$	$\{5\}$
6	$\{1,5\}$	$\{2,10\}$	$\{15,3\}$	$\{1\}$	$\{6\}$	$\{2\}$	$\{3\}$	$\{6\}$
10	$\{1,3\}$	$\{6,2\}$	$\{1\}$	$\{15,5\}$	$\{2,10\}$	$\{10\}$	$\{5\}$	$\{10\}$
15	$\{1,2\}$	$\{1\}$	$\{6,3\}$	$\{10,5\}$	$\{3\}$	$\{5\}$	$\{15\}$	$\{15\}$
30	$\{1\}$	$\{2\}$	$\{3\}$	$\{5\}$	$\{6\}$	$\{10\}$	$\{15\}$	$\{30\}$

$\vee$	1	2	3	5	6	10	15	30
1	1	2	3	5	6	10	15	30
2	a	a	6	10	6	10	30	30
3	3	6	3	15	6	30	15	30
5	5	10	15	5	30	10	15	30
6	6	6	6	30	6	30	30	30
10	10	10	30	10	30	10	30	30
15	15	30	15	15	30	30	15	30
30	30	30	30	30	30	30	30	30

Then $(\mathbb{L}, \sqcap, \vee)$ is a Nakano hyperlattice. The zero-divisor graph of $\mathbb{L}$ with respect to the hyperideals $I = \{1\}$ and $I = \{1, 2\}$ are shown in Figure 9 and Figure 10, respectively.

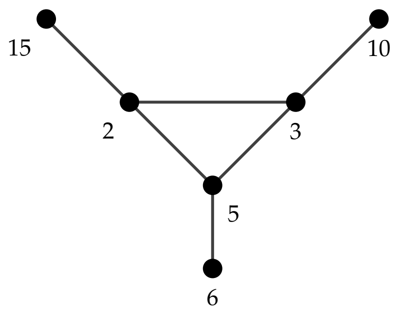

Figure 9: Zero-divisor graph of $\mathbb{L}$ with respect to the hyperideal $I = \{1\}$.

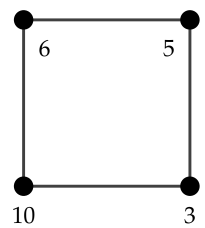

Figure 10: Zero-divisor graph of $\mathbb{L}$ with respect to the hyperideal $I = \{1, 2\}$.

The following Proposition concerns non-primal semiprime hyperideals, which is useful for further studies related to the diameter of zero-divisor graphs.

Proposition 3.16. *Let $J \neq \mathbb{L}$ be a non-primal semiprime hyperideal of $\mathbb{L}$ such that $\mathcal{S}(J)$ is a semi hyperideal. Then there exist $l_1, l_2 \in \mathcal{Z}(J)$ such that $(J : l_1 \vee l_2) = J$.*

Proof. Suppose that $\mathcal{S}(J) \notin I(\mathbb{L})$. Then there exist $l_1, l_2 \in \mathcal{S}(J)$ such that $l_1 \vee l_2 \notin \mathcal{S}(J)$. This means $(J : l_1 \vee l_2) = J$. It remains to show that $l_1, l_2 \notin J$. Clearly, atmost one among l_1 and l_2 belongs to J. Without loss of generality we may assume that $l_1 \in J$ and $l_2 \notin J$. Then $l_2 \in \mathcal{Z}(J)$, which implies there exists $c \in \mathcal{Z}(J)$ such that $(l_2 \bigwedge c) \cap J \neq \emptyset$. By Lemma 2.5, there exists $t \in l_1 \bigwedge c$ such that $t \leq l_2$. As J is a hyperideal, $t \in J$, and so $(l_2 \bigwedge c) \cap J \neq \emptyset$. Now since J is semiprime, it follows that $((l_1 \vee l_2) \bigwedge c) \cap J \neq \emptyset$ and so $c \in (J : l_1 \vee l_2) = J$, a contradiction to $c \in \mathcal{Z}(J)$. Therefore, $l_1, l_2 \notin J$, and hence $l_1, l_2 \in \mathcal{Z}(J)$. ∎

Table 1: Interaction between the oxygen and the hydrogen molecules with stimuli.

$\bigwedge$	H_2	O_2	H_2O	H_2O_2
H_2	$\{H_2\}$	$\mathbb{H}$	$\{H_2, H_2O\}$	$\{H_2, H_2O, H_2, O_2\}$
O_2	$\mathbb{H}$	$\{O_2\}$	$\mathbb{H}$	$\{O_2, H_2O, H_2O_2\}$
H_2O	$\{H_2, H_2O\}$	$\mathbb{H}$	$\{H_2O\}$	$\mathbb{H}$
H_2O_2	$\mathbb{H}$	$\{O_2, H_2O, H_2O_2\}$	$\mathbb{H}$	$\{H_2O, H_2O_2\}$

4 Examples of meet-hyperlattice of chemical compounds

Example 4.1. Let the set $\mathbb{H}$ denote $\{H_2, O_2, H_2O, H_2O_2\}$, representing the dissolution of hydrogen peroxide in water. Define the hyperoperation $\bigwedge$ as the interaction between molecules of hydrogen and oxygen in $\mathbb{H}$ with stimuli (see Table 1) and the binary operation $\vee$ is the interaction of hydrogen and oxygen molecules in $\mathbb{H}$ with and without external stimuli (see Table 2). Then, $(\mathbb{H}, \bigwedge, \vee)$ is a meet-hyperlattice. The possible proper hyperideals of $\mathbb{H}$ are $\{H_2\}, \{O_2\}$, and $\{H_2O, H_2O_2\}$. The zero-divisor graph of $\mathbb{H}$ with respect to $\{H_2\}$ and $\{O_2\}$ are isomorphic, is given in Figure 11, whereas the zero-divisor graph of $\mathbb{H}$ with respect to $\{H_2O, H_2O_2\}$ is an isolated vertex.

Table 2: Interaction between the oxygen and the hydrogen molecules without stimuli.

$\vee$	H_2	O_2	H_2O	H_2O_2
H_2	H_2	H_2O	H_2O	H_2O_2
O_2	H_2O	O_2	H_2O	H_2O_2
H_2O	H_2O	H_2O	H_2O	H_2O_2
H_2O_2	H_2O_2	H_2O_2	H_2O_2	H_2O_2

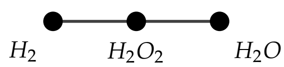

Figure 11: Zero-divisor graph with respect to the hyperideal $J = \{H_2\}$.

Example 4.2. Consider the set $\mathbb{O} = \{O_2, O_3\}$. Define binary operation $\vee$ as the interaction of oxygen in ozone without external stimuli, as given in Table 3 and the hyperoperation $\bigwedge$ as the interaction of oxygen in ozone with any external stimuli, as given in Table 4. Then $\mathbb{O} = \{O_2, O_3\}$, $(\mathbb{O}, \bigwedge, \vee)$ is a meet-hyperlattice. The zero-divisor graph with respect to the hyperideal $J = \{O_2\}$ will be a empty graph (i.e. $\mathcal{Z}(J) = \emptyset$).

Table 3: Interaction between oxygen molecules in ozone without stimuli.

$\vee$	O_2	O_3
O_2	O_2	O_3
O_3	O_3	O_3

Table 4: Interaction between oxygen molecules in ozone with stimuli.

$\bigwedge$	O_2	O_3
O_2	$\{O_2\}$	$\{O_2, O_3\}$
O_3	$\{O_2, O_3\}$	$\{O_3\}$

5 Conclusion

In this work, we have considered the concept of meet-hyperlattices and their zero-divisor graph with respect to hyperideals. As a future scope, one can study hyperlattices from the corresponding zero divisor graphs with respect to hyperideals. The notion of energy of a graph [21, 22] can be extended to zero-divisor graphs of hyperlattices with respect to hyperideals. As an application, we have provided examples of chemical reactions of compounds that lead to a meet-hyperlattice. As a future scope, using Proposition 3.16, we wish to establish the following conclusion:

- For a proper non primal semi-prime hyperideal I of $\mathbb{L}$, which is contained in more than two minimal prime hyperideals, $diam(G^I(\mathbb{L}))$ is equal to 3 and it is equal to 2 if I is contained in exactly two minimal prime hyperideals.

Conflicts of interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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