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Research Paper

Maximum Modified Sombor Index of Unicyclic Graphs with Given Girth

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Abstract

For any graph G, the modified Sombor index is defined as the reciprocal of the well-known Sombor Index. The girth of G, by short $g(G)$, is the length of the smallest cycle in G. A graph with exactly one cycle is a unicyclic graph. If it is further, connected, it is a connected unicyclic graph. In this article, we achieved the modified Sombor index for a collection of graphs, connected unicyclic graphs and also obtained the maximum modified Sombor index for the class of unicyclic graphs based on the restrictions by a fixed girth.

1 Introduction

By a graph G, throughout the paper, we mean only the connected and finite graphs. In this paper, we concentrate only on the connected, finite graphs. A graph G is an ordered pair (V_G, E_G), and the vertices and edges of a graph are the members of the sets V_G and E_G respectively and the set of vertices that are adjacent to a vertex u in G is denoted as $N_G(u)$ and is called the open-neighbourhood of u in G. The closed neighbourhood is $N_G[v] = N_G(v) \cup \{v\}$. A (x, y)-Path $xux_1x_2 \dots y$ is a sequence of distinct members of the set V_G, and the vertices x, y are respectively known as the origin and terminus of the path. The concept of distance between two random vertices $x, y \in V_G$ is defined as the length of the shortest (x, y)-path in G. If $d_G(x) = 1$, then the vertex x is a pendent vertex and its only adjacent vertex in G, say y is a support vertex in G. The graph denoted by $G - w$ is the graph resulted after removal of a vertex $w \in V_G$ and all its incident edges. A connected graph with exactly one cycle is a unicyclic graph. For more on graphs and related works, the reader is referred to [1, 2].

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The Sombor (SO) index for a graph G is defined by Gutman [3] as:

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_G(u)^2 + d_G(v)^2}.$$

A lot of work and modifications on the Sombor (SO) index have been done and received a numerous attention from researchers in the past in the form of surveys [4, 5] and research articles [3, 4, 6–12]. For the applications in the chemical domain of this index, one may refer to the papers [13, 14].
The modified Sombor index is defined as:

$$SO^m(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_G(u)^2 + d_G(v)^2}}.$$

The modified Sombor index is a recently introduced term and some of the works can be found in [15–18].

The modified Sombor index, for a wide collection of graphs, is studied in this article and provides a minimum modified Sombor index for collections of graphs. As the maximum modified Sombor index is not characterized or proposed for the class of unicyclic graphs in the literature so far, hence we characterized the unicyclic graphs with the maximum $SO^m(G)$.

2 Results

The main results on the maximum and minimum modified Sombor index are studied for a variety of classes of graphs extensively. If the order n of G is one, there is no edge in the graph, hence by graph G, throughout this article, we mean a graph with a minimum of two vertices. Also, by graph G, we mean only a connected graph, unless explicitly stated.

2.1 The minimum values of the modified Sombor index of graphs

The results on the minimum modified Sombor index of a variety of classes of graphs are provided here. The results are more generic and pertaining to a clear understanding of the behaviour of the modified Sombor index on different graph structures. Using the behavioural analysis of $SO^m(G)$ on this collection of graphs, further, in the next section, a characterization of the maximum $SO^m(G)$ of unicyclic graphs is provided. The proofs of the results provided in this section are omitted as they are straightforward to see.

Proposition 2.1. *For a graph G, $SO^m(G) \geq \frac{1}{\sqrt{2}}$.*

Proposition 2.2. *For a graph G with k pendents in V_G, $SO^m(G) \geq \frac{k}{\sqrt{2}}$.*

Proposition 2.3. *If G has a path on n vertices, then, $SO^m(G) = \frac{2}{\sqrt{5}} + \frac{n-3}{\sqrt{8}}$ with equality if and only if every vertex of V_G lies in the unique diametrical path of G.*

Proposition 2.4. *If G is a star on n vertices, then, $SO^m(G) = \frac{n-1}{\sqrt{n^2-2n+2}}$.*

Proposition 2.5. *If G is a graph with at least one cycle, then, $SO^m(G) \geq \frac{3}{\sqrt{8}}$.*

Proposition 2.6. *Let the graph G have a cycle of length k. Then, $SO^m(G) \geq \frac{k}{\sqrt{8}}$ with equality if and only if every vertex of G lies in the unique cycle of cycle.*

Proposition 2.7. *If G is a complete graph, then, $SO^m(G) = \frac{n(n-1)}{2\sqrt{8}}$.*

Proposition 2.8. *If G is a complete bipartite graph with a bipartition (X, Y), with $|X| = m$ and $|Y| = n$, then $SO^m(G) = \frac{mn}{\sqrt{m^2+n^2}}$.*

2.2 The maximum values of the modified Sombor index of graphs

A characterization of graphs with the maximum values of $SO^m(G)$ on the class of unicyclic graphs is provided in this section. First, without loss of generality, let the order n of G be at least 3, unless the graph contains no cycle. For any integer n, assuming it is the order of G, we found the unicyclic graphs of order n with maximum $SO^m(G)$. This characterization provides a unique unicyclic graph for any positive integer $n \geq 3$ with maximum $SO^m(G)$.

Lemma 2.9. *Let u be a vertex of a unicyclic graph G such that u lies in the cycle with pendent neighbours u' and v', and v be a non-pendent neighbour of u in the cycle of G. Let H be a graph constructed from G by deleting the edge uv' and attaching it to the cycle by introducing the two new edges uv' and $v'v$. Then, $SO^m(G) < SO^m(H)$.*

Proof. Let u be a vertex of G such that u lies in the cycle of G and u', v' are the pendent vertices adjacent to u. Let v be a neighbour of u in the cycle of G. Note that $d_G(u) = 4$ and $d_G(v) = 2$. Now,

$$\begin{aligned}
SO^m(H) &= SO^m(G) - \frac{1}{\sqrt{d_G(u)^2 + d_G(v')^2}} + \frac{1}{\sqrt{d_H(v)^2 + d_H(v')^2}} + \frac{1}{\sqrt{d_H(u)^2 + d_H(v')^2}} \\
&= SO^m(G) - \frac{1}{\sqrt{d_G(u)^2 + 1}} + \frac{1}{\sqrt{d(v)^2 + 2^2}} + \frac{1}{\sqrt{d_H(u)^2 + 2^2}} \\
&= SO^m(G) - \frac{1}{\sqrt{4^2 + 1}} + \frac{1}{\sqrt{2^2 + 2^2}} + \frac{1}{\sqrt{3^2 + 2^2}} \\
&= SO^m(G) - \frac{1}{\sqrt{16}} + \frac{1}{\sqrt{8}} + \frac{1}{\sqrt{13}} \\
&= SO^m(G) + k, \qquad \text{say} \\
&> SO^m(G), \qquad \text{since } k > 0.
\end{aligned}$$

Hence, it is proved. ■

Lemma 2.10. *Let G be a unicyclic graph with $u_1, u_2, \dots u_k \in V_G$ be the vertices lie on the cycle with pendent neighbours $u'_1, u'_2, \dots u'_k$. Let H' be the graph constructed from G by deleting the edges $u_iu'_i$ for $i \neq 1$ and attaching them to the cycle. Then, $SO^m(G) < SO^m(H)$.*

Proof. The proof is the repeated application of Lemma 2.9 to the graph G. ■

Note: The graphs H and H' in Lemmas 2.9 and 2.10 are of the form $C_{1,n-1}$.

Lemma 2.11. *Let $G = C^*_{n,k}$ be a unicyclic graph such that a vertex v in the cycle with $d_G(v) \geq 3$ and u be the vertex in the cycle of G such that $d_G(u) \geq 2$. Let H' be the graph constructed from G by deleting the vertex u and the edges incident with it, and attaching it to the end-vertex of the path adjacent to v. Then, $SO^m(G) < SO^m(H)$.*

Proof. Let $u, v \in V_G$ be any two vertices in the cycle of G with $d_G(u) \geq 2$ and $d_G(v) \geq 3$. Let $P : v_1, v_2, \dots, v_l$ (where $l \geq 2$) be the path adjacent to v in G. Let uu' and uu'' be the edges removed from G to form H and let H be the graph defined as in the hypothesis. Let the edge

$v_{l-1}v_l$ be the edge removed from H and the edges $u'u''$ and $u'v_l$ be the newly added edges in H. Now,

$$\begin{aligned}
SO^m(H) &= SO^m(G) - \frac{1}{\sqrt{d_G(u)^2 + d_G(u')^2}} - \frac{1}{\sqrt{d_G(u)^2 + d_G(u'')^2}} \\
&\quad + \frac{1}{\sqrt{d_H(u')^2 + d_H(u'')^2}} + \frac{1}{\sqrt{d_H(u)^2 + d_H(v_l)^2}} \\
&= SO^m(G) - \frac{2}{\sqrt{2^2 + 2^2}} + \frac{1}{d_H(u')^2 + d_H(u'')^2} + \\
&\quad \frac{1}{d_H(v_{l-1})^2 + d_H(u)^2} + \frac{1}{\sqrt{d_H(u)^2 + d_H(v_l)^2}} \\
&= SO^m(G) - \frac{2}{\sqrt{8}} + \frac{2}{\sqrt{8}} = SO^m(G).
\end{aligned}$$

Hence, it is proved. ■

The following theorem is a consequence of the lemmas we proved. For any positive integer n, this theorem provides a unique graph (up to isomorphism) of order n with maximum $SO^m(G)$ and hence holds the validity of the existence of the result we provided for unicyclic graphs.

Theorem 2.12. *Given a positive integer n, the graph G is a unicyclic graph of order n and $SO^m(G) \leq \frac{n}{\sqrt{8}}$ with equality if and only if the girth $g(G) = n$.*

Proof. If $g(G) = n$, then G is a cycle on n vertices and hence a unicyclic graph and we have $SO^m(G) = \frac{n}{\sqrt{8}}$. Let us prove the converse of the theorem on induction on the number of vertices n of G. First, if $n = 3$, then $G = C_3$ which is the only unicyclic graph on 3 vertices and $SO^m(G) \leq \frac{3}{\sqrt{8}}$, and hence it is true. If $n = 4$, then either $g = 3$ or $g = 4$. If $g = 3$, then $G = C_1, n-1$ and $SO^m(G) = \frac{1}{\sqrt{8}} + \frac{3}{\sqrt{13}}$. If $g = 4$, then $SO^m(G) = \frac{4}{\sqrt{8}}$. By simple check, G has maximum modified Sombor index if $G = C_4$, that is when $g = 4$. Thus, the induction is true for base cases $n = 3, 4$.

Let us assume that the induction is true for any unicyclic graph with its order not more than $n-1$. Now, assume G is a random unicyclic graph on n-vertices.
Case-1: $g < n$.

Let G be of the form $C_{g,n-g}$. Then, there are g vertices on the cycle of G and the remaining $n-g$ vertices are adjacent to either vertices on the cycle or adjacent to the vertices adjacent to the vertices on the cycles. Let v be a random vertex not on the cycle of G.

Subcase-1: Let v be adjacent to a vertex lying on the cycle of G.
Claim: There exists a unicyclic graph G' of order n such that $SO^m(G) < SO^m(G')$. Let u be a vertex on the cycle of G to which v is adjacent in G. Then, as in Lemma 2.9, the vertex v and the edge uv may be deleted from G and we can construct a graph G' by making this vertex v adjacent to a vertex on the cycle of G. By Lemma 2.9, we have $SO^m(G) < SO^m(G')$.

Subcase-2: Let v be adjacent to a vertex that is not on the cycle of G.
Claim: There exists a unicyclic graph G'' of order n such that $SO^m(G) < SO^m(G'')$. Let v be adjacent to a vertex u' where u' is adjacent to a vertex u lying on the cycle of G. Then, as in Lemma 2.11, the vertex v and the edge $u'v$ may be deleted from G and we can construct a graph G'' by making this vertex v to a vertex on the cycle of G. By Lemma 2.9, we have $SO^m(G) < SO^m(G')$.

By repeatedly applying the Lemmas 2.9 to 2.11 for vertices not lying on the cycle of G, we construct C_n, the cycle of order in a finite number of transformations. Thus, the unicyclic graphs with maximum $SO^m(G)$ are nothing but the cycles of order n. In other words, the graphs with $g(G) = n$. Hence, the theorem is proved. ■

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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