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Research Paper

Retrieving the Transient Temperature Field and Blood Perfusion Coefficient in the Pennes Bioheat Equation Subject to Nonlocal and Convective Boundary Conditions

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Abstract

In this paper, we delve into a coefficient inverse problem linked to the bioheat equation, a pivotal component in medical research concerning phenomena such as temperature response and blood perfusion during surface heating. By considering factors like heat transfer between tissue and blood in capillaries and incorporating the geometric intricacies of the skin, we confine our analysis to a one-dimensional domain. Our approach involves transforming the original problem into one concerning the reconstruction of a multiplicative source term within a parabolic equation. Subsequently, we utilize integral conditions to derive a specific integro-differential equation, accompanied by the requisite initial and boundary conditions. Leveraging a spectral method, we streamline the modified problem into a linear system of algebraic equations. To accomplish this, we employ appropriate regularization algorithms to obtain stable approximations for the derivatives of perturbed boundary data and to effectively solve the resultant system of equations.

1 Introduction

The Pennes bioheat transfer equation is a mathematical model to describe the biological heat transfer of living tissues that concentrates on blood perfusion along the vascular system and metabolic heat generation [1–3]. Assuming $w(\mathbf{x},t)$ as the temperature of a rectangular perfused living tissue D, the Pennes bioheat equation is governed by:

$$\rho_t\gamma_t w_t(\mathbf{x},t) - K\Delta w(\mathbf{x},t) + \beta_p\gamma_b(w(\mathbf{x},t)-w^*) = s(\mathbf{x},t), \quad (\mathbf{x},t)\in D\times(0,T], \tag{1}$$

such that the given parameters β_p γ_b, w^* during thermal treatments stand for the perfusion rate of blood, specific heat of blood, and supplying arterial blood temperature, respectively.

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Moreover, the features corresponding to the tissue such as density, thermal conductivity, and the specific heat of tissue are described by $\rho_t,\ \gamma_t,\ K$, respectively and it is assumed that the whole process is influenced by the external heat source $s(\mathbf{x},t)$. Supposing that the heat transfer occurs between tissue and blood in capillaries and taking the geometrical features of the skin, one can assume that the heat exchanges in a one-dimensional domain [4, 5]. Associated with this fact and considering Equation (1), we aim to approximate the unknown functions $b(t)$ and $w(x,t)$ such that the pair $\{b(t), w(x,t)\}$ satisfies the following equation:

$$w_t(x,t) - w_{xx}(x,t) + b(t)w(x,t) = s(x,t), \quad (x,t) \in \Omega_T, \tag{2}$$

supplied with the initial condition

$$w(x,0) = w_0(x), \quad 0 < x < 1, \tag{3}$$

boundary conditions

$$w(0,t) - w(1,t) = 0, \quad w_x(0,t) + \mu w(0,t) = 0, \quad 0 < t \leq T, \tag{4}$$

and subject to the additional condition in the integral form

$$\int_0^1 w(x,t)dx = E(t), \quad t \in (0,T]. \tag{5}$$

The bounded domain $\Omega_T = (0,1) \times (0,T]$ in $\mathbb{R}^2$, the nonzero number μ and the sufficiently smooth functions $s(x,t)$, and $E(t)$ are given. In addition, some restrictions for the input data are assumed as follows:

$$w_0(0) = w_0(1),\ w_0'(0) + \mu w_0(0) = 0, \quad \int_0^1 w_0(x)dx = E(0), \quad E(t) \neq 0,\ t \in\ (0,T]. \tag{6}$$

When the coefficient term $b(t)$ is given, the problem of finding the function $w(x,t)$ satisfying the system of Equations (2)-(4) is referred to as the forward problem. In contrast to the forward case, the inverse problem consists of recovering the unknown functions $b(t)$ and $w(x,t)$ through Equations (2)-(5) and it belongs to the class of second-order parabolic partial differential equations (PDEs) with nonlocal integral boundary conditions. As a specific feature of problems with nonlocal boundary conditions, it is known that in such problems the corresponding spatial differential operator is nonself-adjoint [6]. The proper way of applying these conditions in numerical methods is also important because classical methods have difficulty in dealing with them. The first part of condition (4) which is referred to as the nonlocal periodic boundary condition expresses that the temperature is kept the same at two tissue extremities (walls) $x = 0$ and $x = 1$ whilst the convective heat transfer stated by equation $\omega_x(0,t) + \mu\omega(0,t) = 0$ simulates the transfer of heat between the boundary $x = 0$ of the tissue and the environment (blood in capillaries) [1].

As highlighted in previous literature [1, 2, 7–15], finding the control parameter [16] $b(t)$ included in the governing equation is important in many branches of science and engineering such as control theory, biochemistry and biological processes, plasma physics, population dynamics, and medical sciences. To mention a few, Equation (2) can be utilized to describe a heat transfer process with a free heat source $s(x,t)$ and unknown source parameter $b(t)$. More accurately, if we represent the temperature distribution by $w(x,t)$, it is possible to design the weighted thermal energy $E(t)$ contained in region $0 < x < 1$, by controlling the heat source $b(t)$. Furthermore, finding the control parameter $b(t)$ has applications in medical sciences. In this direction, considering the term $w(x,t)$ as the temperature of a perfused living tissue and $s(x,t)$ as the volumetric rate of external irradiation heat, we impose the additional condition (5) to obtain the blood perfusion coefficient $b(t)$ uniquely where the initial and boundary conditions are in accordance with Equations (3)-(4).

The inverse problems of finding the lower-order coefficient in a parabolic PDE have been studied in several papers. In [7–10, 14, 15, 17, 18], the authors studied the numerical solution of the parabolic PDEs for computing the coefficient $b(t)$ using several techniques such as the finite difference methods (FDM), the radial basis functions (RBF) methods, the dual reciprocity boundary element method (DRBEM), the Adomain decomposition method (ADM) and the Bernstein-Galerkin technique. The standard initial Dirichlet boundary conditions are associated with the studied problems and extra conditions are supposed to be either the integral condition similar to (5) or the temperature distribution given in a local point of the space. In [1, 2, 11–13], the authors studied the conditions for uniquely recovering the coefficient $b(t)$ from different boundary conditions such as the combination of the integral overdetermination condition (5) with the Iokin boundary condition and Wentzell boundary condition and proposed some numerical algorithms based on FDM to solve the inverse problems. In [19, 20], theoretical results regarding the solvability of a coefficient inverse problem governed by Equation (2) were presented. In [21, 22], the authors applied the method of fundamental solutions (MFS) and the boundary element method (BEM) to recover the perfusion coefficient $b(t)$. A combination of the MFS and the RBF method was proposed to analyze the thermal behavior of skin tissues by [23].

In this paper, we explore the utilization of a novel spectral technique that demonstrates convergence, ease of implementation, and proficiency in handling nonstandard boundary conditions outlined in Equations (2)-(5). Notably, unlike the approach suggested in [2], our method obviates the need to solve the nonlinear system of algebraic equations during implementation.

The structure of this article is as follows: Section 2 introduces a computational framework for addressing the stated problem. Section 3 showcases the outcomes of numerical simulations, while Section 4 offers concluding remarks.

2 Computational scheme

By utilizing the transformation $w(x,t) = v(x,t)e^{-\int_0^t b(s)ds}$, we arrive at the modified version of the problem as follows:

$$v_t(x,t) - v_{xx}(x,t) = s(x,t)\frac{\int_0^1 v(x,t)dx}{E(t)}, \quad (x,t) \in \Omega_T, \tag{7}$$

$$v(x,0) = w_0(x),\ 0 < x < 1, \quad v(0,t) = v(1,t),\ v_x(0,t) = -\mu v(0,t),\ 0 < t \le T. \tag{8}$$

Accordingly, assuming that the conditions (6) hold, the problems (7)-(8) and (2)-(5) are equivalent. Thus, we aim to solve the system of Equations (7)-(8). It is obvious that

$$\int_0^x v_{yy}(y,t)dy = v_x(x,t) - \underbrace{v_x(0,t)}_{=-\mu v(0,t)} \Longrightarrow \int_0^x \int_0^z v_{yy}(y,t)dydz = v(x,t) - v(0,t) + x\mu v(0,t). \tag{9}$$

Setting $x = 1$ in the last argument of Equation (9) and using the condition $v(0,t) = v(1,t)$, we conclude

$$v(0,t) = \frac{1}{\mu}\int_0^1 \int_0^z v_{yy}(y,t)dydz,$$

therefore

$$v(x,t) = \int_0^x \int_0^z v_{yy}(y,t)dydz + \left(\frac{1}{\mu} - x\right) \int_0^1 \int_0^z v_{yy}(y,t)dydz. \tag{10}$$

Now, by defining the new variable $\hat{v}(x,t)$ as:

$$\hat{v}(x,t) := \int_0^x \int_0^z v_{yy}(y,t)dydz + \left(\frac{1}{\mu} - x\right) \int_0^1 \int_0^z v_{yy}(y,t)dydz, \tag{11}$$

it can be found that

$$\hat{v}(0,t) = \hat{v}(1,t), \;\; \hat{v}_x(0,t) + \mu\hat{v}(1,t) = 0. \tag{12}$$

Next, we define

$$v(x,t) := \hat{v}(x,t) + w_0(x) - \hat{v}(x,0), \tag{13}$$

which in view of (6), we can be assured that the conditions (8) are imposed. Following, we take (13) into account and construct an approximation of $v(x,t)$ which approximately satisfies the Equation (7). In this direction, we recall the orthonormal Bernstein basis functions (OBBFs) [24]

$$\{\phi_i(x)\}_{i=0}^{\infty}, \quad x \in [0,1], \qquad \{\psi_i(t)\}_{i=0}^{\infty}, \quad t \in [0,T],$$

and let $\Phi(x)$ and $\Psi(t)$ be vectors given as:

$$\Phi^\top(x) = [\phi_0(x), \phi_1(x), ..., \phi_M(x)], \;\; \Psi^\top(t) = [\psi_0(t), \psi_1(t), ..., \psi_N(t)].$$

We consider the estimation of $\hat{v}_{xx}(x,t)$ as:

$$\hat{v}_{xx}(x,t) \simeq \Phi^\top(x) C \Psi(t), \tag{14}$$

such that the unknown matrix C includes the elements c_{ij} as follows:

$$C = \begin{pmatrix} c_{00} & \cdots & c_{0N} \\ \vdots & & \vdots \\ c_{M0} & \cdots & c_{MN} \end{pmatrix}. \tag{15}$$

By defining

$$\Phi_\#(x) := \int_0^x \int_0^z \Phi(y)dydz = \left[\int_0^x \int_0^z \phi_0(y)dydz, ..., \int_0^x \int_0^z \phi_M(y)dydz\right]^\top,$$

and utilizing Equation (11) we get

$$\hat{v}(x,t) \simeq \left(\Phi_\#^\top(x) + \left(\frac{1}{\mu} - x\right)\Phi_\#^\top(1)\right) C\Psi(t). \tag{16}$$

According to Equation (13), we construct the estimation of $v(x,t)$ as:

$$v(x,t) \simeq \overline{v(x,t)} = w_0(x) + \left(\Phi_\#^\top(x) + \left(\frac{1}{\mu} - x\right)\Phi_\#^\top(1)\right) C\Big(\Psi(t) - \Psi(0)\Big). \tag{17}$$

Moreover, the following estimation is yielded for $v_t(x,t)$

Notation $\top$ is used to represent the transpose operator

$$v_t(x,t) \simeq \overline{v_t(x,t)} = \left(\Phi_{\#}^{\top}(x) + \left(\frac{1}{\mu} - x\right)\Phi_{\#}^{\top}(1)\right)C\Psi_*(t), \tag{18}$$

where $\Psi_*(t) := [\psi_0'(t), ..., \psi_N'(t)]^{\top}$. By defining the following residual function

$$R(x,t,v) := v_t(x,t) - v_{xx}(x,t) - \frac{s(x,t)}{E(t)}\int_0^1 v(x,t)dx, \tag{19}$$

we have

$$R(x,t,\hat{v}) = \left(\Phi_{\#}^{\top}(x) + \left(\frac{1}{\mu} - x\right)\Phi_{\#}^{\top}(1)\right)C\Psi_*(t) - \Phi^{\top}(x)C\Big(\Psi(t) - \Psi(0)\Big) - w_0^{''}(x) \tag{20}$$

$$-\frac{s(x,t)}{E(t)}\left\{\int_0^1 w_0(x)dx + \left(\Phi_{\#\#}^{\top}(1) - \Phi_{\#\#}^{\top}(0) + \left(\frac{1}{\mu} - \frac{1}{2}\right)\Phi_{\#}^{\top}(1)\right)C\Big(\Psi(t) - \Psi(0)\Big)\right\},$$

where $\Phi_{\#\#}(x) := \int_0^x \Phi_{\#}(y)dy$. By using the collocation equations

$$R(x_i, t_j, \hat{v}) = 0, \quad i = 0, ..., M, \ j = 0, ..., N, \tag{21}$$

where t_j and x_i are the roots of the shifted Chebyshev polynomials [25–28] of the first kind of orders $N+1$ and $M+1$ defined over the intervals $[0,T]$ and $[0,1]$, respectively, we can obtain a linear system of algebraic equations, namely $Ac = q$, where the vector c includes the elements of c_{ij}. Tikhonov regularization method is applied to solve $Ac = q$ as follows:

$$c = \left(A^{\top}A + \lambda I\right)^{-1} A^{\top}q,$$

where $\lambda > 0$ is the regularization parameter [29, 30]. Finally, by denoting $G(t) := \frac{\int_0^1 v(x,t)dx}{E(t)}$, the approximation of $b(t)$ is obtained as:

$$b(t) = \frac{G'(t)}{G(t)} = \frac{E(t)\int_0^1 v_t(x,t)dx - E'(t)\int_0^1 v(x,t)dx}{E(t)\int_0^1 v(x,t)dx}, \tag{22}$$

provided that $\int_0^1 v(x,t)dx \neq 0, \ \forall t \in (0,T]$.

Following theorem represents the mean error bound for the approximation of an arbitrary element in $L^2(\Omega_T)$ via the OBBFs.

Theorem 2.1. *Assume that the function $z(x,t) : \Omega_T \longrightarrow \mathbb{R}$ is a sufficiently smooth function and $F = Span\{\phi_i(x)\psi_j(t), \quad i = 0, ..., M, \ j = 0, ..., N\}$ is a complete subspace of the Hilbert space $L^2(\Omega_T)$. Then, considering $\boldsymbol{P}_{M,N}$ as the best approximation to $z(x,t)$ out of F, the mean error bound is*

$$\|\boldsymbol{P}_{M,N} - z(x,t)\|_2 \leq \left(\frac{\theta_1}{(M+1)!2^{2M+1}} + \frac{\theta_2 T^{N+1}}{(N+1)!2^{2N+1}} + \frac{\theta_3 T^{N+1}}{(M+1)!(N+1)!2^{2M+2N+2}}\right)\sqrt{T},$$

where

$$\theta_1 = max_{[0,1]\times[0,T]}\left|\frac{\partial^{M+1}z(x,t)}{\partial x^{M+1}}\right|, \ \theta_2 = max_{[0,1]\times[0,T]}\left|\frac{\partial^{N+1}z(x,t)}{\partial t^{N+1}}\right|,$$

$$\theta_3 = max_{[0,1]\times[0,T]}\left|\frac{\partial^{M+N+2}z(x,t)}{\partial x^{M+1}\partial t^{N+1}}\right|.$$

Proof. See [31]. ■

Remark 1. Calculating the stable numerical derivative to functions that are contaminated with inaccurate data is a challenging problem. In this work, we apply the instructions suggested by [32] and consider $E(t)$ and $E_\xi(t)$ as the exact and perturbed functions such that

$$\frac{1}{M'}\sum_{p=1}^{M'}(E(t_p)-E_\xi(t_p))^2 \le \xi^2,$$

where $t_p \in [0,T]$ and ξ is the noise level of the collected data and M' represents the number of measured data namely, $E_\xi(t_p)$. It has been shown that the stable solution $E_{\lambda^*}(t)$ given by

$$E_{\lambda^*}(t)=\sum_{j=1}^{M'}\theta_j|t-t_j|^{2M''-1}+\sum_{j=1}^{M''}d_j t^{j-1}, \tag{23}$$

is the unique minimizer of the following problem:

$$\min_{E\in\Gamma_{M''}}\Omega(E)=\frac{1}{M'}\sum_{p=1}^{M'}(E(t_p)-E_\xi(t_p))^2+\lambda^*\|\frac{d^{M''}E}{dt^{M''}}\|_{L^2(\mathbf{R})}, \tag{24}$$

such that

$$\Gamma_{M''}=\{E|E\in C^{M''-1}(\mathbf{R}),\ E^{(M")}\in L^2(\mathbf{R})\},$$

and λ^* stands for the regularization parameter. Furthermore, the coefficients $\{\theta_j\}_{j=1}^{M'}$ and $\{d_j\}_{j=1}^{M''}$ satisfy the following system of equations:

$$E_{\lambda^*}(t_i)+2(2M''-1)!(-1)^{M''}\lambda^*M'\theta_i-E_\xi(t_i)=0,\quad i=1,...,M', \tag{25}$$

$$\sum_{j=1}^{M''}d_j t_j^i=0,\quad i=0,...,M''-1. \tag{26}$$

Parameter λ^* can be selected as $\lambda^*=\xi^2$ which is a priori rule [33]. Therefore, taking the natural number M', we set $M''=2$ in the linear system of Equations (23)-(26) and solve it to get $E_{\lambda^*}(t)$. Then, $\frac{d}{dt}\Big(E_{\lambda^*}(t)\Big)$ provides the approximation of $E'_\xi(t)$.

3 Numerical tests

Four numerical examples are solved in this section. We denote the absolute errors of $w(x,t)$ and $b(t)$ by $e(w)$ and $e(b)$, respectively, and the numerical simulations are implemented in the MATHEMATICA. The command *RandomReal*$[-1,1]$ is utilized for producing random numbers in the interval $[-1,1]$ and the system of equations is solved by LinearSolve.

Example 3.1. Consider the problem presented by Equations (2)-(5), such that the domain of the problem is $\Omega_1=[0,1]\times[0,1]$ and the specifications of the input data are given as follows:

$$s(x,t)=6x^2(x-1)(7x^2-8x+2)e^{-t-0.5t^2}, \tag{27}$$

$$\mu=-1,\ E(t)=\frac{1}{280}e^{-t-0.5t^2},\ w_0(x)=x^4(1-x)^3, \tag{28}$$

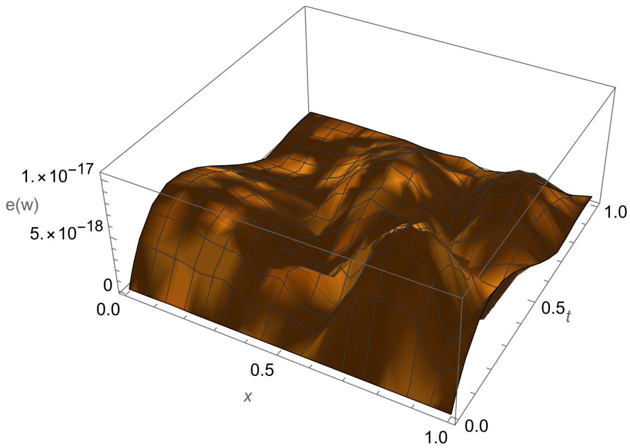

Figure 1: Graph of the absolute error for function $w(x,t)$ with $M = N = 3$, discussed in Example 3.1.

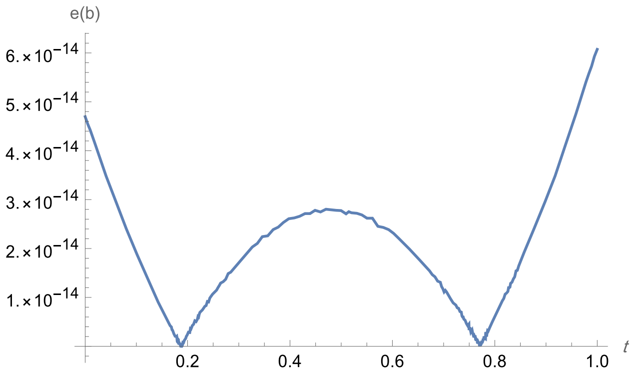

Figure 2: Graph of the absolute error for function $b(t)$ with $M = N = 3$, discussed in Example 3.1.

where the exact solutions are

$$b(t) = 1 + t, \quad w(x,t) = x^4(1-x)^3 e^{-t-0.5t^2}.$$

By using the numerical technique proposed in Section 2, such that

$$M = 3, \ N = 3, \ \lambda = 10^{-4}, \tag{29}$$

we find the results pictured in Figures 1 and 2 implying that the proposed method provides excellent and convergent approximations.

Example 3.2. Consider the inverse problem

$$w_t - w_{xx} + b(t)w = (3 + x - x^2)e^{-t}, \qquad (x,t) \in (0,1) \times (0,1], \tag{30}$$

with initial condition

$$w(x,0) = 1 + x - x^2, \quad 0 < x < 1, \tag{31}$$

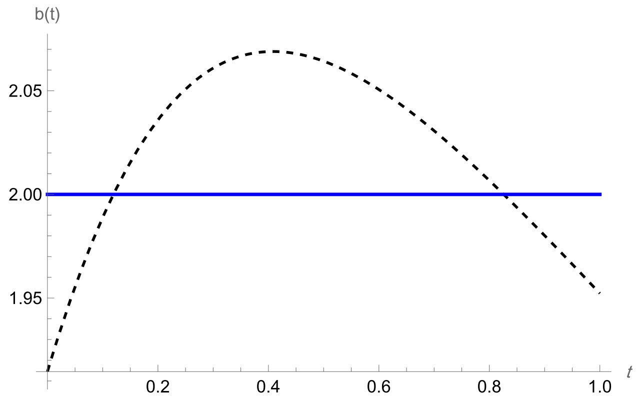

Figure 3: Graph of the exact (thick line) and approximate (dashed line) solutions for $b(t)$ with $M = N = 2$, discussed in Example 3.2.

Table 1: Computational findings for Example 3.2 with the exact boundary conditions.

(M, N)	$\|e(b)\|_2$	$\|e(w)\|_2$	$\|e(R(x,t,\hat{v}))\|_2$	λ
$(2, 2)$	0.04612	0.00018	0.083	0.000001
$(3, 3)$	0.004	0.000026	0.007	0.00001
$(4, 4)$	0.00025	1.58×10^{-6}	0.00044	0.00001

boundary conditions

$$w(0, t) = w(1, t), \quad w_x(0, t) - w(0, t) = 0, \quad 0 < t \leq 1, \tag{32}$$

and integral condition

$$\int_0^1 w(x, t)dx = \frac{7e^{-t}}{6}, \quad t \in (0, 1]. \tag{33}$$

We aim to estimate the functions $b(t) = 2$ and $w(x, t) = (1 + x - x^2)e^{-t}$ as the true solutions, by employing the OBBFs and using the numerical scheme described in Section 2 with

$$M = 2, \ N = 2, \ \lambda = 0.000001, \tag{34}$$

where accurate (free of noise) boundary and initial conditions are considered. Figures 3 to 5 depict the graphs of exact and approximate solutions as well as the absolute errors of unknowns $b(t)$ and $w(x, t)$. Furthermore, we increase the number of basis functions and produce the results shown in Table 1. Numerical findings indicate the convergence of the numerical solution with respect to the number of basis functions.

Example 3.3. Consider the problem given by Equations (2)-(6) defined over the domain $\Omega_1 = [0, 1] \times [0, 1]$ along with the following properties:

$$s(x, t) = \sinh(t)(x^3 - x - 1) + \cosh(t)(-6x + (\sin(t) - t^2)(x^3 - x - 1)), \tag{35}$$

$$\mu = -1, \ E(t) = \frac{-5\cosh(t)}{4}, \ w_0(x) = x^3 - x - 1. \tag{36}$$

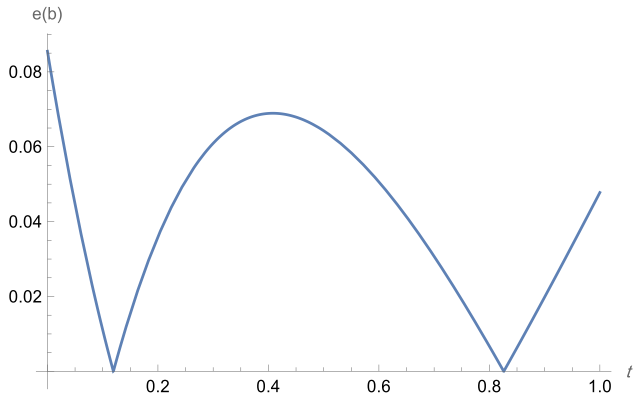

Figure 4: Graph of the absolute error for function $b(t)$ with $M = N = 2$, discussed in Example 3.2.

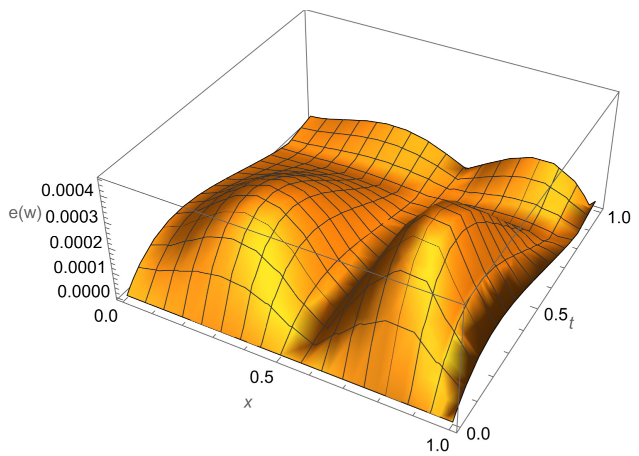

Figure 5: Graph of the absolute error for function $w(x,t)$ with $M = N = 2$, discussed in Example 3.2.

Table 2: Computational findings for Example 3.3 with the exact boundary conditions.

(M, N)	$\|e(b)\|_2$	λ
$(3, 3)$	0.0242	0.0001
$(4, 4)$	0.0064	0.0001

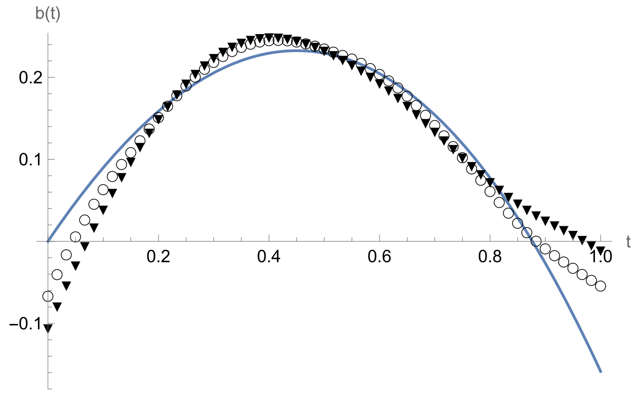

Figure 6: The blue curve represents the exact solution of $b(t)$ whilst the approximate solutions obtained by the suggested method are illustrated by ◦ ◦ ◦ when $\xi = 0.005$ and ▼▼▼ when $\xi = 0.01$.

The exact solutions of this problem are

$$b(t) = \sin(t) - t^2, \quad w(x, t) = (x^3 - x - 1)\cosh(t).$$

The inverse problem is solved by applying the proposed technique and the results of recovering the unknown function $b(t)$ when $M = N \in \{3, 4\}$ are shown in Table 2.

To observe the performance of the method in the presence of inaccurate input data, we consider the perturbed boundary conditions as follows [31, 34, 35]:

$$E_\xi(t_i) = E(t_i) + \xi \times RandomReal[-1, 1], \quad t_i \in [0, 1], \tag{37}$$

subject to $\xi \in \{5 \times 10^{-3}, 10^{-2}\}$ is known as the percentage of the introduced noise. Considering Remark 1, we address the problem by applying the following parameters

$$M = N = 4, \; M' = 25, \; M'' = 2, \; \lambda^* = \xi^2, \tag{38}$$

leading to the results illustrating the approximation of $b(t)$ as shown in Figure 6. When a small amount of noise is introduced, acceptable estimations are observed.

Example 3.4. As the last example, take the analytical solutions to the problem (2)-(5) as

$$\omega(x, t) = (12\pi + \sin(2\pi x))e^{-t}, \quad b(t) = e^t, \tag{39}$$

with the given data

$$\omega_0(x) = 12\pi + \sin(2\pi x), \; \mu = \frac{-1}{6}, \; E(t) = 12\pi e^{-t}, \tag{40}$$

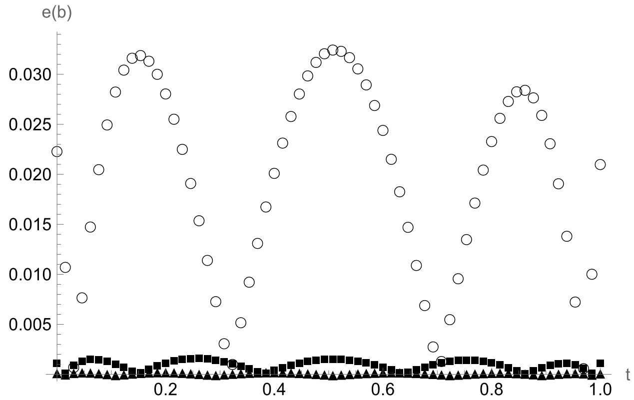

Figure 7: Graphs of the absolute error for function $b(t)$ with different number of basis functions and $\lambda = 10^{-2}$, discussed in Example 3.4: ($\circ\circ\circ$: when $M = N = 3$; ■■■: when $M = N = 6$; ▲▲▲: when $M = N = 7$).

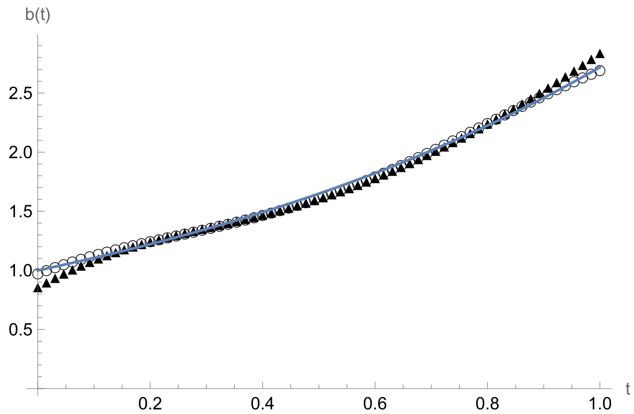

Figure 8: The blue curve represents the exact solution of $b(t)$ whilst the approximate solutions obtained by the suggested method are illustrated by $\circ\circ\circ$ when $\xi = 0$ and ▲▲▲ when $\xi = 0.01$.

$$s(x,t) = 12\pi + \sin(2\pi x) + 4e^{-t}\pi^2 \sin(2\pi x) - e^{-t}(12\pi + \sin(2\pi x)).$$

We use different numbers of basis functions and collocation points to compute the approximations of unknown functions when accurate input data (40) are employed. The results of the absolute error of $b(t)$ are represented in Figure 7. It can be observed that by increasing the bases, the approximations become closer to the analytical solutions. Regarding the noisy data, we use the rule (37) with $\xi \in \{0, 10^{-2}\}$ and consider Remark 1 along with the following parameters

$$M = N = 3,\ M' = 25,\ M'' = 2,\ \lambda^* = \xi^2, \tag{41}$$

and solve the problem to get the findings demonstrated by Figure 8. It is seen that the computed values closely align with the analytical solutions.

4 Concluding remarks

This paper explores the numerical solution of a semilinear inverse heat problem, aiming to approximate a time-dependent control function using integral overdetermination data alongside nonlocal periodic and convective boundary conditions. Initially, the problem is reformulated as a specific integro-differential equation, followed by the application of a spectral technique [25, 31, 34, 36–38] to discretize the modified formulation. Four numerical experiments are conducted, and the issue of numerical stability is addressed, particularly in the presence of slight noise added to the boundary condition (5). The results indicate that employing the proposed method leads to satisfactory outcomes: when exact input data are available, the unknown functions are accurately recovered, and even with noisy input data, the estimated values closely align with the analytical solutions.

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